SVPPLEMENTVM CALCVLI INTEGRALIS PRO INTEGRATIONE FORMVLARVM IRRATIONALIVM.

Auctore

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Problema I.

§. 1.

Si functio X *praeter ipsam variabilem* x *etiam formulam irrationalem* $s = \sqrt{(a + bx)}$ *inuoluat: ita tamen, vt* X *sit functio rationalis binarum quantitatum* x *et* s, *formulam differentialem* X d x *ab irrationalitate liberare.*

Solutio.

Cum irrationalitas tantum in formula $s = \sqrt{(a+bx)}$ insit: hanc tantum ita per idoneam substitutionem tolli

 oportet,

portet, vt inde valor ipsius x non fiat irrationalis. Hoc autem praestabitur, ponendo $a+bx=zz$, vt fiat $s=z$ et $x=\frac{zz-a}{b}$, hincque $dx=\frac{2}{b}z\,dz$; quibus valoribus substitutis tota formula differentialis $X\,dx$, ad rationalem, nouam variabilem z complectens, perducitur.

Exemplum 1.

§. 2. Si fuerit $dy=\frac{dx}{\sqrt{(a+bx)}}$, seu $dy=\frac{dx}{s}$, posito $\sqrt{(a+bx)}=z$, fiet $dy=\frac{2}{b}dz$, et integrando $y=\frac{2z}{b}$, vnde facta substitutione colligitur $y=\frac{2}{b}\sqrt{(a+bx)}+C$.

Exemplum 2.

§. 3. Si fuerit $dy=dx\sqrt{(a+bx)}=s\,dx$, sumto $\sqrt{(a+bx)}=z$ erit $dy=z\,dx=\frac{2}{b}zz\,dz$, vnde integrando fit $y=\frac{2}{3b}z^3$, et facta substitutione prodit

$$y=\frac{2}{3b}(a+bx)^{\frac{3}{2}}+C.$$

Quod integrale si debeat euanescere facto $x=0$, fiet $C=-\frac{2a\sqrt{a}}{3b}$, ideoque

$$y=\frac{2(a+bx)^{\frac{3}{2}}-2a\sqrt{a}}{3b}.$$

Exemplum 3.

§. 4. Si fuerit $dy=\frac{x\,dx}{\sqrt{(a+bx)}}$, facta substitutione $\sqrt{(a+bx)}=z$, erit

$$dy=\frac{2(zz-a)\,dz}{bb}=\frac{2zz\,dz-2a\,dz}{bb},$$

vnde

vnde fit integrando $y = \frac{2}{3bb} z^3 - \frac{2a}{bb} z + C$, et facta restitutione

$$y = \frac{2}{3bb}(a+bx)^{\frac{3}{2}} - \frac{2a}{bb}\sqrt{(a+bx)} + C$$
$$= \frac{2\sqrt{(a+bx)}}{bb}\left(\tfrac{1}{3}bx - \tfrac{2}{3}a\right) + C.$$

Exemplum 4.

§. 5. Si fuerit $dy = \frac{dx}{(a+bx)^{\frac{3}{2}}}$, facta substitutione $\sqrt{(a+bx)} = z$ erit $dy = \frac{dx}{z^3}$; quae formula porro ob $dx = \frac{2zdz}{b}$ abit in $dy = \frac{2dz}{bzz}$, qua integrata fit $y = -\frac{2}{bz}$, seu facta restitutione, $y = \frac{-2}{b\sqrt{(a+bx)}} + C$. Vbi notetur, pro C sumi debere $\frac{2}{b\sqrt{a}}$.

Problema 2.

§. 6. *Si fuerit* X *functio quaecunque rationalis binarum quantitatum* x *et* s, *existente* $s = \sqrt[3]{(a+bx)}$, *formulam differentialem* X dx *ab irrationalitate liberare.*

Solutio.

Ponatur $\sqrt[3]{(a+bx)} = z$, vt fit $s = z$, erit $a+bx = z^3$, hincque $x = \frac{z^3 - a}{b}$ et $dx = \frac{3zzdz}{b}$; quibus valoribus substitutis tota formula fiet rationalis.

Exemplum 1.

§. 7. Si fuerit

$dy = \frac{dx}{\sqrt[3]{(a+bx)}} = \frac{dx}{s}$, posito

 $\sqrt{(a}$

$\sqrt[3]{(a+bx)} = z$ et substituto valore hinc nato $dx = \frac{3zzdz}{b}$, erit $dy = \frac{3zdz}{b}$, vnde integrando fit

$$y = \frac{3}{2b}zz = \frac{3}{2b}\sqrt[3]{(a+bx)^2} + C.$$

Exemplum 2.

§. 8. Si fuerit

$$dy = \frac{dx}{\sqrt[3]{(a+bx)^2}} = \frac{dx}{ss},$$ posito

$\sqrt[3]{(a+bx)} = z$ fiet $dy = \frac{3dz}{b}$, hinc integrando

$$y = \frac{3}{b}z = \frac{3}{b}\sqrt[3]{(a+bx)} + C.$$

Exemplum 3.

§. 9. Si fuerit $dy = dx\sqrt[3]{(a+bx)} = s\,dx$, facta substitutione fit $dy = \frac{3z^3dz}{b}$, hinc integrando

$$y = \frac{3}{4b}z^4 = \frac{3}{4b}(a+bx)\sqrt[3]{(a+bx)} + C.$$

Problema 3.

§. 10. *Si fuerit* X *functio rationalis binarum quantitatum* x *et* s, *existente* $s = \sqrt[n]{(a+bx)}$, *formulam differentialem* X d x *ab irrationalitate liberare.*

Solutio.

Ponatur $\sqrt[n]{(a+bx)} = z$, vt fit $s = z$, erit $a+bx = z^n$, hinc $x = \frac{z^n - a}{b}$ et $dx = \frac{nz^{n-1}dz}{b}$; quibus valoribus

substi-

substitutis formula proposita $X\,dx$ certe fiet rationalis, si modo numerus exponentialis n fuerit integer.

Exemplum 1.

§. 11. Si fuerit

$$dy = \frac{dx}{\sqrt[n]{(a+bx)}} = \frac{dx}{s}, \text{ posito}$$

$\sqrt[n]{(a+bx)} = z$ ob valorem inde natum $dx = \frac{nz^{n-1}dz}{b}$ habebitur $dy = \frac{nz^{n-2}\,dz}{b}$; vnde integrando colligimus

$$y = \frac{n}{b\,(n-1)}\,z^{n-1} + C, \text{ siue restitutis valoribus}$$

$$y = \frac{n}{b\,(n-1)}\,(a+bx)^{\frac{n-1}{n}} + C = \frac{n}{b\,(n-1)}\,\frac{a+bx}{\sqrt[n]{(a+bx)}} + C.$$

Exemplum 2.

§. 12. Si fuerit

$$dy = \frac{dx}{\sqrt[n]{(a+bx)^{\lambda}}} = \frac{dx}{s^{\lambda}}, \text{ posito}$$

$\sqrt[n]{(a+bx)} = z$ et substituto valore $dx = \frac{nz^{n-1}dz}{b}$, fiet

$$dy = \frac{nz^{n-1}\,dz}{bz^{\lambda}} = \frac{n}{b}\,z^{n-\lambda-1}\,dz,$$

cuius integrale dat

$$y = \frac{n}{b\,(n-\lambda)}\,(a+bx)^{\frac{n-\lambda}{n}} + C, \text{ siue}$$

$$y = \frac{n}{b\,(n-\lambda)}\,\frac{a+bx}{\sqrt[n]{(a+bx)^{\lambda}}}.$$

Ex

Ex his autem exemplis iam apparet, integrationem non impediri, etiamsi exponentes n et λ non fuerint numeri integri.

Problema 4.

§. 13. *Si fuerit X functio rationalis binarum quantitatum x et s, existente* $s = \sqrt{(a + b\sqrt{(f + gx)})}$, *quae formula ergo duplicem irrationalitatem inuoluit, formulam differentialem X dx ab hac duplici irrationalitate liberare.*

Solutio.

Ponatur iterum $\sqrt{(a+b\sqrt{(f+gx)})} = z$, vt fit $s = z$, hinc erit sumtis quadratis $a + b\sqrt{(f + gx)} = zz$, hinc $b\sqrt{(f+gx)} = zz - a$: ac sumtis denuo quadratis

$$bb(f + gx) = (zz - a)^2,$$ vnde colligitur

$$x = \frac{(zz - a)^2}{bbg} - \frac{f}{g},$$ hincque

$$dx = \frac{4z\,dz(zz - a)}{bbg}.$$

Quibus valoribus substitutis tota formula reddetur rationalis.

Corollarium.

§. 14. Perspicuum est, eodem modo irrationalitatem tolli posse, si fuerit multo generalius

$$s = \sqrt[n]{(a + b\sqrt[m]{(f + gx)})}.$$

Posita enim hac formula $= z$, fiet

$$a + b\sqrt[m]{(f + gx)} = z^n \text{ et } b\sqrt[m]{(f + gx)} = z^n - a.$$

Porro $b^m(f + gx) = (z^n - a)^m$, et hinc colligitur

$x =$

$$x = \frac{(z^n - a)^m}{b^m g} - \frac{f}{g}, \text{ ideoque}$$

$$dx = \frac{m n z^{n-1} dz (z^n - a)^{m-1}}{b^m g}.$$

Sicque etiam hoc modo tota formula rationalis euadet.

Problema 5.

§. 15. *Si fuerit* X *functio rationalis binarum quantitatum* s *et* x, *existente* $s = \sqrt{\frac{a+bx}{f+gx}}$, *formulam differentialem* X dx *ab irrationalitate liberare.*

Solutio.

Ponatur $\sqrt{\frac{a+bx}{f+gx}} = z$, et sumtis quadratis erit $\frac{a+bx}{f+gx} = zz$, hincque $x = \frac{fzz - a}{b - gzz}$, vnde differentiando colligitur

$$dx = \frac{2bfz\,dz - 2agz\,dz}{(b-gzz)^2}.$$

Hisque valoribus substitutis formula proposita $X\,dx$ ad rationalitatem erit perducta.

Exemplum I.

§. 16. Si fuerit $dy = \frac{dx}{s} = \frac{dx\sqrt{(f+gx)}}{\sqrt{(a+bx)}}$, posito $\sqrt{\frac{a+bx}{f+gx}} = z$ erit $dy = \frac{dx}{z}$, et substituto loco dx valore supra inuento colligitur

$$dy = \frac{2(bf - ag)\,dz}{(b-gzz)^2};$$

quae formula, vti iam satis constat, reduci potest ad talem: $\int \frac{dz}{b-gzz}$, cuius autem integratio vel per logarithmos vel per arcus circulares expedietur.

Exemplum 2.

§. 17. Sit specialius $dy = \frac{dx\sqrt{(1-x)}}{\sqrt{(1+x)}}$, vbi $f = 1$, $g = -1$, $a = 1$ et $b = 1$, ideoque

$$z = \frac{\sqrt{(1+x)}}{\sqrt{(1-x)}} \text{ et } dx = \frac{4zdz}{(1+zz)^2};$$

quibus valoribus substitutis fiet $dy = \frac{4dz}{(1+zz)^2}$. Statuatur ergo:

$$\int \frac{4dz}{(1+zz)^2} = \frac{Az}{1+zz} + B\int \frac{dz}{1+zz} = y,$$

vnde sumtis differentialibus fiet:

$$\frac{4}{(1+zz)^2} = \frac{A-Azz}{(1+zz)^2} + \frac{B}{1+zz} = \frac{A+B+(B-A)zz}{(1+zz)^2}.$$

Oportet igitur esse $A + B = 4$ et $B - A = 0$, ideoque $A = 2$ et $B = 2$; et quia $\int \frac{dz}{1+zz} = A \text{ tang. } z$, adipiscimur $y = \frac{2z}{1+zz} + 2 A \text{ tang. } z$; quocirca facta restitutione, ob $1 + zz = \frac{2}{1-x}$ obtinebitur

$$y = \sqrt{(1-xx)} + 2 A \text{ tang. } \sqrt{\frac{1+x}{1-x}}.$$

Cum igitur huius arcus tangens sit $\sqrt{\frac{1+x}{1-x}}$, erit eius sinus $= \sqrt{\frac{1+x}{2}}$ et cosinus $= \sqrt{\frac{1-x}{2}}$; anguli vero dupli sinus erit $\sqrt{(1-xx)}$ et cosinus $= -x$, vnde fiet

$$2 A \text{ tang. } \sqrt{\frac{1+x}{1-x}} = A \text{ cos. } -x = \frac{\pi}{2} + A \text{ sin. } x;$$

quocirca integrale quaesitum erit

$$y = \sqrt{(1-xx)} + \frac{\pi}{2} + A \text{ sin. } x + C,$$

quod si ita capi debeat, vt euanescat posito $x = 0$, erit $C = -1 - \frac{\pi}{2}$, ideoque $y = \sqrt{(1-xx)} - 1 + A \text{ sin. } x$. Tum igitur, si sumatur $x = 1$, fiet $y = \frac{\pi}{2} - 1$, qui valor in fractionibus decimalibus dat 0, 5707963.

Pro-

Problema 6.

§. 18. *Si fuerit X functio rationalis binarum variabilium x et s, existente $s = \sqrt[n]{\frac{a+bx}{f+gx}}$, formulam differentialem* X d x *ad rationalitatem perducere.*

Solutio.

Posito $s = \sqrt[n]{\frac{a+bx}{f+gx}} = z$, erit $\frac{a+bx}{f+gx} = z^n$, hincque

$$x = \frac{fz^n - a}{b - gz^n}, \text{ consequenter } dx = \frac{n(bf-ag)z^{n-1}dz}{(b-gz)^{2n}};$$

hisque valoribus substitutis tota formula proposita Xdx ad rationalitatem erit perducta.

Problema 7.

§. 19. *Si fuerit X functio binarum quantitatum x et s, existente $s = \sqrt{(a+bxx)}$, formulam differentialem $\frac{Xdx}{x}$ ab irrationalitate liberare.*

Solutio.

Ponamus $s = \sqrt{(a+bxx)} = z$, erit $a+bxx = zz$, hinc $xx = \frac{zz-a}{b}$, et quia in functione X tantum quadratum xx, eiusque ergo potestates pares occurrunt: hac substitutione iam functio X euadet rationalis. Sumtis vero logarithmis $2lx = l(zz-a) - lb$, differentiando fit

$$\frac{2dx}{x} = \frac{2zdz}{zz-a}, \text{ ideoque } \frac{dx}{x} = \frac{zdz}{zz-a}.$$

Hoc ergo modo formula proposita $X . \frac{dx}{x}$ prorsus reddetur rationalis.

Exemplum 1.

§. 20. Si fuerit

$$dy = \frac{x\,dx}{\sqrt{(a+bxx)}},\ \text{erit}\ dy = \frac{dx}{x}\cdot\frac{xx}{\sqrt{(a+bxx)}} = \frac{xx}{s}\cdot\frac{dx}{x}.$$

Posito ergo $\sqrt{(a+bxx)} = z$ erit $dy = \frac{dz}{b}$, vnde colligitur integrando $y = \frac{z}{b} = \frac{\sqrt{(a+bxx)}}{b}$.

Exemplum 2.

§. 21. Si fuerit

$$dy = \frac{x^3\,dx}{\sqrt{(a+bxx)}} = \frac{dx}{x}\cdot\frac{x^4}{s},$$

ponendo $\sqrt{(a+bxx)} = z$, vt fit

$$xx = \frac{zz-a}{b}\ \text{et}\ \frac{dx}{x} = \frac{z\,dz}{zz-a},$$

erit $dy = \frac{1}{bb}\,dz\,(zz-a)$, hincque integrando adipiscimur $y = \frac{z}{3bb}(zz-3a)$; vnde facta restitutione prodibit integrale quaesitum $y = \frac{bxx-2a}{3bb}\sqrt{(a+bxx)} + C$.

Exemplum 3.

§. 22. Si fuerit

$$dy = \frac{x^3\,dx}{\sqrt{(a+bxx)^3}},\ \text{erit}\ dy = \frac{dx}{x}\cdot\frac{x^4}{s^3};$$

hinc posito

$$\sqrt{(a+bxx)} = s = z\ \text{fiet}\ dy = \frac{dz}{bb}\left(\frac{zz-a}{zz}\right),$$

vnde sumto integrali fiet $y = \frac{1}{bb}\left(\frac{zz+a}{z}\right)$, quocirca facta restitutione resultat $y = \frac{2a+bxx}{bb\sqrt{(a+bxx)}} + C$.

Problema 8.

§. 23. *Si fuerit* X *functio rationalis binarum quantitatum* x^n *et* s, *existente* $s = \sqrt[m]{(a+bx^n)}$, *formulam differentialem* $X\frac{dx}{x}$ *ad rationalitatem perducere.*

Solu-

Solutio.

Posito $s = \sqrt[m]{(a + b x^n)} = z$, fiet $a + b x^n = z^m$ et $x^n = \frac{z^m - a}{b}$. Quia igitur in functione X tantum potestas x^n occurrit, ea rationalis reddetur, si hi valores substituantur. Tum vero sumtis logarithmis habebitur

$$n\,l\,x = l\,(z^m - a) - l\,b,$$

et differentiando

$$\frac{d\,x}{x} = \frac{m\,z^{m-1}\,d\,z}{n\,(z^m - a)},$$

sicque tota formula proposita fiet rationalis.

Exemplum.

§. 24. Sit

$$d\,y = \frac{x^{n-1}\,d\,x}{\sqrt[m]{(a + b\,x^n)}} = \frac{d\,x}{x} \cdot \frac{x^n}{s},$$

factaque substitutione orietur haec aequatio:

$$d\,y = \frac{m\,z^{m-2}\,d\,z}{b\,n},$$

qua integrata prodibit

$$y = \frac{m\,z^{m-1}}{n\,b\,(m-1)} = \frac{m}{n\,b\,(m-1)} \sqrt[m]{(a + b\,x^n)^{m-1}} + C, \text{ siue}$$

$$y = \frac{m}{n\,b\,(m-1)} \; \frac{a + b\,x^n}{\sqrt[m]{(a + b\,x^n)}} + C.$$

Problema.

§. 25. *Si fuerit X functio rationalis quantitatum xx et s, existente $s = \sqrt{\frac{a+bxx}{f+gxx}}$, formulam differentialem $X\frac{dx}{x}$ ab irrationalitate liberare.*

Solutio.

Ponatur $s = \sqrt{\frac{a+bxx}{f+gxx}} = z$, eritque $\frac{a+bxx}{f+gxx} = zz$, hinc $xx = \frac{fzz-a}{b-gzz}$, vnde functio X penitus fit rationalis. Porro sumtis logarithmis

$$2\,lx = l(fzz-a) - l(b-gzz)$$

differentietur, vt prodeat

$$\frac{2dx}{x} = \frac{2fzdz}{fzz-a} + \frac{2gzdz}{b-gzz} = \frac{2(bf-ag)zdz}{(fzz-a)(b-gzz)},$$

vnde fit

$$\frac{dx}{x} = \frac{(bf-ag)zdz}{(fzz-a)(b-gzz)};$$

sicque tota formula differentialis fiet rationalis.

Exemplum.

§. 26. Si fuerit $dy = \frac{dx}{\sqrt{(f+gxx)}}$, repraesentemus hanc formulam ita:

$$dy = \frac{dx}{x}\frac{x}{\sqrt{(f+gxx)}} = \frac{dx}{x}\sqrt{\frac{xx}{f+gxx}}.$$

Hic ergo erit $a = 0$, $b = 1$, et

$$z = \frac{x}{\sqrt{f+gxx}}, \text{ ita vt } dy = \frac{zdz}{x};$$

erit autem

$$\frac{dx}{x} = \frac{dz}{z(1-gzz)}, \text{ vnde fit } dy = \frac{dz}{1-gzz},$$

cuius formulae integratio per logarithmos expedietur, si fuerit g numerus positiuus: sin autem fuerit negatiuus per arcus

arcus circulares absoluetur. Sit igitur 1°) $g = + bb$, erit

$$dy = \frac{dz}{1 - bbzz}, \text{ ideoque}$$

$$y = \frac{1}{2b} l \frac{1 + bz}{1 - bz};$$

et restitutis valoribus supra indicatis erit

$$y = \frac{1}{2b} l \left(\frac{\sqrt{(f + bbxx)} + bx}{\sqrt{(f + bbxx)} - bx}\right) = \frac{1}{b} l \frac{\sqrt{(f + bbxx)} + bx}{\sqrt{f}}.$$

Sit 2°) g quantitas negatiua, puta $g = - bb$, erit

$$dy = \frac{dz}{1 + bbzz} = \frac{1}{b} \cdot \frac{bdz}{1 + bbzz},$$

vnde colligitur

$$y = \frac{1}{b} \text{ A tang. } bz = \frac{1}{b} \text{ A tang. } \frac{bx}{\sqrt{(f - bbxx)}}.$$

Vbi manifestum est, f esse debere quantitatem positiuam, quia alioquin formula differentialis esset imaginaria.

Corollarium.

§. 27. Hinc ergo si proponatur formula

$$dy = \sqrt{(1 + xx)}, \text{ ubi } f = 1 \text{ et } g = 1,$$

ex casu priore ob $b = + 1$ erit

$$\int \frac{dx}{\sqrt{(1 + xx)}} = l\,(\sqrt{(1 + xx)} + x).$$

At si fuerit

$$dy = \frac{dx}{\sqrt{(1 - xx)}}, \text{ ubi } f = 1 \text{ et } g = - 1,$$

colligitur ex casu posteriore $y = \text{A tang. } \frac{x}{\sqrt{(1 - xx)}}$, vnde concluditur

$$\int \frac{dx}{\sqrt{(1 - xx)}} = \text{A sin. } x = \text{A cos.} \sqrt{(1 - xx)}.$$

Problema 10.

§. 28. *Si fuerit* X *functio rationalis quantitatum*

x^n

x^n *et* s, *existente* $s = \sqrt[n]{\left(\frac{a + b x^n}{f + g x^n}\right)}$, *formulam differentialem* $X \frac{dx}{x}$ *rationalem efficere.*

Solutio.

Ponatur $s = \sqrt[n]{\left(\frac{a + b x^n}{f + g x^n}\right)} = z$, eritque

$$\frac{a + b x^n}{f + g x^n} = z^n, \text{ hinc } x^n = \frac{f z^n - a}{b - g z^n},$$

tum autem sumtis logarithmis erit

$$n\, l\, x = l\,(f z^n - a) - l\,(b - g z^n)$$

et differentiando

$$\frac{dx}{x} = \frac{f z^{n-1} dz}{f z^n - a} + \frac{g z^{n-1} dz}{b - g z^n} = \frac{(bf - ag) z^{n-1} dz}{(f z^n - a)(b - g z^n)};$$

quibus valoribus substitutis formula proposita fit rationalis.

Problema II.

§. 29. *Si fuerit* X *functio rationalis binarum quantitatum* x^n *et* s, *existente* $s = \sqrt[m]{\left(\frac{a + b x^n}{f + g x^n}\right)}$, *formulam differentialem* $X \frac{dx}{x}$ *ab omni irrationalitate liberare.*

Solutio.

Statuatur $s = \sqrt[m]{\frac{a + b x^n}{f + g x^n}} = z$, eritque

$$\frac{a + b x^n}{f + g x^n} = z^m, \text{ vnde fit } x^n = \frac{f z^m - a}{b - g z^m};$$

hinc

hinc sumtis logarithmis erit

$$n\,l\,x = l\,(f\,z^m - a) - l\,(b - g\,z^m),$$

hinc differentiando

$$\frac{n\,d\,x}{x} = \frac{m\,(b\,f - a\,g)\,z^{m-1}\,d\,z}{(f\,z^m - a)\,(b - g\,z^m)},\ \text{ideoque}$$

$$\frac{d\,x}{x} = \frac{m\,(b\,f - a\,g)\,z^{m-1}\,d\,z}{n\,(f\,z^m - a)\,(b - g\,z^m)},$$

quibus valoribus substitutis irrationalitas formulae propositae penitus tollitur.

Problema 12.

§. 30. *Si fuerit* X *functio rationalis quaecunque binarum quantitatum* x *et* s, *existente* $s = \sqrt{(\alpha + \beta x + \gamma xx)}$, *formulam differentialem* X d x *ad rationalitatem perducere.*

Solutio.

Hic duos casus a se inuicem distingui conuenit, prout γ fuerit vel quantitas positiua vel negatiua.

I. Sit γ quantitas positiua, ac ponatur $\gamma = c\,c$ et $\beta = 2\,b\,c$, vt habeatur

$$s = \sqrt{(\alpha + 2\,b\,c\,x + c\,c\,x\,x)} = \sqrt{(\alpha - b\,b + (b + c\,x)^2)},$$

vbi loco $\alpha - b\,b$ breuitatis ergo scribatur e, vt sit

$$s = \sqrt{(e + (b + c\,x)^2)}.$$

Iam statuatur $s = b + c\,x + z$, eritque

$$s\,s = e + (b + c\,x)^2 = (b + c\,x)^2 + 2\,(b + c\,x)\,z + z\,z,$$

vnde sequitur

$$e - zz = 2z(b + cx), \text{ siue } b + cx = \frac{e - zz}{2z};$$

hincque colligitur

$$x = \frac{e - zz}{2cz} - \frac{b}{c}, \text{ seu } x = \frac{e - 2bz - zz}{2cz}.$$

Aequatio autem $b + cx = \frac{e - zz}{2z}$ differentiata praebet

$$c\,dx = -\frac{e\,dz}{2zz} - \frac{dz}{2} = -\frac{e\,dz - zz\,dz}{2zz},$$

vnde deducitur

$$dx = -\frac{dz\,(e + zz)}{2czz}, \text{ at ob}$$

$$b + cx = \frac{e - zz}{2z} \text{ fiet } s = \frac{e + zz}{2z}.$$

His ergo valoribus substitutis formula nostra $X\,dx$ reddetur rationalis. Postquam igitur eius integrale fuerit inventum loco z valor ante inuentus $\sqrt{(e + (b + cx)^2)} - b - cx$ erit substituendus.

II. Sin autem γ fuerit quantitas negatiua, ponatur $\gamma = -cc$ et $\beta = -2bc$, vt habeatur

$$s = \sqrt{(\alpha - 2bcx - ccxx)} = \sqrt{(\alpha + bb - (b + cx)^2)},$$

vbi euidens est, quantitatem $\alpha + bb$ necessario esse debere positiuam, quia alioquin s euaderet imaginarium. Quamobrem ponamus breuitatis gratia $\alpha + bb = aa$, vt fiat $s = \sqrt{(aa - (b + cx)^2)}$, ad quam formam rationalem efficiendam statuamus

$$\sqrt{(aa - (b + cx)^2)} = a - (b + cx)z;$$

vnde sumtis quadratis erit

$$aa - (b + cx)^2 = aa - 2az(b + cx) + (b + cx)^2 zz$$

quae aequatio reducitur ad hanc:

$$-(b + cx) = -2az + (b + cx)zz,$$

vnde reperitur

$$b + c x = \frac{2 a z}{1 + z z}, \text{ ideoque}$$

$$x = \frac{2 a z - b - b z z}{c (1 + z z)}.$$

Illa autem aequatio differentiata dat

$$c d x = \frac{2 a d z (1 + z z) - 4 a z z d z}{(1 + z z)^2} = \frac{2 a d z (1 - z z)}{(1 + z z)^2};$$

vnde fit

$$d x = \frac{2 a d z (1 - z z)}{c (1 + z z)^2}.$$

Porro autem, cum fit

$$s = a - (b + c x) z, \text{ ob } b + c x = \frac{2 a z}{1 + z z}$$

erit $s = \frac{a (1 - z z)}{1 + z z}$, quocirca, fi loco x, s et $d x$ inuenti hi valores fubftituantur, formula propofita differentialis $X d x$ euadet rationalis, et per variabilem z exprimetur, cuius integrale poftquam fuerit inuentum, loco z vbique eius reftituatur valor affumtus

$$z = a - \sqrt{(a a - (b + c x)^2)},$$

et integrale obtinebitur per folam variabilem x expreffum.

Exemplum I.

§. 31. Si fuerit

$$d y = \frac{d x}{\sqrt{(e + (b + c x)^2)}},$$

quae formula ad cafum priorem pertinet, erit

$$d y = \frac{d x}{s} = - \frac{d z}{c z}, \text{ ob } d x = - \frac{d z (e + z z)}{2 e z z} \text{ et } s = \frac{e + z z}{2 z},$$

cuius integrale eft $y = - \frac{1}{c} l z$; reftituto ergo valore

$$z = \sqrt{(e + (b + c x)^2)} - b - c x, \text{ erit}$$

$$y = - \frac{1}{c} l (\sqrt{(e + (b + c x)^2)} - b - c x) + C,$$

quod integrale fi euanefcere debeat pofito $x = 0$, fiet

$$C = \frac{1}{c} l (\sqrt{(e + b b)} - b).$$

Corollarium.

§. 32. Si ponatur $b=0$ et $c=1$, siue

$$dy=\frac{dx}{\sqrt{(e+xx)}},$$ erit integrale

$$y=-l\left(\sqrt{(e+xx)}-x\right)+l\sqrt{e}=l\frac{\sqrt{e}}{\sqrt{(e+xx)}-x}$$

quae formula reducitur ad hanc:

$$y=l\frac{\sqrt{(e+xx)}+x}{\sqrt{e}}.$$

Cum vero porro fit

$$d.\sqrt{(e+xx)}=\frac{xdx}{\sqrt{(e+xx)}}$$ erit

$$\int\frac{xdx}{\sqrt{(e+xx)}}=\sqrt{(e+xx)}.$$

Si igitur hae duae formulae combinentur, habebitur ista integratio notatu digna:

$$\int\frac{A\,dx+B\,x\,dx}{\sqrt{(e+xx)}}=A\,l\frac{\sqrt{(e+xx)}+x}{\sqrt{e}}+B\sqrt{(e+xx)}.$$

Exemplum II.

§. 33. Sit $dy=\frac{dx}{\sqrt{(aa-(b+cx)^2)}}$, quae formula ad casum secundum est referenda, ita vt fit $dy=\frac{dx}{s}$. Cum igitur fit

$$dx=\frac{2a\,dz\,(1-zz)}{c(1+zz)^2}$$ et $s=\frac{a(1-zz)}{1+zz}$, erit

$$y=\frac{dx}{s}=\frac{2}{c}\cdot\frac{dz}{1+zz},$$

vnde fit integrando $y=\frac{2}{c}$ A tang. z.

Quia igitur est

$$z=\frac{a-\sqrt{(aa-(b+cx)^2)}}{b+cx},$$ erit

$$y=\frac{2}{c}\text{ A tang. }\frac{a-\sqrt{(aa-(b+cx)^2}}{b+cx}+C.$$

Corollarium.

§. 34. Sit igitur iterum $b=0$ et $c=1$, seu formu-

formula differentialis proposita $dy = \frac{dx}{\sqrt{(aa - xx)}}$, reperieturque

$$y = 2\,\text{A tang.}\,\frac{a - \sqrt{(aa - xx)}}{x} + C.$$

Quia igitur tangens huius arcus est $\frac{a - \sqrt{(aa - xx)}}{x}$, tangens dupli arcus erit $\frac{x}{\sqrt{(aa - xx)}}$, ita ut sit

$$y = \text{A tang.}\,\frac{x}{\sqrt{(aa - xx)}}:$$

huius autem arcus sinus erit $\frac{x}{a}$; sicque integrale quaesitum

$$\int \frac{dx}{\sqrt{(aa - xx)}} = \text{A sin.}\,\frac{x}{a}.$$

Quia porro

$$d.\sqrt{(aa - xx)} = -\frac{x\,dx}{\sqrt{(aa - xx)}},\ \text{erit}$$

$$\int \frac{x\,dx}{\sqrt{(aa - xx)}} = -\sqrt{(aa - xx)},$$

quocirca ista generalior conficitur integratio:

$$\int \frac{\text{A}\,dx + \text{B}\,x\,dx}{\sqrt{(aa - xx)}} = \text{A tang.}\,\frac{x}{a} - \text{B}\sqrt{(aa - xx)}.$$

Problema 13.

§. 35. *Si fuerit* V *functio rationalis binarum quantitatum* v^n *et* s, *existente* $s = \sqrt{(\alpha + \beta v^n + \gamma v^{2n})}$, *formulam differentialem* $V v^{n-1} dv$ *ab irrationalitate liberare.*

Solutio.

Ponatur $v^n = x$, erit

$$s = \sqrt{(\alpha + \beta x + xx)}\ \text{et}\ v^{n-1}\,dv = \frac{dx}{n};$$

hic ergo iam erit V functio rationalis binarum quantitatum x et s, existente

$$s = \sqrt{(\alpha + \beta x + \gamma xx)}$$

 et

et formula ab irrationalitate liberanda erit $\frac{V\,dx}{n}$; qui casus prorsus conuenit cum problemate praecedente, ideoque eandem habebit solutionem.

Scholion.

§. 36. Praecepta hactenus tradita ad omnes fere formulas differentiales, quae quidem adhuc tractari potuerunt, extenduntur. Interim tamen eiusmodi casus occurrere possunt, quibus idonea substitutio, ad irrationalitatem tollendam necessaria, non tam facile perspicitur: sed acri iudicio demum inuestigare licet, in quo negotio cum praecepta generalia tradere nondum liceat, exempla quaedam particularia speciminis loco in medium afferamus.

Exemplum 1.

§. 37. *Si proposita fuerit haec formula irrationalis*: $dP = \frac{dx(1+xx)}{(1-xx)\sqrt{(1+x^4)}}$, *eius integrale* P *inuestigare.*

Si quis hic eiusmodi vti vellet substitutione, qua formula $\sqrt{(1+x^4)}$ ad rationalitatem perduceretur, oleum et operam esset perditurus, interim tamen singulari artificio sequens substitutio negotium conficere poterit. Statuatur $\frac{x\sqrt{2}}{1-xx} = p$, eritque $1+pp = \frac{1+x^4}{(1-xx)^2}$, hinc

$$\sqrt{1+pp} = \frac{\sqrt{1+x^4}}{1-xx};$$

tum vero erit differentiando

$$dp = \frac{dx\sqrt{2}(1+xx)}{(1-xx)^2},$$

ex quibus valoribus colligitur

$$\frac{dp}{\sqrt{1+pp}} = \frac{dx\sqrt{2}(1+xx)}{(1-xx)\sqrt{(1+x^4)}},$$

quae

quae feliciter cum formula ipsa proposita conuenit, ita vt sit

$$\frac{dp}{\sqrt{(1+pp)}} = dP\sqrt{2}, \text{ siue } dP = \frac{1}{\sqrt{2}} \frac{dp}{\sqrt{1+pp}},$$

vnde colligitur integrando

$$P = \frac{1}{\sqrt{2}} l(\sqrt{(1+pp)}+p).$$

Quare si loco p et $\sqrt{(1+pp)}$ valores dati substituantur, haec obtinetur integratio satis memorabilis:

$$P = \int \frac{dx(1+xx)}{(1-xx)\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}} l \frac{\sqrt{(1+x^4)}+\sqrt{2}}{1+xx}.$$

Exemplum 2.

§. 38. *Si proposita fuerit haec formula irrationalis:* $\frac{dx(1-xx)}{(1+xx)\sqrt{(1+x^4)}}$, *eius integrale* Q *inuestigare.*

Ad hoc praestandum fiat $\frac{x\sqrt{2}}{1+xx} = q$, eritque

$$\sqrt{(1-qq)} = \frac{\sqrt{(1+x^4)}}{1+xx};$$

tum vero erit $dq = \frac{dx(1-xx)\sqrt{2}}{(1+xx)^2}$, atque hinc colligitur

$$\frac{dq}{\sqrt{(1-qq)}} = \frac{dx(1-xx)\sqrt{2}}{(1+xx)\sqrt{(1+x^4)}} = dQ\sqrt{2},$$

vnde fit

$$Q = \frac{1}{\sqrt{2}} \int \frac{dq}{\sqrt{(1-qq)}} = \frac{1}{\sqrt{2}} \text{ A sin. } q.$$

Restituto ergo pro q valore assumto ista obtinebitur integratio:

$$Q = \int \frac{dx(1-xx)}{(1+xx)\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}} \text{ A sin. } \frac{x\sqrt{2}}{1+xx}.$$

Scholion.

§. 39. Cum istae duae formulae:

$$\frac{dx(1+xx)\sqrt{2}}{(1-xx)\sqrt{(1+x^4)}} \text{ et } \frac{dx(1-xx)\sqrt{2}}{(1+xx)\sqrt{(1+x^4)}},$$

per-

perductae sint ad has simplices:

$$\frac{dp}{\sqrt{(1+pp)}} \text{ et } \frac{dq}{\sqrt{(1-qq)}};$$

quarum vtraque facile ab irrationalitate liberatur, istae ipsae formulae propositae ope idoneae substitutionis ab irrationalitate liberari possunt; vnde mirum non est, earum integralia siue per logarithmum siue per arcum circularem exhiberi potuisse. Satis enim iam est ostensum: omnium formularum differentialium rationalium integralia semper vel per logarithmos et arcus circulares, vel adeo algebraice exhiberi posse; quod igitur etiam de illis formulis irrationalibus est tenendum, quas certae substitutionis ope ad rationalitatem perducere licet. Vnde vicissim plures Geometrae concluserunt: si quae formula differentialis nullo plane modo ab irrationalitate liberari queat, tum eius integrale etiam neque per logarithmos nec arcus circulares, multo minus algebraice exprimi posse, sed ad aliud genus quantitatum transcendentium referri oportere. Ceterum combinatio duorum praecedentium exemplorum manuducit ab solutionem sequentium.

Exemplum 3.

§. 40. *Si proposita fuerit haec formula differentialis*: $dy = \frac{dx\sqrt{(1+x^4)}}{1-x^4}$, *eius integrale inuenire.*

Hanc formulam per neutram substitutionem ante vsurpatam rationalem reddere licet: vtraque tamen iuncta negotium confici poterit; namque eius integrale per logarithmos et arcus circulares sequenti artificio expedietur: Formula enim proposita in binas sequentes partes discerpi potest, quae sunt

dy

$$dy = \frac{\frac{1}{2} dx (1 + xx)}{(1 - xx) \sqrt{(1 + x^4)}} + \frac{\frac{1}{2} dx}{(1 + xx) \sqrt{(1 + x^4)}},$$

quippe quarum ſumma ipſam formulam noſtram propoſitam producit; prodit enim

$$dy = \frac{\frac{1}{2} dx (1 + xx)^2 + \frac{1}{2} dx (1 - xx)^2}{(1 - x^4) \sqrt{(1 + x^4)}}$$

$$= \frac{dx (1 + x^4)}{(1 - x^4) \sqrt{(1 + x^4)}} = \frac{dx \sqrt{(1 + x^4)}}{1 - x^4}.$$

Quod ſi ergo duo praecedentia exempla in ſubſidium vocentur, manifeſto fiet $dy = \frac{1}{2} dP + \frac{1}{2} dQ$, conſequenter integrale quaeſitum erit $y = \frac{1}{2} P + \frac{1}{2} Q$, quod ſequenti modo exprimere licebit:

$$\int \frac{dx \sqrt{(1 + x^4)}}{1 - x^4} = \frac{1}{2\sqrt{2}} l \frac{\sqrt{(1 + x^4)} + x\sqrt{2}}{1 - xx} + \frac{1}{2\sqrt{2}} \text{A ſin.} \frac{x\sqrt{2}}{1 + xx}.$$

Exemplum. 4.

§. 41. *Si propoſita fuerit haec formula differentialis:* $dy = \frac{xx\,dx}{(1 - x^4) \sqrt{(1 + x^4)}}$, *eius integrale inueſtigare.*

Haec formula ſimili modo ac praecedens tractari poteſt; diſcerpatur enim in ſequentes duas partes:

$$\frac{\frac{1}{4} dx (1 + xx)}{(1 - xx) \sqrt{(1 + x^4)}} - \frac{\frac{1}{4} dx (1 - xx)}{(1 + xx) (\sqrt{1 + x^4})},$$

quippe quae coniunctae producunt

$$dy = \frac{\frac{1}{4} dx (1 + xx)^2 - \frac{1}{4} dx (1 - xx)^2}{(1 - x^4) \sqrt{(1 + x^4)}}$$

$$= \frac{\frac{1}{4} dx. 4xx}{(1 - x^4) \sqrt{(1 + x^4)}} = \frac{xx\,dx}{(1 - x^4) \sqrt{(1 + x^4)}},$$

quae cum sit ipsa formula proposita, erit ex praecedentibus exemplis $dy = \frac{1}{4} dP - \frac{1}{4} dQ$, consequenter $y = \frac{1}{4} P - \frac{1}{4} Q$, hinc integrale quaesitum ita reperietur expressum:

$$\int \frac{xx\,dx}{(1-x^4)\sqrt{(1+x^4)}} = \frac{1}{4\sqrt{2}}\, l\, \frac{\sqrt{(1+x^4)}+x\sqrt{2}}{1-xx} - \frac{1}{4\sqrt{2}}\, A\, \sin. \frac{x\sqrt{2}}{1+xx}.$$

Scholion.

§. 42. Haec duo postrema exempla si nullo plane modo ope cuiuspiam substitutionis ad rationalitatem perduci possent, insigne praeberent documentum, quod conclusio supra memorata quandoque fallere possit: Re autem attentius perpensa inueni, omnia haec quatuor exempla ope vnica substitutione immediate ad rationalitatem perduci ideoque integrari posse; id quod ostendisse vtique operae erit pretium.

Alia resolutio quatuor postremorum exemplorum.

§. 43. Statuatur pro primo exemplo

$$v = \frac{x\sqrt{2}}{\sqrt{(1+x^4)}}, \text{ eritque } \sqrt{(1+vv)} = \frac{1+xx}{\sqrt{(1+x^4)}},$$

tum vero $\sqrt{(1-vv)} = \frac{1-xx}{\sqrt{(1+x^4)}}$, vnde fit

$$\sqrt{\frac{1+vv}{1-vv}} = \frac{1+xx}{1-xx} \text{ et } \sqrt{(1-v)^4} = \frac{1-x^4}{1+x^4}.$$

At differentiando adipiscimur

$$dv = \frac{dx(1-x^4)\sqrt{2}}{(1+x^4)\sqrt{(1+x^4)}}.$$

Cum nunc sit $\frac{1-x^4}{1+x^4} = \sqrt{(1-v^4)}$; erit

$$dv = \frac{dx\sqrt{2}\sqrt{(1-v^4)}}{\sqrt{(1+x^4)}}, \text{ siue } \frac{dv}{\sqrt{(1-v^4)}} = \frac{dx\sqrt{2}}{\sqrt{(1+x^4)}};$$

quae aequalitas maxime est notatu digna. Quod si iam haec

haec aequatio multiplicetur per $\sqrt{\frac{1+vv}{1-vv}} = \frac{1+xx}{1-xx}$, nascetur haec aequatio:

$$\frac{dv}{1-vv} = \frac{dx(1+xx)\sqrt{2}}{(1-xx)(1+x^4)},$$

sicque erit

$$\int \frac{dx(1+xx)}{(1-xx)\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}} \int \frac{dv}{1-vv} = \frac{1}{2\sqrt{2}} l \frac{1+v}{1-v}.$$

Deinde aequatio

$$\frac{1}{\sqrt{2}} \cdot \frac{dv}{\sqrt{(1-v^4)}} = \frac{dx}{\sqrt{(1+x^4)}}$$

multiplicetur per

$$\sqrt{\frac{1-vv}{1+vv}} = \frac{1-xx}{1+xx},$$

ac prodibit formula exempli secundi

$$\int \frac{dx(1-xx)}{(1+xx)\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}} \int \frac{dv}{1+vv} = \frac{1}{\sqrt{2}} \text{A tang. } v.$$

Porro eadem aequatio

$$\frac{1}{\sqrt{2}} \cdot \frac{dv}{\sqrt{(1-v^4)}} = \frac{dx}{\sqrt{(1+x^4)}}$$

diuidatur per

$\sqrt{(1-v^4)} = \frac{1-x^4}{1+x^4}$ et prodibit

$$\frac{1}{\sqrt{2}} \cdot \frac{dv}{1-v^4} = \frac{dx\sqrt{(1+x^4)}}{1-x^4};$$

quae est ipsa formula exempli tertii, ita vt iam sit

$$\int \frac{dx\sqrt{(1+x^4)}}{1-x^4} = \frac{1}{\sqrt{2}} \int \frac{dv}{1-v^4} = \frac{1}{2\sqrt{2}} \int \frac{dv}{1+vv} + \frac{1}{2\sqrt{2}} \int \frac{dv}{1-vv},$$

quod integrale cum ante inuento egregie conuenit. Tandem postrema aequatio hic inuenta:

$$\frac{1}{\sqrt{2}} \cdot \frac{dv}{1-v^4} = \frac{dx\sqrt{(1+x^4)}}{1-x^4}$$

ducatur in $vv = \frac{2xx}{1+x^4}$, vt prodeat

$$\frac{1}{\sqrt{2}} \cdot \frac{vvdv}{1-v^4} = \frac{2xxdx\sqrt{(1+x^4)}}{(1-x^4)(1+x^4)} = \frac{2xxdx}{(1-x^4)\sqrt{(1+x^4)}},$$

vnde pro exemplo quarto colligitur

$$\int \frac{xxdx}{(1-x^4)\sqrt{(1+x^4)}} = \frac{1}{2\sqrt{2}} \int \frac{vvdv}{1-v^4} = -\frac{1}{4\sqrt{2}} \int \frac{dv}{1+vv} + \frac{1}{4\sqrt{2}} \int \frac{dv}{1-vv},$$

 vnde

vnde cum sit $v = \frac{x\sqrt{2}}{\sqrt{(1+x^4)}}$, erit

$$\int \frac{dv}{1-vv} = \frac{1}{2}\, l\, \frac{1+v}{1-v} = \frac{1}{2}\, l\, \frac{\sqrt{(1+x^4)}+x\sqrt{2}}{\sqrt{(1+x^4)}-x\sqrt{2}}$$
$$= \frac{1}{2}\, l\, \frac{(\sqrt{(1+x^4)}+x\sqrt{2})^2}{(1-xx)^2} = l\, \frac{\sqrt{(1+x^4)}+x\sqrt{2}}{1-xx}.$$

Deinde vero est

$$\int \frac{dv}{1+vv} = \text{A tang.}\, v = \text{A sin.}\, \frac{v}{\sqrt{(1+vv)}} = \text{A sin.}\, \frac{x\sqrt{2}}{1+xx}.$$

Scholion.

§. 44. Quanquam autem haec quatuor exempla ad rationalitatem reducere licuit: tamen conclusio supra memorata, quod omnes formulae integrales, quae nullo modo rationales effici queant, ad aliud pertineant transcendentium genus, neque per solos logarithmos et arcus circulares expediri possint, non solum manet suspecta, sed etiam falsitas eius euidenter ob oculos poni potest. Sit enim functio

$$X = \frac{a}{\sqrt{(1+xx)}} + \frac{b}{\sqrt[3]{(1+x^3)}} + \frac{c}{\sqrt[4]{(1+x^4)}};$$

tum certe formula differentialis $X\,dx$ nullo modo ad rationalitatem perduci poterit; interim tamen singulos eius partes

$$\frac{a\,dx}{\sqrt{(1+xx)}},\quad \frac{b\,dx}{\sqrt[3]{(1+x^3)}} \quad\text{et}\quad \frac{c\,dx}{\sqrt[4]{(1+x^4)}}$$

seorsim rationales effici et integralia per logarithmos et arcus circulares exhiberi possunt. Coronidis loco hic sequens problema notatu dignum adiungamus.

Pro-

Problema.

§. 45. *Formularum integralium* $\int \frac{dx}{\sqrt{(1+x^4)}}$ *et* $\int \frac{dv}{\sqrt{(1+v^4)}}$ *valores per series inuestigare, pro casibus, quibus ponitur tam* $v = 1$ *quam* $x = 1$.

Solutio.

Cum posito $v = \frac{x\sqrt{2}}{\sqrt{(1+x^4)}}$, vt supra fecimus, euidens sit, sumto $x = 0$ fore etiam $v = 0$, et sumto $x = 1$ fore $v = 1$, ita vt hae duae quantitates x et v simul euanescant et simul vnitati aequentur: hinc deducimus istam aequationem differentialem attentione dignissimam:

$$\frac{1}{\sqrt{2}} \cdot \frac{dv}{\sqrt{(1-v^4)}} = \frac{dx}{\sqrt{(1+x^4)}},$$

quas ergo ambas formulas in series conuerti oportet; erit autem

$$\frac{1}{\sqrt{(1-v^4)}} = (1-v^4)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}v^4 + \tfrac{1.3}{2.4}v^8 + \tfrac{1.3.5}{2.4.6}v^{12} + \text{etc. et}$$

$$\frac{1}{\sqrt{(1+x^4)}} = (1+x^4)^{-\frac{1}{2}} = 1 - \tfrac{1}{2}x^4 + \tfrac{1.3}{2.4}x^8 - \tfrac{1.3.5}{2.4.6}x^{12} + \text{etc.}$$

Illa iam per dv multiplicata et integrata praebet

$$\int \frac{dv}{\sqrt{(1-v^4)}} = v + \tfrac{1}{2.5}v^5 + \tfrac{1.3}{2.4.9}v^9 + \tfrac{1.3.5}{2.4.6.13}v^{13} + \text{etc.}$$

vnde posito $v = 1$ valor huius integralis erit

$$1 + \tfrac{1}{2.5} + \tfrac{1.3}{2.4.9} + \tfrac{1.3.5}{2.4.6.13} + \tfrac{1.3.5.7}{2.4.6.8.17} + \text{etc.}$$

quam seriem littera A indicemus. Simili modo altera series in dx ducta et integrata producit

$$\int \frac{dx}{\sqrt{(1+x^4)}} = x - \tfrac{1}{2.5}x^5 + \tfrac{1.3}{2.4.9}x^9 - \tfrac{1.3.5}{2.4.6.13}x^{13} + \text{etc.}$$

 cuius

cuius valor facto $x = 1$ erit

$$1 - \frac{1}{2.5} + \frac{1.3}{2.4.9} - \frac{1.3.5}{2.4.6.13} + \frac{1.3.5.7}{2.4.6.8.17} - \text{etc.}$$

quem littera B designemus, ita vt sit $B = \frac{A}{\sqrt{2}}$, siue $A = B\sqrt{2}$; vnde patet, priorem seriem se habere ad posteriorem vt $\sqrt{2} : 1$.

Scholion.

§. 46. Valor formulae integralis $\int \frac{dv}{\sqrt{(1-v^4)}}$ etiam hoc modo per seriem inuestigari potest. Cum sit

$$\frac{1}{\sqrt{(1-v^4)}} = \frac{(1+vv)^{-\frac{1}{2}}}{\sqrt{(1-vv)}} \text{ et}$$

$$(1+vv)^{-\frac{1}{2}} = 1 - \tfrac{1}{2} vv + \tfrac{1.3}{2.4} v^4 - \tfrac{1.3.5}{2.4.6} v^6 + \text{etc.}$$

notetur esse $\int \frac{dv}{\sqrt{(1-vv)}} = \frac{\pi}{2}$. Deinde pro integratione reliquorum terminorum ponatur

$$\int \frac{v^{n+2}\,dv}{\sqrt{(1-vv)}} = A v^{n+1} \sqrt{(1-vv)} + B \int \frac{v^n\,dv}{\sqrt{(1-vv)}},$$

quae aequatio differentiata dat

$$\frac{v^{n+2}}{\sqrt{(1-vv)}} = (n+1) A v^n \sqrt{1-vv} - \frac{A v^{n+2}}{\sqrt{(1-vv)}} + \frac{B v^n}{\sqrt{(1-vv)}},$$

vnde per $\sqrt{(1-vv)}$ multiplicando prodit

$$v^{n+2} = (n+1) A v^n - (n+1) A v^{n+2} - A v^{n+2} + B v^n.$$

Hinc termini in quibus inest v^{n+2} inter se aequati praebent $1 = -(n+2) A$, ideoque $A = -\frac{1}{n+2}$; termini vero v^n continentes praebent $0 = (n+1) A + B$, vnde fit $B = \frac{n+1}{n+2}$, ita vt in genere fit

$$\int \frac{v^{n+2}\,dv}{\sqrt{(1-vv)}} = -\frac{1}{n+2} v^{n+1} \sqrt{(1-vv)} + \frac{n+1}{n+2} \int \frac{v^n\,dv}{\sqrt{(1-vv)}},$$

quod

quod integrale vti requiritur euanefcit pofito $v = 0$. Ponatur nunc $v = 1$, eritque:

$$\int \frac{v^{n+2}\,dv}{\sqrt{(1 - vv)}} = \frac{n+1}{n+2} \int \frac{v^n\,dv}{\sqrt{(1 - vv)}};$$

hinc ergo pro n fcribendo fucceffiue valores 0, 2, 4, 6, 8, etc. erit:

I. $\int \frac{vv\,dv}{\sqrt{(1 - vv)}} = \frac{1}{2} \cdot \frac{\pi}{2},$

II. $\int \frac{v^4\,dv}{\sqrt{(1 - vv)}} = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2},$

III. $\int \frac{v^6\,dv}{\sqrt{(1 - vv)}} = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2},$

etc. etc.

quibus valoribus adhibitis erit cafu $v = 1$

$$\int \frac{dv}{\sqrt{(1 - v^4)}} = \frac{\pi}{2} - \frac{1}{2^2} \cdot \frac{\pi}{2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \cdot \frac{\pi}{2} - \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \cdot \frac{\pi}{2} + \text{etc.}$$

$$= \frac{\pi}{2}\left(1 - \frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} - \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \text{etc.}\right)$$

ita vt fit ex problemate praecedente

$$1 - \frac{1}{2 \cdot 5} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 9} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 9 \cdot 13} - \text{etc.}$$

$$= \frac{\pi}{2}\left(1 - \frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} - \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \text{etc.}\right)$$

vnde fit

$$\frac{\pi}{2} = \frac{1 - \frac{1}{2 \cdot 5} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 9} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 9 \cdot 13} + \text{etc.}}{1 - \frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} - \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \text{etc.}}$$

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