NOVA METHODVS FRACTIONES QVASCVNQVE RATIONALES IN FRACTIONES SIMPLICES RESOLVENDI.

Auctore

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§. 1.

Sit $\frac{P}{Q}$ fractio quaecunque propofita, cuius tam numerator P quam denominator Q fint functiones rationales integrae quantitatis variabilis z, denominator autem Q fit productum ex quotcunque factoribus fimplicibus formae $z \pm a$, fiue aequalibus fiue inaequalibus inter fe: et notum eft, iftam fractionem femper refolui poffe in fractiones fimplices, quarum denominatores finguli formentur ex factoribus ipfius Q, numeratores vero fint quantitates conftantes, fiquidem variabilis z in numeratore P ad pauciores dimenfiones affurgat quam in denominatore Q, quoniam aliter praeter iftas fractiones fimplices infuper partes integrae effent adiiciendae. Qui cafus cum nulla laboret difficultate, propterea quod partes iftae integrae facile reperiuntur, dum numerator P per denominatorem Q actu diuiditur, fufficiet eiusmodi tantum fractiones confideraffe, in quarum denominatore Q variabilis z ad altiores poteftates afcendit quam in numeratore P. Tum igitur,

igitur, quando pro singulis factoribus denominatoris Q inventae fuerint fractiones simplices ipsis respondentes, summa omnium harum fractionum aequabitur fractioni propositae $\frac{P}{Q}$. Primus quidem in Introductione mea ad Analysin infinitorum methodum tradidi satis simplicem, cuius ope omnes istae fractiones partiales pro singulis denominatoris factoribus reperiri queant, sine vllo respectu ad reliquas habito, quarum ratio antehac teneri debebat. Postmodum vero istam methodum magis excolui, et quemadmodum ope calculi differentialis facilius ad quosuis vsus accommodari possit, vberius ostendi. Nunc autem penitus noua Idea sese mihi obtulit eandem resolutionem perficiendi, quae plerumque negotium non mediocriter subleuare videtur. Imprimis autem ad functiones transcendentes mira facilitate accommodari potest, vnde operae pretium fore existimo, si istam nouam methodum adcuratius exposuero.

§. 2. Sit igitur $z-a$ factor simplex denominatoris Q, siue solitarius, siue quotcunque sibi aequales admittens. Ac priori casu inde fractio simplex oriunda erit $\frac{\alpha}{z-a}$. Sin autem denominator binos huiusmodi factores aequales inuoluat, scilicet $(z-a)^2$, tum resolutio binas dabit fractiones simplices $\frac{\alpha}{(z-a)^2}+\frac{\beta}{z-a}$; at si factorem habeat cubicum $(z-a)^3$, fractiones simplices inde ortae erunt $\frac{\alpha}{(z-a)^3}+\frac{\beta}{(z-a)^2}+\frac{\gamma}{z-a}$, et ita porro pro altioribus potestatibus. Totum igitur negotium huc redit, vt pro singulis huiusmodi factoribus numeratores α, β, γ, etc. definiantur, pro qua inuestigatione iam olim praecepta dedi.

dedi. Noua autem methodus, quam hic sum traditurus, huic innititur principio, quod posito $z = a$ omnes istae fractiones partiales euadant infinitae, dum reliquae omnes manent finitae magnitudinis, ideoque prae illis quasi euanescant. Hinc si in fractione proposita $\frac{P}{Q}$ statuatur $z = a$, ea vtique etiam in infinitum excrescet, eiusque valor debite euolutus praebebit ipsas illas fractiones simplices in infinitum abeuntes, id quod hic accuratius sum persecuturus.

§. 3. Ne igitur hic consideratio infiniti moram facessat, statuamus non $z - a = 0$, sed $z - a = \omega$, denotante ω quantitatem infinite paruam atque adeo ipsam euanescentem, ac ponamus tam in numeratore P quam in denominatore Q vbique $z = a + \omega$, quo facto numerator P euoluatur in huiusmodi formam:

$$P = A + B\omega + C\omega\omega + D\omega^3 + \text{etc.}$$

denominator vero Q, quia per hypothesin euanescit posito $z = a$, talem induet formam:

$$Q = \mathfrak{A}\omega + \mathfrak{B}\omega\omega + \mathfrak{C}\omega^3 + \mathfrak{D}\omega^4 + \text{etc.}$$

vbi si factor $z - a$ fuerit solitarius, primus terminus $\mathfrak{A}\omega$ necessario aderit. Sin autem denominator Q factorem habeat $(z - a)^2$, erit $\mathfrak{A} = 0$, ac denominator a termino $\mathfrak{B}\omega^2$ incipiet. Quodsi vero denominatoris factor fuerit $(z - a)^3$, primus terminus in denominatore erit $\mathfrak{C}\omega^3$, et ita porro, ita vt si in genere factor fuerit $(z - a)^n$, infima potestas in denominatore sit $\mathfrak{N}\omega^n$.

§. 4. Haec quidem substitutio, ponendo $z = a + \omega$, operationes tantum vulgares algebraicas postulat: interim tamen

tamen per nota principia differentialium mirifice fubleuari poteft. Nam fi in genere loco z fcribatur $z + \omega$, functio ipfius z quaecunque P accipiet iftum valorem:

$$P + \frac{\omega\, dP}{1 . dz} + \frac{\omega\omega\, ddP}{1 . 2\, dz^2} + \frac{\omega^3\, d^3P}{1 . 2 . 3\, dz^3} + \frac{\omega^4\, d^4P}{1 . 2 . 3 . 4\, dz^4} + \text{etc.}$$

Hoc igitur modo ftatim forma tam numeratoris P quam denominatoris Q fecundum poteftates ipfius ω difpofita reperietur, tantumque opus eft, vt in fingulis terminis loco z vbique fcribatur a. Quousque autem iftas expreffiones per poteftates ipfius ω continuari oporteat, ex primo denominatoris termino, feu infima poteftate ipfius ω facile diiudicabitur, vnde fequentes cafus euoluamus.

Cafus I.

Quo denominatoris Q factor eft $z - a$.

§. 5. Hoc igitur cafu non erit $\mathfrak{A} = 0$, vnde fractio noftra $\frac{P}{Q}$, facto $z = a + \omega$, induet hanc formam

$$\frac{1}{\omega} \cdot \frac{A + B\omega + C\omega^2 + D\omega^3 + etc.}{\mathfrak{A} + \mathfrak{B}\omega + \mathfrak{C}\omega^2 + \mathfrak{D}\omega^3 + etc.}$$

vbi fractio ifta per diuifionem euoluatur tantum vsque ad primam poteftatem ω, propterea quod in fractione praefixa $\frac{1}{\omega}$ haec littera vnicam tantum habet dimenfionem, vnde hoc cafu tam numeratorem P quam denominatorem Q ad duos tantum terminos extendiffe fufficit, ita vt fit $\frac{1}{\omega} \cdot \frac{A + B\omega}{\mathfrak{A} + \mathfrak{B}\omega}$. Nunc igitur ex euolutione iftius fractionis $\frac{A + B\omega}{\mathfrak{A} + \mathfrak{B}\omega}$, oriatur quotus $\alpha + \beta\omega$, eritque

$$\alpha = \frac{A}{\mathfrak{A}};\ \beta = \frac{B}{\mathfrak{A}} - \frac{A\mathfrak{B}}{\mathfrak{A}^2}.$$

His autem valoribus inuentis fractio noftra difcerpitur in has partes: $\frac{\alpha}{\omega} + \alpha$, quarum cum prima tantum fiat infini-

 ta,

ta, si loco ω restituamus valorem $z-a$, fractio simplex hinc resultans erit $\frac{\alpha}{z-a}$. Hoc igitur modo facillime fractiones simplices ex singulis denominatoris Q factoribus solitariis formae $z-a$ obtinentur; neque adeo opus est, valorem ipsius β nosse, vnde sufficere potuisset tantum primos terminos A et 𝔄 indagasse. Oritur autem A ex numeratore P, posito $z=a$; at 𝔄 oritur ex formula $\frac{dQ}{dz}$, posito itidem $z=a$. Cum enim posito $z=a$ fiat $Q=0$, si loco z scribatur $a+\omega$, prodibit $\mathfrak{A}=\frac{dQ}{dz}$.

§. 6. Interim tamen bonum est, etiam valorem ipsius β nosse, quoniam inde quaestio non parum curiosa facile potest resolui. Cum enim ex factore $z-a$ deducta sit fractio $\frac{\alpha}{z-a}$, si pro reliquis omnibus terminis scribamus litteram R, erit vtique $\frac{P}{Q}=\frac{\alpha}{z-a}+R$. Quod si ergo desideretur summa omnium reliquorum terminorum R, casu quo ponitur $z=a$, siue $z=a+\omega$, quippe quae summa est finita, ex aequatione modo inuenta fiet $R=\frac{P}{Q}-\frac{\alpha}{z-a}$, ideoque posito $z=a+\omega$, erit

$$R=\frac{\alpha}{\omega}+\beta-\frac{\alpha}{\omega}=\beta;$$

sicque valor litterae β, quem inuenimus, exprimit summam omnium reliquorum terminorum pro casu $z=a$. Erat autem $\beta=\frac{B}{\mathfrak{A}}-\frac{A\mathfrak{B}}{\mathfrak{A}^2}$.

§. 7. Facile autem patet, hoc modo pro omnibus factoribus simplicibus denominatoris Q easdem prodire fractiones partiales, ad quas methodus antehac exposita deducit. Si enim ponamus $\frac{P}{Q}=\frac{\alpha}{z-a}+R$, ac per $z-a$ multiplicemus, fiet

P (z

$$\frac{P(z-a)}{Q} = \alpha + R(z-a).$$

Quia nunc nouimus, numeratorem quaesitum α esse constantem, pro eo semper idem valor prodire debet, quicquid pro z scribatur. Ponatur igitur $z = a$, vt ratio reliquorum terminorum R ex calculo egrediatur, fietque $\alpha = \frac{P(z-a)}{Q}$, posito scilicet $z = a$; tum autem tam numerator quam denominator euanescit, vnde si eorum loco sua differentialia ponantur, fiet

$$\alpha = \frac{(z-a)\,dP + P\,dz}{dQ}.$$

Posito igitur $z = a$ erit $\alpha = \frac{P\,dz}{dQ}$. At vero supra assumsimus, casu $z = a$ fieri $P = A$ et $\frac{dQ}{dz} = \mathfrak{A}$, ita vt et hinc etiam prodeat $\alpha = \frac{A}{\mathfrak{A}}$.

Casus II.

Quo denominatoris Q factor est $(z-a)^2$.

§. 8. Hic igitur in forma, ad quam nostram fractionem $\frac{P}{Q}$, posito $z = a + \omega$, conuertimus, erit $\mathfrak{A} = 0$, vnde fractio pro hoc casu ita referri poterit:

$$\frac{1}{\omega\omega} \cdot \frac{A + B\omega + C\omega\omega}{\mathfrak{B} + \mathfrak{C}\omega + \mathfrak{D}\omega\omega}.$$

Hic scilicet potestates ipsius ω non vltra secundam protendimus. Nunc illa expressio in hanc formam:

$$\frac{1}{\omega^2}(\alpha + \beta\omega + \gamma\omega\omega),$$

reducatur, et reperietur calculo subducto

$$\alpha = \frac{A}{\mathfrak{B}};\ \beta = \frac{B}{\mathfrak{B}} - \frac{A\mathfrak{C}}{\mathfrak{B}^2};\ \gamma = \frac{C}{\mathfrak{B}} - \frac{B\mathfrak{C}}{\mathfrak{B}^2} - \frac{A\mathfrak{D}}{\mathfrak{B}^2} + \frac{A\mathfrak{C}^2}{\mathfrak{B}^3}.$$

His igitur valoribus inuentis nostra fractio discerpetur in has partes: $\frac{\alpha}{\omega^2} + \frac{\beta}{\omega} + \gamma$, quarum binae priores, ob $\omega = z - a$, praebent istas fractiones partiales: $\frac{\alpha}{(z-a)^2} + \frac{\beta}{z-a}$: at quantitas γ aequabitur summae omnium reliquorum terminorum, siquidem in illis statuatur $z = a$.

Casus III.

Quo denominatoris factor est $(z-a)^3$.

§. 9. Hic igitur ob $\mathfrak{A} = 0$ et $\mathfrak{B} = 0$ fractio evoluenda erit

$$\frac{1}{\omega^3} \cdot \frac{A + B\omega + C\omega\omega + D\omega^3}{\mathfrak{C} + \mathfrak{D}\omega + \mathfrak{E}\omega\omega + \mathfrak{F}\omega^3},$$

quae reducatur ad hanc formam:

$$\frac{1}{\omega^3}(\alpha + \beta\omega + \gamma\omega\omega + \delta\omega^3),$$

ope harum aequalitatum:

$$A = \alpha\mathfrak{C};\ B = \alpha\mathfrak{D} + \beta\mathfrak{C};\ C = \alpha\mathfrak{E} + \beta\mathfrak{D} + \gamma\mathfrak{C};$$
$$D = \alpha\mathfrak{F} + \beta\mathfrak{E} + \gamma\mathfrak{D} + \delta\mathfrak{C},$$

quibus valoribus inuentis ex denominatoris Q factore cubico $(z-a)^3$ obtinentur istae fractiones partiales:

$$\frac{\alpha}{(z-a)^3} + \frac{\beta}{(z-a)^2} + \frac{\gamma}{z-a}.$$

At vero δ exhibet summam omnium reliquorum terminorum, si in ipsis vbique scribatur $z = a$. Facile autem intelligitur, hoc modo etiam ad altiores potestates procedi posse.

§. 10. Haec methodus etiam succedit, si factores denominatoris fuerint imaginarii, scilicet formae

$$z - a + b\sqrt{-1},$$

tum autem, quoniam etiam factor erit $z - a - b\sqrt{-1}$, binae fractiones partiales hinc ortae

a

$$\frac{\alpha}{z-a+b\sqrt{-1}}+\frac{\beta}{z-a-b\sqrt{-1}},$$

facile in factorem duplicatum realem contrahentur, cuius denominator erit $(z-a)^2+bb$. Hoc igitur sequenti exemplo ostendisse iuuabit.

Exemplum.

Si fractio proposita resoluenda fuerit $\frac{P}{Q}=\frac{\sin.\Phi}{\tang.\Phi-\cos.\Phi}$, *eam in fractiones simplices resoluere.*

Solutio.

§. 11. Hic igitur primo omnes angulos Φ quaeri oportet, quibus denominator $\tang.\Phi-\cos.\Phi$ euanescit. Ponatur igitur

$$\tang.\Phi-\cos.\Phi=0,\ \text{siue}\ \sin.\Phi-\cos.\Phi^2=0,$$

ita vt sit

$$\sin.\Phi^2+\sin.\Phi=1,\ \text{vnde colligitur}$$

$$\sin.\Phi=-\frac{1\pm\sqrt{5}}{2},$$

sicque pro $\sin.\Phi$ duo habentur valores:

$$\sin.\Phi=-\frac{1+\sqrt{5}}{2}\ \text{et}\ \sin.\Phi=-\frac{1-\sqrt{5}}{2},$$

quorum prior cum sit vnitate minor, dabit valorem realem pro angulo Φ. Cum enim sit proxime

$$\frac{\sqrt{5}-1}{2}=0,618034,\ \text{erit}\ \Phi=38^\circ.\ 10'.\ 22'',$$

quem angulum breuitatis gratia ponamus $=\zeta$, ita vt sit

$$\sin.\zeta=0,618034\ \text{et}\ \cos.\zeta=0,786151$$

atque $\tang.\zeta=0,786154$, ideoque vti posuimus

$$\cos.\zeta=\tang.\zeta.$$

§. 12.

§. 12. Alter autem valor pro fin. Φ inuentus dat fin. $\Phi = -1,618034$, qui cum fit vnitate maior, monftrat hunc angulum effe imaginarium, ad quem definiendum notetur effe $\text{cof.}\,\theta\sqrt{-1} = \frac{e^{-\theta} + e^{+\theta}}{2}$, qui valor cum manifefto maior fit vnitate, et quidem pofitiuus vtamur hac formula:

$$\text{cof.}(\pi + \theta\sqrt{-1}) = \frac{-e^{\theta} - e^{-\theta}}{2}.$$

Quodfi ergo faciamus

$$\Phi = 90^{\circ} - \pi - \theta\sqrt{-1} = \theta\sqrt{-1} - \tfrac{\pi}{2}, \text{ erit}$$

$$\text{fin.}\,\Phi = -\frac{e^{-\theta} - e^{+\theta}}{2},$$

quamobrem debet effe

$$\frac{e^{-\theta} + e^{\theta}}{2} = +1,618034 = \frac{1 + \sqrt{5}}{2}$$

pro quo numero breuitatis gratia fcribamus ε, vt fit $e^{\theta} + e^{-\theta} = 2\varepsilon$, vnde colligitur

$$e^{\theta} = \varepsilon + \sqrt{(\varepsilon\varepsilon - 1)} = \tfrac{1+\sqrt{5}}{2} + \sqrt{\tfrac{1+\sqrt{5}}{2}} = \varepsilon + \sqrt{\varepsilon},$$

et fubftituto valore fit $e^{\theta} = 2,0581710$. Hinc igitur fiet $\theta = l.\,2,0581710$, fumendo fcilicet logarithmum hyperbolicum, qui reperitur fi logarithmus vulgaris multiplicetur per $2,3025851$. Cum igitur logarithmus vulgaris fit $0,3134816$ erit

$$\theta = 0,3134816 \cdot 2,3025851 = 0,7218177.$$

§. 13. Inventis igitur his valoribus pro ζ et θ, ex priore $\Phi = \zeta$ omnes anguli Φ, quibus nofter denominator

nator tang. Φ − cof. Φ euanefcit, funt in genere

$$2 i \pi + \zeta \text{ et } (2 i + 1) \pi - \zeta,$$

quippe qui anguli omnes eundem habent finum, vnde colliguntur omnes factores fimplices reales. Pro imaginariis autem tantum loco ζ fcribi oportet $\theta \sqrt{-1} - \frac{1}{2}\pi$, ficque fimul obtinentur omnes factores imaginarii.

§. 14. Primo igitur denominatoris noftri factor fit $\Phi - 2 i \pi - \zeta$, ponaturque hic factor $= \omega$, ita vt fit

$$\Phi = 2 i \pi + \zeta + \omega, \text{ eritque fin. } \Phi = \text{fin. } (\zeta + \omega)$$
$$= \text{fin. } \zeta \text{ cof. } \omega + \text{cof. } \zeta \text{ fin. } \omega = \text{fin. } \zeta + \omega \text{ cof. } \zeta,$$

quoniam non vltra primam dimenfionem ipfius ω progredi neceffe eft. Deinde vero erit

$$\text{cof. } \Phi = \text{cof. } (\zeta + \omega) = \text{cof. } \zeta - \omega \text{ fin. } \zeta,$$

denique

$$\text{tang. } \Phi = \text{tang. } (\zeta + \omega) = \text{tang. } \zeta + \frac{\omega}{\text{cof. } \zeta^2}.$$

Hinc igitur denominator erit

$$\text{tang. } \zeta - \text{cof. } \zeta + \omega \left(\text{fin. } \zeta + \frac{1}{\text{cof. } \zeta^2}\right);$$

at vero per hypothefin eft tang. ζ − cof. $\zeta = 0$, vnde denominator ifte erit $\omega \left(\text{fin. } \zeta + \frac{1}{\text{cof. } \zeta^2}\right)$. Vbi notetur, fi accuratius procedere voluiffemus, in denominatorem infuper terminum ω^3 ingreffurum fuiffe, quem autem hic negligere licet, quia nullus factor bis occurrit. Hinc ergo nafcitur valor infinitus noftrae fractionis

$$\frac{\text{fin. } \zeta}{\omega \left(\text{fin. } \zeta + \frac{1}{\text{cof. } \zeta^2}\right)} = \frac{\text{fin. } \zeta \text{ cof. } \zeta^2}{\omega \left(\text{fin. } \zeta \text{ cof. } \zeta^2 + 1\right)}$$

ex quo haec fractio partialis deducitur:

$$\frac{\sin.\zeta\cos.\zeta^2}{\sin.\zeta\cos.\zeta^2+1}\cdot\frac{1}{\Phi-2i\pi-\zeta},$$

quare si pro i omnes numeros tam positiuos quam negatiuos statuamus, prodibit ista series fractionum:

$$\frac{\sin.\zeta\cos.\zeta^2}{1+\sin.\zeta\cos.\zeta^2}\left(\frac{1}{\Phi-\zeta}+\frac{1}{\Phi-2\pi-\zeta}+\frac{1}{\Phi+2\pi-\zeta}\right.$$
$$\left.+\frac{1}{\Phi-4\pi-\zeta}+\frac{1}{\Phi+4\pi-\zeta}+\text{etc.}\right)$$

At si pro ζ scribamus $\theta\sqrt{-1}-\frac{1}{2}\pi$, series fractionum imaginariarum erit

$$\frac{\sin.\zeta\cos.\zeta^2}{1+\sin.\zeta\cos.\zeta^2}\left(\frac{1}{\Phi+\frac{1}{2}\pi-\theta\sqrt{-1}}+\frac{1}{\Phi-\frac{3}{2}\pi\Phi-\theta\sqrt{-1}}\right.$$
$$\left.+\frac{1}{\Phi+\frac{5}{2}\pi-\theta\sqrt{-1}}+\frac{1}{\Phi-\frac{7}{2}\pi-\theta\sqrt{-1}}\right).$$

§. 15. Pro altero casu, quo in genere erat factor

$$\Phi-(2i+1)\pi+\zeta=\omega,\ \text{erit}$$
$$\Phi=(2i+1)\pi-\zeta+\omega$$

hincque

$$\sin.\Phi=\sin.(\zeta-\omega)=\sin.\zeta-\omega\cos.\zeta,$$

tum vero

$$\cos.\Phi=-\cos.(\zeta-\omega)=-\cos.\zeta-\omega\sin.\zeta\ \text{et}$$
$$\text{tang}.\Phi=-\text{tang}.(\zeta-\omega)=-\text{tang}.\zeta+\frac{\omega}{\cos.\zeta^2},$$

quare totus denominator erit

$$-\text{tang}.\zeta+\cos.\zeta+\omega\left(\frac{1}{\cos.\zeta^2}+\sin.\zeta\right)$$
$$=\omega\left(\sin.\zeta+\frac{1}{\cos.\zeta^2}\right)\ \text{ob}\ -\text{tang}.\zeta+\cos.\zeta=0,$$

vnde pars infinita nostrae fractionis erit

$$\frac{\sin.\zeta}{\omega\left(\sin.\zeta+\frac{1}{\cos.\zeta^2}\right)}=\frac{\sin.\zeta\cos.\zeta^2}{\omega(1+\sin.\zeta\cos.\zeta^2)}.$$

Nunc

Nunc igitur, si loco ω scribamus valorem assumtum

$$\Phi - (2i + 1)\pi + \zeta,$$

orietur forma generalis fractionum simplicium

$$\frac{\sin.\zeta\cos.\zeta^2}{(1 + \sin.\zeta\cos.\zeta^2)}\cdot\frac{1}{\Phi - (2i + 1)\pi + \zeta}.$$

Loco i ergo successiue scribamus omnes valores

$$0,\ \pm 1,\ \pm 2,\ \pm 3,\ \pm 4 \text{ etc.}$$

et colligetur sequens series fractionum:

$$\frac{\sin.\zeta\cos.\zeta^2}{1 + \sin.\zeta\cos.\zeta^2}\left(\frac{1}{\Phi - \pi + \zeta} + \frac{1}{\Phi + \pi + \zeta} + \frac{1}{\Phi - 3\pi + \zeta}\right.$$

$$\left. + \frac{1}{\Phi + 3\pi + \zeta} + \frac{1}{\Phi - 5\pi + \zeta} + \text{etc.}\right)$$

Quodsi iam hic pro ζ scribamus $\theta\sqrt{-1} - \frac{1}{2}\pi$, orientur fractiones imaginariae, quae erunt

$$\frac{\sin.\zeta\cos.\zeta^2}{1 + \sin.\zeta\cos.\zeta^2}\left(\frac{1}{\Phi + \frac{1}{2}\pi + \theta\sqrt{-1}} + \frac{1}{\Phi - \frac{3}{2}\pi + \theta\sqrt{-1}}\right.$$

$$\left. + \frac{1}{\Phi + \frac{5}{2}\pi + \theta\sqrt{-1}} + \frac{1}{\Phi - \frac{7}{2}\pi + \theta\sqrt{-1}} + \text{etc.}\right)$$

§. 16. Colligamus nunc primo seorsim omnes fractiones reales, et cum omnes habeant eundem coefficientem constantem $\frac{\sin.\zeta\cos.\zeta^2}{1 + \sin.\zeta\cos.\zeta^2}$, ante omnia in eius valorem numericum inquiramus. Primo igitur cum fuisset

$$\sin.\zeta - \cos.\zeta^2 = 0, \text{ erit } \cos.\zeta^2 = \sin.\zeta,$$

ideoque iste coefficiens $= \frac{\sin.\zeta^2}{1 + \sin.\zeta^2}$; porro vero erat

$$\sin.\zeta^2 + \sin.\zeta = 1, \text{ ideoque } \sin.\zeta^2 = 1 - \sin.\zeta$$

vnde fit coefficiens $\frac{1 - \sin.\zeta}{2 - \sin.\zeta}$. Denique vero pro factoribus realibus eruimus $\sin.\zeta = \frac{\sqrt{5} - 1}{2}$, vnde noster coefficiens euadet $\frac{3 - \sqrt{5}}{5 - \sqrt{5}} = \frac{5 - \sqrt{5}}{10}$, cuius ergo valor erit 0, 2763932,

 pro

pro quo breuitatis gratia fcribamus α, et omnes fractiones fimplices reales in ordinem redactae erunt

$$\frac{\alpha}{\Phi-\zeta}+\frac{\alpha}{\Phi-\zeta-2\pi}+\frac{\alpha}{\Phi-\zeta+2\pi}+\frac{\alpha}{\Phi-\zeta-4\pi}+\frac{\alpha}{\Phi-\zeta+4\pi}+\text{etc.}$$

$$\frac{\alpha}{\Phi+\zeta-\pi}+\frac{\alpha}{\Phi+\zeta+\pi}+\frac{\alpha}{\Phi+\zeta-3\pi}+\frac{\alpha}{\Phi+\zeta+3\pi}+\frac{\alpha}{\Phi+\zeta-5\pi}+\text{etc.}$$

§. 17. Pro partibus autem imaginariis idem coefficiens communis

$\frac{\text{fin.}\,\zeta\,\text{cof.}\,\zeta^2}{1+\text{fin.}\,\zeta\,\text{cof.}\,\zeta^2}$, ob $\text{cof.}\,\zeta^2=\text{fin.}\,\zeta$ et $\text{fin.}\,\zeta^2=1-\text{fin.}\,\zeta$, transmutatur vt ante in hanc formam: $\frac{1-\text{fin.}\,\zeta}{2-\text{fin.}\,\zeta}$. At vero pro partibus imaginariis inuenimus $\text{fin.}\,\zeta=-\frac{1-\sqrt{5}}{2}$, in quo iam inuoluitur fubftitutio ante memorata $\zeta=\theta\sqrt{-1}-\frac{1}{2}\pi$. Hoc ergo valore fubftituto coefficiens communis erit

$$\frac{3+\sqrt{5}}{5+\sqrt{5}}=\frac{5+\sqrt{5}}{10},$$

ideoque in numeris 0, 7236068, pro quo numero fcribamus β, ita vt fit $\alpha+\beta=1$, hanc ob rem bini ordines fractionum imaginariarum erunt

$$\frac{\beta}{\Phi-\theta\sqrt{-1}+\frac{1}{2}\pi}+\frac{\beta}{\Phi-\theta\sqrt{-1}-\frac{3}{2}\pi}+\frac{\beta}{\Phi-\theta\sqrt{-1}+\frac{5}{2}\pi}$$

$$+\frac{\beta}{\Phi-\theta\sqrt{-1}-\frac{7}{2}\pi}\ \text{etc.}$$

$$\frac{\beta}{\Phi+\theta\sqrt{-1}+\frac{1}{2}\pi}+\frac{\beta}{\Phi+\theta\sqrt{-1}-\frac{3}{2}\pi}+\frac{\beta}{\Phi+\theta\sqrt{-1}+\frac{5}{2}\pi}$$

$$+\frac{\beta}{\Phi+\theta\sqrt{-1}-\frac{7}{2}\pi}\ \text{etc.}$$

Hinc ergo fi binae harum fractionum in vnam fummam colligantur, imaginaria fe mutuo deftruent, ac prodibit fequens feries:

β

$$\frac{\beta(2\Phi+\pi)}{(\Phi+\frac{1}{2}\pi)^2+\theta\theta}+\frac{\beta(2\Phi-3\pi)}{(\Phi-\frac{3}{2}\pi)^2+\theta\theta}$$
$$+\frac{\beta(2\Phi+5\pi)}{(\Phi+\frac{5}{2}\pi)^2+\theta\theta}+\frac{\beta(2\Phi-7\pi)}{(\Phi-\frac{7}{2}\pi)^2+\theta\theta}, \text{ etc.}$$

vbi notetur esse $\theta\theta = 0,5210210$.

§. 18. Quoniam partes imaginariae commode se contrahi sunt passae, vt similis contractio in partibus realibus succedat statuamus $\zeta = \frac{1}{2}\pi + \eta$ et ambae series ita se habebunt:

$$\frac{\alpha}{\Phi-\frac{1}{2}\pi-\eta}+\frac{\alpha}{\Phi-\frac{5}{2}\pi-\eta}+\frac{\alpha}{\Phi+\frac{3}{2}\pi-\eta}$$
$$+\frac{\alpha}{\Phi-\frac{5}{2}\pi-\eta}+\frac{\alpha}{\Phi+\frac{7}{2}\pi-\eta}+\text{ etc.}$$

$$\frac{\alpha}{\Phi-\frac{1}{2}\pi+\eta}+\frac{\alpha}{\Phi+\frac{3}{2}\pi+\eta}+\frac{\alpha}{\Phi-\frac{5}{2}\pi+\eta}$$
$$+\frac{\alpha}{\Phi+\frac{7}{2}\pi+\eta}+\frac{\alpha}{\Phi-\frac{9}{2}\pi+\eta}+\text{ etc.}$$

Hic ergo cuilibet termino conuenit quasi socius, binisque contractis orietur sequens series:

$$\frac{\alpha(2\Phi-\pi)}{(\Phi-\frac{1}{2}\pi)^2-\eta\eta}+\frac{\alpha(2\Phi+3\pi)}{(\Phi+\frac{3}{2}\pi)^2-\eta\eta}$$
$$+\frac{\alpha(2\Phi-5\pi)}{(\Phi-\frac{5}{2}\pi)^2-\eta\eta}+\frac{\alpha(2\Phi+7\pi)}{(\Phi+\frac{7}{2}\pi)^2-\eta\eta}+\text{ etc.}$$

vbi notetur, cum sit $\eta = \zeta - \frac{1}{2}\pi$, quoniam in partibus radii est $\zeta = 0,6662405$ fore $\eta = -0,9045558$, ideoque $\eta\eta = 0,0182214$, cum ante fuisset $\theta\theta = 0,5210210$.

§. 19. Quae igitur hactenus ſunt allata huc redeunt, vt fractio propoſita $\frac{\sin.\,\Phi}{\tang.\,\Phi - \cos.\,\Phi}$, aequetur binis ſequentibus ſeriebus iunctim ſumtis:

$$\frac{\alpha(2\Phi - \pi)}{(\Phi - \frac{1}{2}\pi)^2 - \eta\eta} + \frac{\alpha(2\Phi + 3\pi)}{(\Phi + \frac{3}{2}\pi)^2 - \eta\eta} + \frac{\alpha(2\Phi - 5\pi)}{(\Phi - \frac{5}{2}\pi)^2 - \eta\eta} + \frac{\alpha(2\Phi + 7\pi)}{(\Phi + \frac{7}{2}\pi)^2 - \eta\eta} + \text{etc.}$$

$$\frac{\beta(2\Phi + \pi)}{(\Phi + \frac{1}{2}\pi)^2 + \theta\theta} + \frac{\beta(2\Phi - 3\pi)}{(\Phi - \frac{3}{2}\pi)^2 + \theta\theta} + \frac{\beta(2\Phi + 5\pi)}{(\Phi + \frac{5}{2}\pi)^2 + \theta\theta} + \frac{\beta(2\Phi - 7\pi)}{(\Phi - \frac{7}{2}\pi)^2 + \theta\theta} + \text{etc.}$$

Hinc igitur ſequitur, ſi ſumatur $\Phi = 0$, quo caſu fractio ipſa in nihilum abit, fore

$$0 = \begin{cases} -\frac{4.1\,\alpha\pi}{\pi\pi - 4\eta\eta} + \frac{4.3.\,\alpha\pi}{9\pi\pi - 4\eta\eta} - \frac{4.5\,\alpha\pi}{25\pi\pi - 4\eta\eta} + \frac{4.7\,\alpha\pi}{49\pi\pi - 4\eta\eta} - \text{etc.} \\ +\frac{4.1\,\beta\pi}{\pi\pi + 4\theta\theta} - \frac{4.3.\,\beta\pi}{9\pi\pi + 4\theta\theta} + \frac{4.5\,\beta\pi}{25\pi\pi + 4\theta\theta} - \frac{4.7\,\beta\pi}{49\pi\pi + 4\theta\theta} + \text{etc.} \end{cases}$$

Ceterum hoc exemplum perquam idoneum eſt viſum, quo applicatio ad factores imaginarios illuſtraretur.

EVO-