EVOLVTIO
PRODVCTI INFINITI
$(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$
IN SERIEM SIMPLICEM.

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§. 1.

Posito $s=(1-x)(1-xx)(1-x^3)(1-x^4)$ etc. facile patet fore:

$$s=1-x-xx(1-x)-x^3(1-x)(1-xx)-x^4(1-x)(1-xx)(1-x^3)-\text{etc.}$$

quae series cum iam sit infinita, quaeritur, si singuli eius termini euoluantur, qualis series secundum simplices potestates ipsius x sit proditura. Cum igitur duo primi termini $1-x$ iam sint euoluti, loco reliquorum omnium scribatur littera A, ita vt sit $s=1-x-A$, ideoque

$$A=xx(1-x)+x^3(1-x)(1-xx)+x^4(1-x)(1-xx)(1-x^3)\text{etc.}$$

§. 2. Quoniam hi termini omnes factorem habent communem $1-x$, eo euoluto singuli termini discerpentur in binas partes quas ita repraesentemus:

$$A=xx+x^3(1-xx)+x^4(1-xx)(1-x^3)+x^5(1-xx)(1-x^3)(1-x^4)$$
$$-x^3-x^4(1-xx)-x^5(1-xx)(1-x^3)-x^6(1-xx)(1-x^3)(1-x^4)$$

Hinc

Hinc iam binae partes eadem potestate ipsius x affectae in vnam contrahantur, ac resultabit pro A sequens forma:

$$A = xx - x^5 - x^7(1-xx) - x^9(1-xx)(1-x^3) - x^{11}(1-x^2)(1-x^3)(1-x^4) - \text{etc.}$$

vbi duo termini primi $xx - x^5$ iam sunt euoluti; sequentes autem procedunt per has potestates: $x^7, x^9, x^{11}, x^{13}, x^{15}$ quarum exponentes binario crescunt.

§. 3. Ponamus nunc simili modo vt ante $A = xx - x^5 - B$, ita vt sit

$$B = +x^7(1-xx) + x^9(1-xx)(1-x^3) + x^{11}(1-xx)(1-x^3)(1-x^4) + \text{etc.}$$

cuius omnes termini habent factorem communem $1-xx$, quo euoluto singuli termini in binas partes discerpantur, vti sequitur:

$$B = x^7 + x^9(1-x^3) + x^{11}(1-x^3)(1-x^4) + x^{13}(1-x^3)(1-x^4)(1-x^5) \text{ etc.}$$
$$-x^9 - x^{11}(1-x^3) - x^{13}(1-x^3)(1-x^4) - x^{15}(1-x^3)(1-x^4)(1-x^5) \text{ etc.}$$

Hic iterum bini termini, qui eandem potestatem ipsius x habent praefixam, in vnam colligantur et prodibit:

$$B = x^7 - x^{12} - x^{15}(1-x^3) - x^{18}(1-x^3)(1-x^4) - x^{21}(1-x^3)(1-x^4)(1-x^5) - \text{etc.}$$

vbi iam potestates ipsius x crescunt ternario.

§. 4. Statuatur nunc porro $B = x^7 - x^{12} - C$, ita vt sit

$$C = x^{15}(1-x^3) + x^{18}(1-x^3)(1-x^4) + x^{21}(1-x^3)(1-x^4)(1-x^5) + \text{etc.}$$

et iam singuli termini per euolutionem factoris $1-x^3$ in binas partes resoluantur, fietque:

C =

$$C = x^{15} + x^{18}(1-x^4) + x^{21}(1-x^4)(1-x^5) + x^{24}(1-x^4)(1-x^5)(1-x^6)$$
$$- x^{18} - x^{21}(1-x^4) - x^{24}(1-x^4)(1-x^5) - x^{27}(1-x^4)(1-x^5)(1-x^6) \text{ etc.}$$

vbi denuo membra, quibus eadem poteſtas ipſius x praefixa, in vnum contracta praebebunt

$$C = x^{15} - x^{22} - x^{26}(1-x^4) - x^{30}(1-x^4)(1-x^5)$$
$$- x^{34}(1-x^4)(1-x^5)(1-x^6) \text{ etc.}$$

vbi poteſtates praefixae quaternario creſcunt.

§. 5. Statuatur $C = x^{15} - x^{22} - D$, vt ſit

$$D = x^{26}(1-x^4) + x^{30}(1-x^4)(1-x^5)$$
$$+ x^{34}(1-x^4)(1-x^5)(1-x^6) \text{ etc.}$$

qui termini per euolutionem factoris $1-x^4$ in binos diſcerpantur hoc modo:

$$D = x^{26} + x^{30}(1-x^5) + x^{34}(1-x^5)(1-x^6) + x^{38}(1-x^5)(1-x^6)(1-x^7)$$
$$- x^{30} - x^{34}(1-x^5) - x^{38}(1-x^5)(1-x^6) - x^{42}(1-x^5)(1-x^6)(1-x^7) \text{ etc.}$$

Nunc binis vt hactenus contrahendis colligitur

$$D = x^{26} - x^{35} - x^{40}(1-x^5) - x^{45}(1-x^5)(1-x^6)$$
$$- x^{50}(1-x^5)(1-x^6)(1-x^7) \text{ etc.}$$

Hic igitur poteſtates ipſius x quinario creſcunt.

§. 6. Statuatur $D = x^{26} - x^{35} - E$, ita vt ſit

$$E = x^{40}(1-x^5) + x^{45}(1-x^5)(1-x^6)$$
$$+ x^{50}(1-x^5)(1-x^6)(1-x^7) \text{ etc.}$$

ac reſolutione in binas partes vt hactenus inſtituta prodit

$$E = x^{40} + x^{45}(1-x^6) + x^{50}(1-x^6)(1-x^7) + x^{55}(1-x^6)(1-x^7)(1-x^8)$$
$$- x^{45} - x^{50}(1-x^6) - x^{55}(1-x^6)(1-x^7) - x^{60}(1-x^6)(1-x^7)(1-x^8) \text{ etc.}$$

Contractis vero binis terminis in vnum prodibit

$$E = x^{40} - x^{51} - x^{57}(1-x^6) - x^{63}(1-x^6)(1-x^7) \\ - x^{69}(1-x^6)(1-x^7)(1-x^8) \text{ etc.}$$

vbi poteſtates ipſius x ſenario creſcunt.

§. 7. Cum lex, qua iſtae operationes vlterius ſunt continuandae ſatis ſit perſpicua, ſi poſtremi valores pro ſingulis litteris A, B, C, D, inuenti ordine ſubſtituantur, pro ſerie quaeſita reperiemus ſequentem formam:

$$S = 1 - x, - xx + x^5, + x^7 - x^{12}, - x^{15} + x^{22}, \\ + x^{26} - x^{35}, - x^{40} + x^{51}, \text{ etc.}$$

Hic igitur tota quaeſtio huc reducitur, vt ordo definiatur, quo exponentes poteſtatum ipſius x continuo vlterius augentur, quandoquidem ex operationibus inſtitutis iam ſatis eſt manifeſtum ſigna terminorum + et − ita alternatim ſe excipere, vt ambo geminentur.

§. 8. Quo igitur in hanc legem inquiramus, videamus quomodo in ſingulis litteris iſti numeri ſint orti. Hunc in finem primos ſaltem cuiusque litterae terminos in eius forma prima exhibitos ordine diſponamus

$A = xx(1-x)$	$7 = 3 + 4 = 3 + 1 + 3 = 3 + 1 + 1 + 2$
$B = x^7(1-xx)$	$15 = 4 + 11 = 4 + 2 + 9 = 4 + 2 + 2 + 7$
$C = x^{15}(1-x^3)$	$26 = 5 + 21 = 5 + 3 + 18 = 5 + 3 + 3 + 15$
$D = x^{26}(1-x^4)$	$40 = 6 + 34 = 6 + 4 + 30 = 6 + 4 + 4 + 26$
$E = x^{40}(1-x^5)$	$57 = 7 + 50 = 7 + 5 + 45 = 7 + 5 + 5 + 40$
etc.	etc.

Hic ſcilicet ex euolutione litterae A vidimus, numerum 7 oriri ex aggregato 3 + 4, tum vero 4 oriri ex 1 + 3, ac denique 3 ex 1 + 2, quae ergo reſolutio dabit

$$7 = 3 + 4 = 3 + 1 + 3 = 3 + 1 + 1 + 2.$$

At-

Atque idem ordo in sequentibus litteris est obseruatus, vbi vltimi numeri procedunt ordine 2, 7, 15, 26, 40.

§. 9. Ex his iam manifestum est, numerorum 2, 7, 15, 26, 40, 57, etc. differentias progressionem arithmeticam constituere, vnde horum numerorum terminus generalis erit:

$$2 + 5(n-1) + \frac{3(n-1)(n-2)}{1.2} = \frac{3nn+n}{2}.$$

Exponentes autem, qui hos antecedunt, erant 1, 5, 12, 22, 35, 51 ab illis numeris 1, 2, 3, 4, 5, et in genere ipso numero n, ita vt exponens, qui formulam $\frac{3nn+n}{2}$ praecedit, futurus sit $\frac{3nn-n}{2}$.

§. 10. Nunc igitur seriem simplicem inuentam quae aequalis est producto infinito proposito

$$(1-x)(1-xx)(1-x^3)(1-x^4) \text{ etc.}$$

perfecte cognoscimus. Cum enim haec series inuenta sit:

$$s = 1 - x^1 - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26}$$
$$- x^{35} - x^{40} + x^{51} + \text{ etc.}$$

certi nunc sumus, in ea alias potestates ipsius x non occurrere, nisi quarum exponentes contineantur in hac formula generali: $\frac{3nn \pm n}{2}$, et quidem ita, vt si n fuerit numerus impar, bini termini inde nati habituri sint signum $-$, qui autem ex paribus oriuntur signum $+$.

Alia inuestigatio eiusdem seriei.

§. 11. Eadem series secundum potestates ipsius x procedens etiam sequenti modo inuestigari potest. Cum scilicet sit

$$s = 1 - x - xx(1-x) - x^3(1-x)(1-xx)$$
$$- x^4(1-x)(1-x^2)(1-x^3) \text{ etc.}$$

euoluatur ſtatim ſecundum membrum $-xx(1-x)$, vt fiat

$$s = 1 - x - xx + x^3 - x^3(1-x)(1-xx)$$
$$- x^4(1-x)(1-xx)(1-x^3) \text{ etc.}$$

ac ſtatuatur $s = 1 - x - xx + A$ vt ſit

$$A = x^3 - x^3(1-x)(1-xx)$$
$$- x^4(1-x)(1-xx)(1-x^3) - \text{ etc.}$$

cuius ſingula membra per euolutionem factoris $1-x$ in duas partes diſcerpantur, vt prodeat

$$A = x^3 - x^3(1-xx) - x^4(1-xx)(1-x^3) - x^5(1-x^2)(1-x^3)(1-x^4)$$
$$+ x^4(1-xx) + x^5(1-xx)(1-x^3) + x^6(1-x^2)(1-x^3)(1-x^4)$$

Hic iterum bina membra eadem poteſtate ipſius x affecta contracta praebebunt:

$$A = + x^5 + x^7(1-xx) + x^9(1-xx)(1-x^3)$$
$$+ x^{11}(1-x^2)(1-x^3)(1-x^4) \text{ etc.}$$

§. 12. Hic nunc iterum ſecundum membrum euoluatur, vt prodeat:

$$A = x^5 + x^7 - x^9 + x^9(1-xx)(1-x^3)$$
$$+ x^{11}(1-x^2)(1-x^3)(1-x^4) \text{ etc.}$$

Iam ponatur $A = x^5 + x^7 - B$, vt ſit

$$B = x^9 - x^9(1-xx)(1-x^3) - x^{11}(1-xx)(1-x^3)(1-x^4) \text{ etc.}$$

quare ſi vbique factor $1-xx$ euoluatur, obtinebitur

$$B = x^5 - x^9(1-x^3) - x^{11}(1-x^3)(1-x^4) - x^{13}(1-x^3)(1-x^4)(1-x^5)$$
$$+ x^{11}(1-x^3) + x^{13}(1-x^3)(1-x^4) + x^{15}(1-x^3)(1-x^4)(1-x^5)$$

tum vero contrahendis binis membris orietur

$$B = x^{12} + x^{15}(1-x^3) + x^{18}(1-x^3)(1-x^4)$$
$$+ x^{21}(1-x^3)(1-x^4)(1-x^5) \text{ etc.}$$

§. 13.

§. 13. Euoluatur pariter secundum membrum ac statuatur $B = x^{12} + x^{15} - C$; eritque

$$C = x^{18} - x^{18}(1-x^3)(1-x^4) - x^{21}(1-x^3)(1-x^4)(1-x^5) - \text{etc.}$$

Nunc termini euoluantur secundum factorem $1 - x^3$, fietque

$$C = x^{18} - x^{18}(1-x^4) - x^{21}(1-x^4)(1-x^5) - x^{24}(1-x^4)(1-x^5)(1-x^6)$$
$$+ x^{21}(1-x^4) + x^{24}(1-x^4)(1-x^5) + x^{27}(1-x^4)(1-x^5)(1-x^6)$$

Hinc binis membris contrahendis fiet

$$C = x^{22} + x^{26}(1-x^4) + x^{30}(1-x^4)(1-x^5)$$
$$+ x^{34}(1-x^4)(1-x^5)(1-x^6) \text{ etc.}$$

§. 14. Euoluto nunc hic iterum secundo membro statuatur $C = x^{22} + x^{26} - D$, eritque

$$D = x^{30} - x^{30}(1-x^4)(1-x^5) - x^{34}(1-x^4)(1-x^5)(1-x^6) \text{ etc.}$$

vbi euolutio factoris $1 - x^4$ producet

$$D = x^{30} - x^{30}(1-x^5) - x^{34}(1-x^5)(1-x^6) - x^{38}(1-x^5)(1-x^6)(1-x^7)$$
$$+ x^{34}(1-x^5) + x^{38}(1-x^5)(1-x^6) + x^{42}(1-x^5)(1-x^6)(1-\ldots)$$

Hinc binis membris contractis fiet $(1-x^5)(1-x^6)$

$$D = x^{35} + x^{40}(1-x^5)(1-x^6)(1-x^7) \text{ etc.}$$

§. 15. Euoluto secundo membro statuatur denuo $D = x^{35} + x^{40} - E$, eritque

$$E = x^{45} - x^{45}(1-x^5)(1-x^6) - x^{50}(1-x^5)(1-x^6)(1-x^7) \text{ etc.}$$

et euoluto factore secundo $1 - x^5$ fiet

$$E = x^{45} - x^{45}(1-x^6) - x^{50}(1-x^6)(1-x^7) - x^{55}(1-x^6)(1-x^7)(1-x^8)$$
$$+ x^{50}(1-x^6) + x^{55}(1-x^6)(1-x^7) + x^{60}(1-x^6)(1-x^7)(1-x^8) \text{ etc.}$$

 binis-

binisque terminis collectis elicitur

$$E = x^{51} + x^{57}(1 - x^{6}) + x^{63}(1 - x^{6})(1 - x^{7}) + x^{69}(1 - x^{6})(1 - x^{7})(1 - x^{8}) \text{ etc.}$$

§. 16. Inuentis igitur his valoribus litterarum A, B, C, D, E, si singuli successiue substituantur, resultabit ista series:

$$1 - x - xx, + x^{5} + x^{7}, - x^{12} - x^{15}, + x^{22} + x^{26}, - x^{35} - x^{40}, + \text{etc.}$$

Hic autem ordo exponentium facilius perspicitur. Cum enim in valoribus litterarum A, B, C, D, primo constitutis primi termini simplices essent x^{3}, x^{9}, x^{18}, x^{30}, x^{45}, exponentes manifesto sunt numeri trigonales triplicati, vnde generatim pro numero n erit iste exponens $\frac{3nn+3n}{2}$. Verum hi termini sequuntur binas potestates ipsius x procedentes per eandem differentiam n, vnde numerum n ab hac formula bis subtrahendo orientur binae potestates in quaesitam ingredientes, quarum exponentes consequenter erunt $\frac{3nn+n}{2}$ et $\frac{3nn-n}{2}$.

§. 17. Hinc igitur vicissim patet, seriem hanc

$$s = 1 - x - xx + x^{5} + x^{7} - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - \text{etc.}$$

in infinitum continuatam habere infinitos factores, qui scilicet erunt $(1 - x)$, $1 - xx$, $1 - x^{3}$, $1 - x^{4}$, $1 - x^{5}$, etc. ita vt si primo diuidatur per $1 - x$, tum vero quotus per $1 - xx$, iste quotus porro per $1 - x^{3}$, hocque modo in infinitum diuisio continuetur, vltimum quotum resultantem vnitati aequalem esse oportebit.

§. 18. Quod si ergo proposita fuerit ista aequatio in infinitum excurrens:

$$1 - x - xx + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \text{etc.} = 0,$$

eius omnes radices facile assignari possunt. Primum enim radix erit $x = 1$, deinde binae radices quadratae ex vnitate, tum vero ternae radices cubicae ex vnitate, porro quaternae radices biquadratae ex vnitate, similique modo quinae radices potestatis quintae ex vnitate, et ita porro, inter quas igitur ipsa vnitas infinities occurrit; at vero -1 ibi reperietur, vbi radix potestatis paris est extrahenda.

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