

DE  
**ELLISSI MINIMA**  
 DATO PARALLELOGRAMMO RECTANGVLO  
 CIRCVMSCRIBENDA.

Auctore  
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**C**um omni rectangulo infinitae ellipses circumscribi queant, quaestiones non parum curiosae videntur, quibus inter has ellipses vel ea quaeritur, cuius area futura sit minima, vel ea, quae minimam habitura sit perimetrum. Prior quidem nulla prorsus laborat difficultate; solutio tamen nihilominus attentione digna videtur; verum altera quaestio de perimetro maxime est ardua, ita ut eius vix solutionem perfectam exspectare liceat. Eo magis igitur utile erit, conatus eam resoluendi in medium adferre.

Problema I.

§. I. Circa datum rectangulum  $MmNn$  eam ellipsem  $AaBb$  describere, cuius area sit omnium minima.

Tab. I.  
Fig. I.

Solutio.

Posito centro tam rectanguli quam ellipsis in punto **C** vocentur semides laterum rectanguli  $CF=f$  et  $CG=g$ ;

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semi-

semiaxes vero ellipsis sint  $CA = a$  et  $CB = b$ ; et quoniam punctum N in ipsa ellipsi est situm, ex elementis constat fore  $\frac{f^2}{a^2} + \frac{g^2}{b^2} = 1$ . Cum nunc area ellipsis sit  $= \pi ab$ ; quantitates  $a$  et  $b$  ita definiri oportet, vt differentiale areae euanescat, sumtis scilicet semiaxisbus  $a$  et  $b$  variabilibus; unde nanciscimur  $da \cdot db + b \cdot da = 0$ , siue  $da : db = a : -b$ , quare ipsa illa aequatio  $\frac{f^2}{a^2} + \frac{g^2}{b^2} = 1$ , differentietur et loco  $da$  et  $db$  scribantur proportionalia  $a$  et  $-b$ , ac prodibit  $\frac{g^2}{b^2} = \frac{f^2}{a^2}$ , hincque deducimus  $\frac{2f^2}{a^2} = 1$ , ideoque  $a = f\sqrt{2}$  et  $b = g\sqrt{2}$ . Definitis autem semiaxisbus ipsa ellipsis facilime describitur.

### Corollarium.

§. 2. Cum hinc semiaxes lateribus rectanguli prodiere proportionales, sitque  $a:b = f:g$ ; euidens est, si ducantur rectae A B et F G, eas inter se fore parallelas; qua conditione ellipsis iam determinatur.

### Scholion.

§. 3. Expedita igitur priore quaestione, alteram ne suscipere quidem licet, nisi ante in genere cuiusque ellipsis perimeter per seriem infinitam ita commode exprimitur, quae pro omnibus speciebus quam maxime conuergat. Etsi autem iam plures huiusmodi series inueniri queant; tamen vix villa reperietur, quae sequenti, quam sum daturus, palmam praeripiatur.

### Lemta.

§. 4. Inuenire seriem maxime conuergentem, quae datae ellipsis perimetrum exhibeat.

Sit-

Sit quadrans ellipsis propositae A M B, cuius centrum in C et semiaxes CA =  $a$  et CB =  $b$ . Ex centro C semiaxe maiore CA superstruatur quadrans circuli A L D, cuius ergo radius AC =  $a$ , ducaturque radius quicunque CL; tum vero ducatur applicata LP, ellipsis in puncto M intersecans, pro quo vocentur coordinatae CP =  $x$  et PM =  $y$ ; et posito angulo A CL =  $\phi$  colligimus CP =  $x = a \cdot \cos \phi$  et PL =  $a \cdot \sin \phi$ . Quia vero est CD : CB = PL : PM =  $a : b$ , habebimus  $y = b \sin \phi$ , vnde fit  $dx = -a d\phi \cdot \sin \phi$  et  $dy = b d\phi \cdot \cos \phi$ , hincque elementum ellipticum

$$d\phi \sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi};$$

quocirca integrando arcus ellipticus A M erit

$$\int d\phi \sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi},$$

integrali ita sumto, vt evanescat posito  $\phi = 0$ . Hinc vero ipse quadrans ellipticus A M B reperietur, statuendo  $\phi = 90^\circ$ . Sicque totum negotium ad idoneam integrationem formulae

$$d\phi \sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}$$

reducitur. Cum autem sit

$$\sin \phi^2 = \frac{1 - \cos 2\phi}{2} \text{ et } \cos \phi^2 = \frac{1 + \cos 2\phi}{2}$$

nostra formula integranda abibit in hanc:

$$\int d\phi \sqrt{\frac{a^2 + b^2}{2} - \frac{(a^2 - b^2)}{2} \cos 2\phi}$$

quam concinniorem reddemus, ponendo

$$a^2 + b^2 = c^2 \text{ et } \frac{a^2 - b^2}{a^2 + b^2} = n;$$

tum enim erit arcus

$$AM = \frac{c}{\sqrt{2}} \int d\phi \sqrt{1 - n \cos 2\phi}$$

a qua igitur simplici formula rectificatio ellipsis pendet. Conuertamus ergo hanc formulam irrationalem in seriem,

quae erit

$$\begin{aligned} \sqrt{(1 - n \cos 2\Phi)} &= 1 - \frac{1}{2} n \cos 2\Phi \\ &- \frac{1 \cdot 3}{2 \cdot 4} n^2 \cos^2 2\Phi - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} n^3 \cos^3 2\Phi \\ &- \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} n^4 \cos^4 2\Phi - \text{etc.} \end{aligned}$$

Ad has integrationes peragendas in subsidium vocemus  
hanc reductionem:

$$\begin{aligned} \int d\Phi \cos 2\Phi^{\lambda+2} &= \frac{\lambda+1}{\lambda+2} \int d\Phi \cos 2\Phi^\lambda \\ &+ \frac{1}{2\lambda+4} \sin 2\Phi \cos 2\Phi^{\lambda+1}, \end{aligned}$$

vbi terminus postremus casu  $\Phi = 0$  sponte evanescit, ita  
ut nulla constante opus sit adiicienda. Extendamus autem  
pro instituto nostro hanc integrationem usque ad  $\Phi = 90^\circ$ ,  
quo fit  $2\Phi = 180^\circ$ , ac denuo evanescet terminus ille ab-  
solutus; quo circa reductio nostra erit

$$\int d\Phi \cos 2\Phi^{\lambda+2} = \frac{\lambda+1}{\lambda+2} \int d\Phi \cos 2\Phi^\lambda$$

cuius ope ex binis terminis primoribus nostrae seriei se-  
quentes omnes facilime integrabuntur, a termino scilicet  
 $\Phi = 0$  usque ad  $\Phi = 90^\circ = \frac{\pi}{2}$ ; quemadmodum sequens  
tabula declarat.

$\int d\Phi = \Phi$	$=$	$\frac{\pi}{2}$
$\int d\Phi \cos 2\Phi$	$=$	0
$\int d\Phi \cos 2\Phi^2$	$=$	$\frac{1}{2} + \frac{\pi}{2}$
$\int d\Phi \cos 2\Phi^3$	$=$	0
$\int d\Phi \cos 2\Phi^4$	$=$	$\frac{1}{2} + \frac{3}{4} + \frac{\pi}{2}$
$\int d\Phi \cos 2\Phi^5$	$=$	0
$\int d\Phi \cos 2\Phi^6$	$=$	$\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} + \frac{\pi}{2}$
$\int d\Phi \cos 2\Phi^7$	$=$	0
$\int d\Phi \cos 2\Phi^8$	$=$	$\frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} + \frac{\pi}{2}$

etc.

His

Hic igitur valoribus introductis quadrans noster ellipticus prodibit

$$AMB = \frac{\pi c}{2\sqrt{2}} \left( 1 - \frac{1 \cdot 1}{2 \cdot 4} \cdot \frac{1}{2} n^2 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{1 \cdot 3}{2 \cdot 4} n^4 - \right.$$

$$\left. - \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} n^6 - \text{etc.} \right).$$

Quia hic singuli coëfficientes numerici praecedentes in se complectuntur, pro hac serie sequentem formam scribamus:

$$AMB = \frac{\pi c}{2\sqrt{2}} (1 - \alpha \cdot n^2 - \beta \cdot n^4 - \gamma \cdot n^6 - \delta \cdot n^8 - \text{etc.})$$

quarum litterarum valores ita progredientur:

$$\alpha = \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 2} = \frac{1 \cdot 3}{4 \cdot 4}$$

$$\beta = \frac{3 \cdot 5}{6 \cdot 8} \cdot \frac{3}{4} = \frac{3 \cdot 5}{8 \cdot 8}$$

$$\gamma = \frac{7 \cdot 9}{10 \cdot 12} \cdot \frac{5}{6} = \frac{7 \cdot 9}{12 \cdot 12}$$

$$\delta = \frac{11 \cdot 13}{14 \cdot 16} \cdot \frac{7}{8} = \frac{11 \cdot 13}{16 \cdot 16}$$

Pro ellipsi igitur, cuius semiaxes sunt  $a$  et  $b$ , ponendo breuitatis gratia

$$a^2 + b^2 = c^2 \text{ et } \frac{a^2 - b^2}{a^2 + b^2} = n,$$

quadrans perimetri exprimetur sequenti serie:

$$\frac{\pi c}{2\sqrt{2}} \left( 1 - \frac{1 \cdot 1}{4 \cdot 4} \cdot n^2 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{4 \cdot 4 \cdot 6 \cdot 8} n^4 - \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{4 \cdot 4 \cdot 8 \cdot 8 \cdot 12 \cdot 12} n^6 - \text{etc.} \right)$$

quae series semper admodum conuergit, quantumuis axes ellipsis fuerint inter se diuersi, quia semper  $n$  est vnitate minor, ac praeterea coëfficientes numerici vehementer decrescent.

### Corollarium.

§. 5. Casus, quo haec series minime conuergit, est, quo  $n=1$ , quod evenit, vbi  $b=0$ , seu vbi axis conjugatus euaneat; tum autem manifestum est, quadrantem ellipti-

ellipticum ipsi semiaxi  $A \cdot C = a$  aequalem fore; unde quia etiam  $c = a$ , habebimus pro hoc casu

$$\frac{2\sqrt{2}}{\pi} = 1 - \frac{1 \cdot 1}{4 \cdot 4} + \frac{1 \cdot 1 \cdot 3 \cdot 5}{4 \cdot 4 \cdot 8 \cdot 8} - \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{4 \cdot 4 \cdot 8 \cdot 8 \cdot 12 \cdot 12} \text{ etc.}$$

quae series utique attentione digna videtur, idque eo magis, quod terminis continuo colligendis praebat sequentem insignem aequalitatem:

$$\frac{3 \cdot 5}{4 \cdot 4} \cdot \frac{7 \cdot 9}{8 \cdot 8} \cdot \frac{11 \cdot 13}{12 \cdot 12} \cdot \frac{15 \cdot 17}{16 \cdot 16} \text{ etc.} = \frac{2\sqrt{2}}{\pi}$$

ideoque

$$\frac{\pi}{2\sqrt{2}} = \frac{4 \cdot 4}{3 \cdot 3} \cdot \frac{8 \cdot 8}{7 \cdot 9} \cdot \frac{12 \cdot 12}{11 \cdot 13} \text{ etc.}$$

cuius veritas ex iis, quae olim de productis infinitis protulii, facile elucet.

### Corollarium 2.

§. 6. Si seriem ante inuenitam statuamus

$$1 - \frac{1 \cdot 1}{4 \cdot 4} n^2 + \frac{1 \cdot 1 \cdot 3 \cdot 5}{4 \cdot 4 \cdot 8 \cdot 8} n^4 - \text{etc.} = s$$

per ea, quae olim de summatione huiusmodi serierum ostendi, reperitur ista aequatio differentialis secundi gradus:

$$\frac{+n d d s}{d n^2} + \frac{+d s}{d n} + \frac{n s}{1 - n n} = 0$$

vbi  $d n$  pro constante est sumtum. Cum enim hinc sit

$$\frac{+n d d s}{d n^2} (1 - n n) + \frac{+d s}{d n} (1 - n n) + n s = 0,$$

si fingamus

$$s = 1 + A n n + B n^4 + C n^6 + \text{etc.}$$

erit vt sequitur:

$$\frac{+n d d s}{d n^2} = 4 \cdot 2 \cdot 1 \cdot A \cdot n + 4 \cdot 4 \cdot 3 \cdot B \cdot n^3$$

$$= 4 \cdot 2 \cdot 1 \cdot A \cdot n^5$$

$$+ 4 \cdot 6 \cdot 5 \cdot C \cdot n^5 + 4 \cdot 8 \cdot 7 \cdot D \cdot n^7 \text{ etc.}$$

$$+ 4 \cdot 4 \cdot 3 \cdot B \cdot n^5 - 4 \cdot 6 \cdot 5 \cdot C \cdot n^7 \text{ etc.}$$

$$4 \cdot d s$$

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$$\begin{aligned} \frac{4 \cdot 2 \cdot 3}{n^4} (1 - n^4) &= 4 \cdot 2 \cdot A \cdot n + 4 \cdot 4 \cdot B \cdot n^3 + 4 \cdot 6 \cdot C \cdot n^5 \\ &\quad - 4 \cdot 2 \cdot A \cdot n^2 - 4 \cdot 4 \cdot B \cdot n^4 \\ &\quad + 4 \cdot 8 \cdot D \cdot n^7 \text{ etc.} \\ &\quad - 4 \cdot 6 \cdot C \cdot n^9 \text{ etc.} \end{aligned}$$

$$n^4 = n + A \cdot n^3 + B \cdot n^5 + C \cdot n^7 \text{ etc.}$$

vnde sequentes determinations oriuntur:

- $\equiv 4 \cdot 4 \cdot A + 1$ ; ergo  $A = \frac{-1}{4 \cdot 4}$ ;
- $\equiv 4 \cdot 4 \cdot 4 \cdot B - 3 \cdot 5 \cdot A$ ; ergo  $B = \frac{5 \cdot 5}{4 \cdot 4} A$ ;
- $\equiv 4 \cdot 6 \cdot 6 \cdot C - 7 \cdot 9 \cdot B$ ; ergo  $C = \frac{7 \cdot 9}{12 \cdot 12} B$ ;
- $\equiv 4 \cdot 8 \cdot 8 \cdot D - 11 \cdot 13 \cdot C$ ; ergo  $D = \frac{11 \cdot 13}{16 \cdot 16} C$ ;

vnde eadem series resultat, quam supra inuenimus.

## Problema 2.

§. 7. Circa datum rectangulum  $MmNn$  eam ellipsis describere, cuius perimeter sit minima.

### Solutio.

Positis vt ante semilateribus rectanguli dati  $CF=f$ , Tab. I.  $CG=g$ , et semiaxibus ellipsis quae fitae  $CA=a$  et  $CB=b$  Fig. 1. habemus primo  $\frac{f^2}{a^2} + \frac{g^2}{b^2} = 1$ . Tum vero si ponamus

$$a^2 + b^2 = c^2 \text{ et } \frac{a^2 - b^2}{a^2 + b^2} = n,$$

quadrantem perimetri modo ante inuenimus

$$= \frac{\pi \cdot c}{2\sqrt{2}} (1 - \alpha n^2 - \alpha \beta n^4 - \alpha \beta \gamma n^6 - \alpha \beta \gamma \delta n^8 - \text{etc.}),$$

existente

$$\alpha = \frac{1 \cdot 3 \cdot 5}{4 \cdot 4 \cdot 6}, \beta = \frac{5 \cdot 7 \cdot 9}{8 \cdot 8 \cdot 10}, \gamma = \frac{7 \cdot 9}{12 \cdot 12}, \delta = \frac{11 \cdot 13}{16 \cdot 16}, \text{ etc.}$$

quae quantitas, ut fiat minima, eius differentiale nihilo aequari debet, dum scilicet litterae  $c$  et  $n$  tanquam variabiles tractantur, unde dividendo per  $\frac{\pi}{2\sqrt{2}}$  sequens nascentur aequatio:

$$\left. \begin{aligned} & d c (1 - \alpha n^2 - \alpha \beta n^4 - \alpha \beta \gamma n^6 - \alpha \beta \gamma \delta n^8 - \text{etc.}) \\ & - c d n (2 \alpha n + 4 \alpha \beta n^3 + 6 \alpha \beta \gamma n^5 + 8 \alpha \beta \gamma \delta n^7 + \text{etc.}) \end{aligned} \right\} = 0$$

quae hoc modo repraesentetur:

$$\begin{aligned} & \frac{dc}{c} (1 - \alpha n^2 - \alpha \beta n^4 - \alpha \beta \gamma n^6 - \alpha \beta \gamma \delta n^8 - \text{etc.}) \\ & = \frac{dn}{n} (2 \alpha n + 4 \alpha \beta n^3 + 6 \alpha \beta \gamma n^5 + 8 \alpha \beta \gamma \delta n^7 + \text{etc.}) \end{aligned}$$

Hic autem differentialia  $\frac{dc}{c}$  et  $\frac{dn}{n}$  certam inter se tenere debent relationem, quam ex aequatione fundamentali  $\frac{f^2}{a^2} + \frac{g^2}{b^2} = 1$ , peti oportet. Hic igitur ante omnia loco  $a$  et  $b$  litteras  $c$  et  $n$  introduci conuenit. Cum enim sit

$$a^2 + b^2 = c^2 \text{ et } a^2 - b^2 = \frac{c^2}{2} n, \text{ erit}$$

$$a^2 = \frac{1}{2} c^2 (1 + n) \text{ et } b^2 = \frac{1}{2} c^2 (1 - n);$$

unde nostra aequatio abibit in haec:

$$\frac{2f^2}{1+n} + \frac{2g^2}{1-n} = c^2.$$

Quia  $f$  et  $g$  dantur, ponamus breuitatis gratia

$$f^2 + g^2 = b^2 \text{ et } \frac{f^2 - g^2}{f^2 + g^2} = i, \text{ siue } f^2 - g^2 = b^2 \cdot i;$$

quo facto nostra aequatio erit  $\frac{2b^2(1+i)}{1-n} = c^2$ , cuius surmanus logarithmos, eritque

$$l 2 b^2 + l(1-i) = l(1-n) = 2 \cdot l c,$$

quae aequatio differentiata dat

$$\frac{i d n}{1-n} + \frac{2n d n}{1-n} = \frac{2 d c}{c},$$

unde fit

$$\frac{d c}{c} = \frac{n d n}{1-n} - \frac{i d n}{2(1-i)} = \frac{2n - i - i n n}{2(1-i)(1-n)} \cdot d n$$

quo

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quo valore substituto nostra aequatio erit

$$(2n^2 - in - in^3)(1 - \alpha n^2 - \alpha\beta n^4 - \alpha\beta\gamma n^5 - \alpha\beta\gamma\delta n^6 \text{ etc.}) \\ = 2(1 - in)(1 - nn)(2\alpha n^2 + 4\alpha\beta n^4 + 6\alpha\beta\gamma n^6 \text{ etc.})$$

in qua aequatione tantum duae insunt quantitates  $i$  et  $n$ , quarum illa ex rectangulo datur, haec autem  $n$  ex illa debet definiri; quod ergo non aliter nisi resolutio aequationis infinitae fieri potest. Conueniet igitur quantitatem  $n$  tanquam cognitam spectare indeque vicissim  $i$  definire, quod si pro pluribus casibus instituatur, facile iudicare licet, quinam valor ipsius  $n$  cuius valori dato  $i$  respondeat.

Spectemus primo quantitatem  $n$  ut minimam, quippe cui etiam valor minimus ipsius  $i$  respondebit; rejectis ergo terminis  $\beta, \gamma, \delta, \text{ etc.}$  continentibus, habebimus

$$(2n^2 - in - in^3)(1 - \alpha n^2) = 2(1 - in)(1 - nn) \cdot 2\alpha n^2$$

quae aequatio, rejectis potestatisbus ipsius  $n$  tertio altioribus, abit in hanc:

$$2n^2 - 4\alpha n^3 - in - in^3 + 5\alpha in^3 = 0,$$

vnde prodit

$$i = \frac{2n(1 - 2\alpha)}{1 + n^2 - 5\alpha n^2}.$$

Quare si  $n$  fuerit fractio valde parua, erit proxime

$$i = 2n(1 - 2\alpha) = \frac{2}{7}n$$

vnde vicissim concludere licet, si  $i$  fuerit fractio quam minima, fore  $n = \frac{2}{7}i$ .

Nunc igitur aliquanto propius ad veritatem accedamus, reificiendo potestates ipsius  $n$  quinta superiores, eritque

$$2n^2 - 4\alpha n^3 + 2\alpha n^4 - 8\alpha\beta n^5 - in - in^3 + 5\alpha in^3 = 0,$$

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vnde

vnde elicitur

$$i = \frac{2n(1-2\alpha) + 2\alpha n^3(1-\beta)}{1+n^2-5\alpha n^2},$$

et diuidendo

$$i = 2n(1-2\alpha) + 2\alpha n^3(1-\beta) \\ - 2n^3(1-2\alpha)(1-5\alpha),$$

qui valor, ob  $\alpha = \frac{1}{16}$  et  $\beta = \frac{15}{64}$ , praebet

$$i = \frac{7}{4} \cdot n - \frac{153}{128} \cdot n^3,$$

vnde vicissim pro dato  $i$  concludimus

$$n = \frac{4}{7}i + \frac{306}{257} \cdot i^3$$

sicque vterius ad veritatem accedere licebit.

Enoluamus adhuc casum, quo  $n$  maximum obtinet valorem, qui vnitati est aequalis et ponamus breuitatis gratia

$$1 - \alpha - \alpha\beta - \alpha\beta\gamma \dots = s \text{ et}$$

$$2\alpha + 4\alpha\beta + 6\alpha\beta\gamma \dots = t$$

vt habeamus hanc aequationem:

$$2(1-i)s = 2(1-i)(1-nn)t, \text{ siue}$$

$$2(1-i)(s - (1-nn)t) = 0$$

vnde prior factor manifesto dat  $i = 1$ , id quod rei natura postulat; si enim latitudo ellipsis euaneat, hoc est si  $b = 0$ , tum etiam latitudo rectanguli debet euanesce. Cum igitur evoluerimus tam casus, quibus litterae  $i$  et  $n$  sunt quam minimae, quam eum, vbi maximum sortiuntur valorem  $= 1$ , pro reliquis casibus iudicium haud erit difficile. Cum enim pro exiguis valoribus habeamus  $n = \frac{4}{7}i$ , pro maximo autem  $n = i$ ; in genere non multum a veritate aberrabimus, si statuamus

$$n =$$

$$n = \frac{\frac{4}{7}i}{1 - \frac{3}{7}ii} = \frac{4i}{7 - 3ii}.$$

Quamdiu enim  $i$  est fractio valde parua, erit  $n = \frac{4}{7}i$ ; et si aliquanto maius fuerit, erit  $n = \frac{4}{7}i + \frac{12}{49}i^3$ , qui valor illo  $n = \frac{4}{7}i + \frac{506}{2401}i^3$  aliquantillum maior est; verum ubi fit  $i = 1$ , iterum prodit  $n = 1$ .

### Corollarium.

§. 8. Quo naturam huius solutionis penitus perspiciamus, ponamus in genere:

$$1 - \alpha n^2 - \alpha \beta n^4 - \alpha \beta \gamma n^6 \dots = s,$$

$$2 \alpha n^2 + 4 \alpha \beta n^4 + 6 \alpha \beta \gamma n^6 \dots = t$$

vt nostra aequatio fiat

$$(2n^2 - in - in^3)s = 2(1 - in - n^2 + in^3)t,$$

vnde sequitur

$$i = \frac{2n^2s - 2(1 - n^2)t}{n(1 + n^2)s - 2n(1 - n^2)t},$$

vnde casu  $n = 1$  manifesto fit  $i = 1$ . Pro alio vero casu quocunque pro  $n$  assumto, quia ambas series  $s$  et  $t$  satis expedite summare licet, verus valor ipsius  $i$  satis exacte definiri poterit.

### Corollarium 2.

§. 9. Ambae series, quae in nostram solutionem ingrediuntur, manifesto ita a se inuicem pendent, vt sit  $t = -\frac{n ds}{dn}$ . Nunc autem statuamus  $t = sz$ , vt fiat

$$i = \frac{2n^2 - 2(1 - n^2)z}{n(1 + n^2) - 2n(1 - n^2)z},$$

ita vt hic vnicum valorem  $z$  quaeri oporteat, id quod

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sequent modo fieri poterit. Cum sit  $t = -\frac{nd s}{dn}$ , erit nunc  
 $s z = -\frac{nd s}{dn}$ , hincque  $\frac{ds}{s} = -\frac{z dn}{n}$ , et differentiando  
 $\frac{d ds}{s} = \frac{d s^2}{s^2} = +\frac{z dn^2}{n^2} = \frac{dz dn}{n}$ . Addatur  
 $\frac{d s^2}{s^2} = \frac{z^2 dn^2}{n^2}$ , fietque  
 $\frac{d ds}{s} = \frac{z dn^2}{n^2} + \frac{z^2 dn^2}{n^2} = \frac{dz dn}{n}$ .

Supra autem dedimus hanc inter  $s$  et  $n$  aequationem:

$$\frac{4n \cdot d ds}{dn^2} + \frac{4nd s}{dn} + \frac{ns}{(1-n)n} = 0,$$

quae per  $s$  diuisa praebet

$$\frac{4n}{dn^2} \cdot \frac{d ds}{s} + \frac{4}{dn} \cdot \frac{ds}{s} + \frac{n}{(1-n)n} = 0;$$

hic autem valores modo inuenti substituti producunt

$$\frac{4z}{n} + \frac{4z^2}{n} - \frac{4dz}{dn} - \frac{4z}{n} + \frac{n}{(1-n)n} = 0, \text{ unde fit}$$

$$dz = \frac{n dn}{4(1-n)n} + \frac{z^2 dn}{n},$$

quae est aequatio differentialis tantum primi gradus, similis  
 formae, qua olim aequatio Riccatiana referri est solita.  
 Semper autem resolutionem ita institui decet, ut littera  $n$   
 tanquam cognita spectetur, et pro ea debitus valor ipsius  $z$   
 eruatur; tum vero inde valor litterae  $i$  determinetur. Sic  
 enim vicissim affirmare poterimus, huic ipsi valori ipsius  $i$   
 assumptum valorem litterae  $n$  conuenire; quo inuento, cum  
 sit  $c^2 = \frac{z b^2 (1-in)}{1-nn}$ , etiam innoscit quantitates  $c$ ; at deni-  
 que ex  $c$  et  $n$  ipsi semiaxes  $a$  et  $b$  definiuntur.

### Corollarium 3.

§. 10. Quo omnes isti calculi facilius expediri  
 queant, valores coëfficientium  $\alpha, \beta, \gamma, \delta$ , in fractionibus  
 decimalibus. exhibeamus, una cum earum logarithmis:

la =

$\alpha = 8.7958800;$	$\alpha = 0.0625000.$
$\alpha\beta = 8.1657913;$	$\alpha\beta = 0.0146484.$
$\alpha\beta\gamma = 7.8067693;$	$\alpha\beta\gamma = 0.0064087.$
$\alpha\beta\gamma\delta = 7.5538653;$	$\alpha\beta\gamma\delta = 0.0035799.$
$\alpha\beta\gamma\delta\epsilon = 7.3583455;$	$\alpha\beta\gamma\delta\epsilon = 0.0022821.$
$\alpha\beta\gamma\delta\epsilon\zeta = 7.1988959;$	$\alpha\beta\gamma\delta\epsilon\zeta = 0.0015808.$

### Exemplum.

§. II. Euoluamus casum, quo semiaxes ellipsis  $a$  et  $b$  sunt in ratione dupla, sive  $a:b = z:1$ , unde fit

$$n = \frac{a^2 - b^2}{a^2 + b^2} = \frac{z}{z+1}, \text{ hinc } n^2 = \frac{z}{2z+1},$$

vnde valores singulorum terminorum ita se habebunt pro triaque serie  $s$  et  $t$ :

$\alpha n^2 = 0.0225000$	$2\alpha n^2 = 0.0450000$
$\alpha\beta n^4 = 0.0018984$	$4\alpha\beta n^4 = 0.0073934$
$\alpha\beta\gamma n^6 = 0.0002990$	$6\alpha\beta\gamma n^6 = 0.0017940$
$\alpha\beta\gamma\delta n^8 = 0.0000601$	$8\alpha\beta\gamma\delta n^8 = 0.0004808$
$\alpha\beta\gamma\delta\epsilon n^{10} = 0.0000138$	$10\alpha\beta\gamma\delta\epsilon n^{10} = 0.0001380$
$\alpha\beta\gamma\delta\epsilon\zeta n^{12} = 0.0000034$	$12\alpha\beta\gamma\delta\epsilon\zeta n^{12} = 0.0000408$
$\dots$	$\dots$
$10$	$215$

$$0.0247757 \quad t = 0.0550685$$

$$\text{ergo } s = 0.9752242$$

Hinc reperitur  $\log. z = 8.7517967$ . Cum nunc sit

$$ni = \frac{z}{z+1} i = \frac{z n^2 - z(1-n^2)z}{1+n^2-2(1-n^2)z} = \frac{18}{34} - \frac{32}{34} \cdot \frac{z}{z+1},$$

reperiemus

$$\frac{z}{z+1} i = 0.50300, \text{ hincque } i = 0.838333.$$

Hinc ergo vicissim, si pro dato rectangulo fuerit

$$i = \frac{f^2 - g^2}{f^2 + g^2} = 0.838333,$$

tum

tum pro ellipsi satisfaciente erit

$$n = \frac{z}{s}, \text{ siue } a:b = z:1.$$

### Problema 3.

§. 12. Si data fuerit species rectanguli, cui ellipsis minimae perimetri circumscribi oporteat, eius speciem per aequationem finitam definire.

### Solutio.

Cum species rectanguli continetur ratione inter eius latera, littera nostra  $i$  eius speciem declarat, cum sit  $i = \frac{f^2 - g^2}{f^2 + g^2}$ ; deinde quia species ellipsis ratione inter eius axes indicatur, ea in nostra littera  $n$  comprehendetur, cum sit  $n = \frac{a^2 - b^2}{a^2 + b^2}$ . Requiritur ergo aequatio finitis terminis expressa, quae relationem inter has quantitates  $i$  et  $n$  exhibeat. Nunc autem vidimus esse

$$in = \frac{2n^2 - 2(1-n^2)x}{1+n^2 - 2(1-n^2)x},$$

vbi  $x$  per hanc aequationem differentialem determinatur:

$$dx = \frac{n dn}{1-n n} + \frac{x^2 dn}{n}.$$

Nihil aliud igitur supereft, nisi vt hinc littera  $x$  eliminetur. Quod quo facilius fieri possit, ponamus  $2(1-n^2)x = y$ , vt fiat  $in = \frac{2n^2 - y}{1+n^2 - y}$ ; at ob

$$x = \frac{y}{2(1-n n)} \text{ erit } dy = \frac{dn}{2(1-n n)} + \frac{nx dn}{(1-n n)^2};$$

vnde orietur

$$2(1-n n)dx + 4nx dn = (1-n n)ndn + \frac{x^2 dn}{n}.$$

Ex illa autem aequatione, ponendo  $v$  loco  $in$ , nascitur

$$x = \frac{2n^2 - (1+n^2)v}{1-v}, \text{ hincque}$$

$$dx =$$

$$dx = -\frac{dv(1-nn)}{(1-v)^2} + \frac{4ndn}{1-v} - \frac{2nvdn}{1-v},$$

quo substituto prodit

$$-\frac{2dv(1-nn)^2}{(1-v)^2} + 8ndn = (1-nn)ndn \\ + \frac{4n^4 - 4n^2(1+n^2)v + (1+n^2)^2v^2}{(1-v)^2} \cdot \frac{dn}{n},$$

et fractionibus sublatis hinc colligitur

$$-\frac{2dv(1-nn)^2}{dn} = -7n + 3n^3 + 10nv + \frac{1}{n}v^2 \\ - 2n^3 \cdot v - 5n \cdot v^2.$$

Nunc denique loco  $v$  suum valorem  $i n$  substituamus, ac prodit

$$-\frac{2n(1-nn)^2 \cdot di}{dn} = -7n + 3n^3 \\ + 2i(1+3nn) \\ + i^2 \cdot n(1-5n^2).$$

Quodsi igitur constructio huius aequationis concedatur, non solum pro singulis valoribus ipsius  $n$  conuenientes valores pro  $i$ , sed etiam vicissim pro singulis valoribus ipsius  $i$  respondentes ipsius  $n$  elicere licebit. Haecque profecto aequatio multo simplicior evasit, quam initio sperare licuerat. Quodsi eam adeo integrare, vel saltem construere liceret, analysi insigne incrementum accessisse foret censendum.