# DILVCIDATIONES DE MOTV CHORDARVM INAEQVALITER CRASSARVM.

Auctore

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Cum mihi nuper praelegeretur egregia dissertatio Illustr. *Bernoulli*, de motu chordarum inaequaliter crassarum, *Actis Academiae Berolinensis pro Anno* 1767 *inserta*, mirifice mihi placuit formula generalis, quam dedit pro crassitie variabili chordae, vt eius motus saltem regularis per sinus cuiuspiam anguli exprimi posset. Statim igitur in mentem mihi venit, num pro his casibus etiam motus in genere definiri, ac solutio perfecta exhiberi queat. Saepius enim iam monui, solutionem perfectam huiusmodi quaestionum sollicite distingui opportere a solutionibus generalibus, quae quidem omnes motus possibiles in se complectuntur, sed quia numero terminorum infinito constant, ad eos casus, quibus status chordae initialis praescribitur,

 neuti-

neutiquam accommodari possunt: si quidem infinitos coëfficientes definiri oporteret, id quod omnes Analyseos vires superaret. Ad solutionem scilicet perfectam huiusmodi Problematum requiritur, vt motus per quampiam formulam finitam exprimatur, cuius applicationem ad quosuis status initiales exsequi liceat, qualem solutionem Celeberr. *la Grange* et ego iam dudum dedimus pro chordis vniformiter crassis, quae functionibus generalissimis siue continuis siue discontinuis continetur; cuius ope pro quouis statu initiali totus chordae motus facillime determinari potest, id quod per alias solutiones, etiamsi omnes motus possibiles in se complectantur, nullo modo praestare licet. Huius rei exemplum iam pridem proposui circa chordam, cuius aliqua tantum portio initio de statu naturali deturbetur, dum reliqua pars in directum maneat extensa, quem casum nemo adhuc per formulas illas sinuum in infinitum progredientes assignare valuit. Hanc ob caussam etiam determinatio motuum, quibus laminae elasticae cieri possunt, nequaquam pro solutione completa haberi potest: propterea quod eam non ad quemuis statum initialem accommodare licet. Multo minus, quae de oscillationibus funium perfecte flexibilium sunt prolata, solutionem perfectam continent; propterea quod formulae non solum in infinitum progrediuntur, sed etiam ne omnes quidem motus regulares assignare licet. His igitur praemissis inuestigationem meam de motu chordarum inaequaliter crassarum aggredior.

Tab. IV. Fig. 1. §. 1. Repraesentet igitur recta A B chordam vtcunque inaequaliter crassam et in punctis A et B fixam, cuius longitudo sit $AB = a$; tum sumta eius portione qua-

quacunque indefinita A X $= x$, sit hoc loco crassities sectio chordae transuersa $= uu$, vt volumen huius portionis AX sit $\int uu\,dx$, quo simul eius massa et pondus exprimatur; vis autem, qua haec chorda tenditur, aequalis sit ponderi, quod massa eiusdem materiae, ex qua chorda constat, cuius volumen sit $= c^{3}$, esset habeturum. His positis si elapso tempore $= t''$ punctum chordae X reperiatur in Y, et haec applicata quasi infinite parua ponatur XY $= y$, ex iis, quae de motu chordarum iam saepius sunt tradita, constat, omnes motus exprimi hac aequatione differentiali secundi gradus:

$$\frac{uu}{2g}\left(\frac{ddy}{dt^2}\right) = c^3\left(\frac{ddy}{dx^2}\right), \text{ siue } \frac{1}{2g}\left(\frac{ddy}{dt^2}\right) = \frac{c^3}{uu}\left(\frac{ddy}{dx^2}\right),$$

cuius integrale completum si exhiberi posset, praeberet ad quoduis tempus quantitatem applicatae X Y $= y$. Facile autem intelligitur, tale integrale completum in genere pro quacunque crassitiei lege nequaquam sperari posse, ex quo in eiusmodi leges crassitiei $uu$ inquiri conuenit, pro quibus solutio perfecta obtineri queat.

§. 2. Hic autem primo sum contemplaturus casum, quem Illustris *Bernoulli* tractauit, vbi inuestigauit, cuiusmodi functio ipsius $x$ statui debeat crassities $uu$, vt saltem motum chordae regularem et quasi principalem per sinum cuiuspiam anguli exprimere liceat. Hunc in finem sit $k$ longitudo penduli simplicis, cuius oscillationes cum motu chordae congruant: et iam satis constat esse debere $\frac{1}{2g}\left(\frac{ddy}{dt^2}\right) = -\frac{y}{k}$, cuius aequationis integrale completum, sumto $x$ constante, est $y = F : \sin. (\zeta + t\sqrt{\frac{2g}{k}})$, vbi litera F functionem quamcunque ipsius $x$ significare potest; tum igitur etiam necesse est fieri

$$\frac{c^4}{uu}\left(\frac{ddy}{dx^2}\right) = -\frac{y}{k}, \text{ siue } y + \frac{c^4 k}{uu}\left(\frac{ddy}{dx^2}\right) = 0,$$

Atque hic iam quaeritur, qualis functio crassities $uu$ ipsius $x$ esse debeat, vt integrale huius aequationis per sinum cuiuspiam anguli exprimi queat, ita vt sit $y = p \sin. q$, vbi $p$ et $q$ certas functiones ipsius $x$ exprimant. Quia autem applicata $y$ euanescere debet, tam posito $x = 0$ quam $x = a$, primo functio $u$ ita debet esse comparata, vt euanescat posito $x = 0$; tum vero vt posito $x = a$ fiat $q = \pi$; denotante $\pi$ semiperipheriam circuli, cuius radius $= 1$. Hanc ob rem denotet X talem functionem ipsius $x$, quae facto $x = 0$ euanescat; tum vero posito $x = a$ haec functio X abeat in A, ac manifestum est huic duplici conditioni satisfieri, si sumatur $q = \frac{\pi X}{A}$, ita vt sit $y = p \sin. \frac{\pi X}{A}$.

§. 3. Quoniam autem in nostra inuestigatione tantum differentiale huius anguli $\frac{\pi X}{A}$ in computum ingreditur: ponamus $\frac{\pi X}{A} = \int v\, dx$, vt sit $y = \sin. \int v\, dx$; hinc igitur differentiando erit

$$\frac{dy}{dx} = \frac{dp}{dx} \sin. \int v\, dx + p v \cos. \int v\, dx,$$

hincque porro

$$\frac{ddy}{dx^2} = \frac{ddp}{dx^2} \sin. \int v\, dx + \frac{v\, dp}{dx} \cos. \int v\, dx$$
$$+ d.\frac{pv}{dx} \cos. \int v\, dx - p v v \sin. \int v\, dx.$$

Quia autem in valore $y$ cosinus istius anguli non occurrit, necesse est vt etiam hic termini cosinum continentes se destruant, vnde fiet

$$v\, dp + d.pv = 0, \text{ siue } 2v\, dp + p\, dv = 0,$$

quae aequatio ducta in $p$ et integrata praebet: $ppv = \alpha$, ideo-

ideoque $v = \frac{\alpha}{pp}$, vbi $\alpha$ denotat constantem quamcunque. Hinc igitur erit

$$\frac{ddy}{dx^2} = \left(\frac{ddp}{dx^2} - \frac{\alpha\alpha}{p^3}\right) \text{fin.} \int v\, dx;$$

quo valore fubftituto aequatio noftra $y - \frac{c^3 k}{uu}\left(\frac{ddy}{dx^2}\right) = 0$, nobis fuppeditat iftam relationem:

$$p + \frac{c^3 k}{uu}\left(\frac{ddp}{dx^2} - \frac{\alpha\alpha}{p^3}\right) = 0,$$

vnde deducitur craffities quaefita

$$uu = \frac{c^3 k}{p}\left(\frac{\alpha\alpha}{p^3} - \frac{ddp}{dx^2}\right).$$

Videri hic quidem poffet ex hac aequatione etiam viciffim $p$ per $uu$ definiri potuiffe, ita vt ipfa craffities $uu$ arbitrio noftro relinqueretur. Verum tentanti mox patebit, illam aequationem differentialem fecundi gradus in genere nullo modo refolutionem admittere.

§. 4. Cum igitur iam pro $p$ functio quaecunque ipfius $x$ accipi queat, erit $\int v\, dx = \alpha \int \frac{dx}{pp}$, quod integrale ita eft capiendum, vt euanefcat pofito $x = 0$; tum vero fieri oportet $\frac{\pi X}{A} = \alpha \int \frac{dx}{pp}$; vnde, quia pofito $x = a$ fit $X = A$, hinc noua determinatio conftantium eft petenda. Deinde quia in hac integratione tempus $t$ pro conftante eft habitum, valor ipfius $y = p$ fin $\int v\, dx$ infuper per functionem quamcunque temporis, quae fit T, multiplicari poteft, ita vt fit $y = T p\, \text{fin.} \int v\, dx$. Ex confideratione vero temporis habebamus $y = F : \text{fin.} (\zeta + t \sqrt{\frac{2g}{k}})$, quae ergo duae aequationes identicae funt reddendae, quod fiet fi fumamus $F = C p\, \text{fin.} \int v\, dx$ et $T = C\, \text{fin.} (\zeta + t \sqrt{\frac{2g}{k}})$, ita vt motus ifte regularis noftrae chordae definiatur hac

aequa-

aequatione:

$$y = C p \sin. \int v\, dx \sin. (\zeta + t \sqrt{\tfrac{2g}{k}}),$$

dummodo craſſities chordae ita fuerit comparata, vt ſit

$$u u = \frac{c^5 k}{p} \left(\frac{\alpha\alpha}{p^4} - \frac{d d p}{d x^2}\right)$$

exiſtente $v = \frac{\alpha}{p p}$.

§. 5. Hoc autem modo vnica tantum ſpecies vibrationum regularis refertur. Vt igitur plures ſimul complectamur loco $\pi$ ſcribamus $i \pi$; denotante $i$ numerum integrum quemcunque, et faciamus $\int v\, d x = \frac{i \pi X}{A}$; tum vero ſumta pro $p$ functione quacunque ipſius $x$, quia invenimus $v = \frac{\alpha}{p p}$, ſtatim ſtatuere licet $X = \int \frac{d x}{p p}$, fietque

$$\int v\, d x = \alpha X = \frac{i \pi X}{A},$$

ſicque erit conſtans $\alpha = \frac{i \pi}{A}$; quo valore ſuſtituto nanciſcimur craſſitiem chordae $u u = c^5 k \left(\frac{i i \pi \pi}{A A\, p^4} - \frac{d d p}{p d x^2}\right)$; vbi notandum, hunc valorem neutiquam a numero $i$ pendere debere; quia alioquin quaelibet alia vibrationum ſpecies aliam craſſitiem requireret, vnde pro $k$ talis functio ipſius $i$ accipi debet, vt numerus $i$ penitus ex iſta formula excludatur; quo obſeruato motus chordae iſta aequatione exprimetur:

$$y = C p \sin. \frac{i \pi X}{A} \sin. (\zeta + t \sqrt{\tfrac{2g}{k}}),$$

hincque tempus vnius vibrationis erit $\pi \sqrt{\frac{k}{2g}}$ min. ſec. et ſonus hinc oriundas $\frac{\sqrt{2g}}{\pi \sqrt{k}}$. Quo autem melius indolem harum formularum perſpiciamus, aliquot caſus particulares euoluamus.

Caſus

## Casus primus.

quo $p = \lambda x + \mu$.

§. 6. Hinc igitur erit

$$X = \int \frac{dx}{(\lambda x + \mu)^2} = \frac{-1}{\lambda(\lambda x + \mu)} + C.$$

quae constans C cum ita debeat sumi, vt posito $x = 0$ fiat etiam $X = 0$, erit $X = \frac{x}{\mu(\lambda x + \mu)}$; vnde facto $x = a$ erit $A = \frac{a}{\mu(\lambda a + \mu)}$, sicque $\frac{X}{A} = \frac{x(\lambda a + \mu)}{a(\lambda x + \mu)}$. Porro pro crassitie, quoniam $ddp = 0$, habebimus

$$uu = \frac{c^3 k.\, ii\, \pi\pi\mu\mu(\lambda a + \mu)^2}{aa(\lambda x + \mu)^4}.$$

Vt igitur hinc numerum $i$ excludamus, statuamus $iik = f$, vt sit $k = \frac{f}{ii}$, eritque

$$uu = c^3 f. \frac{\pi\pi\mu\mu(\lambda a + \mu)^2}{aa(\lambda x + \mu)^4}.$$

Hinc tempus vnius vibrationis fiet $\frac{\pi}{i}\sqrt{\frac{f}{2g}}$ et sonus ipse $= \frac{i\sqrt{2g}}{\pi\sqrt{f}}$.

§. 7. Quia inuenimus

$$uu = c^3 f. \frac{\pi\pi\mu\mu(\lambda a + \mu)^2}{aa(\lambda x + \mu)^4}$$

in ipso termino A, vbi $x = 0$, erit crassities $= c^3 f. \frac{\pi\pi(\lambda a + \mu)^2}{aa\mu\mu}$, in altero vero termino, vbi $x = a$, ea erit $c^3 f \frac{\pi\pi\mu\mu}{aa(\lambda a + \mu)^2}$; Intra hos autem terminos crassities variatur in ratione reciproca quadruplicata formulae $\lambda x + \mu$. Hinc igitur quantitates constantes $\lambda$ et $\mu$ ita accipi possunt, vt in vtroque termino A et B chorda datam obtineat crassitiem. Veluti si in termino A crassities ponatur $= \alpha\alpha$, pro altero vero termino $B = \beta\beta$, quantitates $\lambda$ et $\mu$ sequenti modo determinabuntur. Ponamus breuitatis gratia $c^3 f \frac{\pi\pi}{aa} = ee$, vt

 fiat

fiat $\alpha\alpha = ee\frac{(\lambda a + \mu)^2}{\mu\mu}$ et $\beta\beta = ee\frac{\mu\mu}{(\lambda a + \mu)^2}$, vnde extracta radice erit $\alpha = \frac{e(\lambda a + \mu)}{\mu}$ et $\beta\beta = \frac{e\mu}{\lambda a + \mu}$. Ex priore expressione fit $\frac{\lambda}{\mu} = \frac{(\alpha - e)}{ae}$, ex posteriore vero $\frac{\lambda}{\mu} = \frac{e - \beta}{\beta a}$; vnde patet, crassities $\alpha\alpha$ et $\beta\beta$ non pro lubitu assumi posse, siquidem quantitati $e$ valor datus tribuatur. At si literis $\alpha$ et $\beta$ dati valores dentur, litera $e$ ex hac aequatione definietur: $\frac{\alpha - e}{e} = \frac{e - \beta}{\beta}$, ex qua fit $e = \sqrt{\alpha\beta}$, hincque porro fit

$$\frac{\lambda}{\mu} = \frac{\alpha - \sqrt{\alpha\beta}}{a\sqrt{\alpha\beta}} = \frac{\sqrt{\alpha} - \sqrt{\beta}}{a\sqrt{\beta}}.$$

Hinc si capiatur $\mu = a$, erit

$$\lambda = \frac{\sqrt{\alpha} - \sqrt{\beta}}{\sqrt{\beta}} = \sqrt{\frac{\alpha}{\beta}} - 1.$$

Quia autem requiritur vt sit $ee = \alpha\beta$, erit

$$\alpha\beta = c^3 f\frac{\pi\pi}{aa}, \text{ hincque } c^3 f = \frac{aa\alpha\beta}{\pi\pi}$$

hincque porro $f = \frac{aa\alpha\beta}{c^3\pi\pi}$, vbi $c^3$ denotat tensionem nostrae chordae, quae manente chorda vtcunque variari potest. Hinc etiam patet, quomodo vibrationes mutata tensione se sint habiturae: Erit enim tempus cuiusque vibrationis $= \frac{a}{i\pi}\sqrt{\frac{\alpha\beta}{2g\,c^3}}$ et ipse sonus $= \frac{i\pi}{a}\sqrt{\frac{2g\,c^3}{\alpha\beta}}$; vnde patet, sonum sequi rationem subduplicatam tensionis $c^3$, prorsus vti in chordis vniformiter crassis.

§. 8. Quod si ergo pro chorda nostra detur crassities in termino $A = \alpha\alpha$, in altero vero termino $B = \beta\beta$: ob $\lambda = \sqrt{\frac{\alpha}{\beta}} - 1$ et $\mu = a$, erit crassities $uu = \frac{a^4\alpha\alpha}{(\lambda x + a)^4}$. Pro ipso autem motu prodibit ista expressio:

$$y = C(\lambda x + a)\,\text{sin.}\frac{i\pi x}{\lambda x + a}\sqrt{\frac{\alpha}{\beta}}\,\text{sin.}\left(\zeta + \frac{i\pi}{a}t\sqrt{\frac{2g\,c^3}{\alpha\beta}}\right).$$

Sic igitur manifestum est, eandem chordam, quam descrip-

simus

ſimus, infinitos motus regulares recipere poſſe, et quidem pro eadem tenſione $c^3$; vnde pro ſingulis valoribus numeri $i$ ſoni ſequentibus formulis exprimentur:

$$\frac{1}{a}\sqrt{\frac{2gc^3}{\alpha\beta}},\ \frac{2}{a}\sqrt{\frac{2gc^3}{\alpha\beta}};\ \frac{3}{a}\sqrt{\frac{2gc^3}{\alpha\beta}},\ \text{etc.}$$

quae formulae numerum vibrationum vno minuto ſecundo editarum indicant. Euidens igitur eſt hos diuerſos ſonos ſecundum ſeriem numerorum naturalium 1, 2, 3, 4, etc. progredi, prorſus vt euenit in chordis vniformiter craſſis.

§. 9. Cum igitur innumerabiles prodierint ſoni ſimplices, quos noſtra chorda edere poterit, iis vtcunque inter ſe permiſcendis orientur omnes plane ſoni mixti, qui produci poſſunt; atque adeo aequatio generalis exhiberi poteſt, quae omnes iſtos motus poſſibiles in ſe complectatur, quae erit

$$\begin{aligned} y = {} & C(\lambda x + a)\,\text{ſin.}\,\frac{\pi x}{\lambda x + a}\sqrt{\frac{\alpha}{\beta}}\,\text{ſin.}\left(\zeta + \frac{\pi}{a}t\sqrt{\frac{2gc^3}{\alpha\beta}}\right) \\ & + C'(\lambda x + a)\,\text{ſin.}\,\frac{2\pi x}{\lambda x + a}\sqrt{\frac{\alpha}{\beta}}\,\text{ſin.}\left(\zeta' + \frac{2\pi}{a}t\sqrt{\frac{2gc^3}{\alpha\beta}}\right) \\ & + C''(\lambda x + a)\,\text{ſin.}\,\frac{3\pi x}{\lambda x + a}\sqrt{\frac{\alpha}{\beta}}\,\text{ſin.}\left(\zeta'' + \frac{3\pi}{a}t\sqrt{\frac{2gc^3}{\alpha\beta}}\right) \\ & + C'''(\lambda x + a)\,\text{ſin.}\,\frac{4\pi x}{\lambda x + a}\sqrt{\frac{\alpha}{\beta}}\,\text{ſin.}\left(\zeta''' + \frac{4\pi}{a}t\sqrt{\frac{2gc^3}{\alpha\beta}}\right) \\ & + \text{etc.} \end{aligned}$$

vbi tam literae $C$, $C'$, $C''$, quam $\zeta$, $\zeta'$, $\zeta''$ etc. penitus pro arbitrio accipi poſſunt. Quia vero haec expreſſio in infinitum extenditur, maxime etiamnunc deſideratur formula finita omnes pariter motus poſſibiles comprehendens, qualis pro chordis vniformiter craſſis eſt inuenta. Niſi enim talis expreſſio habeatur, quaeſtio, qua ex dato ſtatu initiali motus ſecuturus poſtulatur, ne ſuſcipi quidem poteſt. Nemo autem profecto negabit, quin etiam his chordis initio figura quaecunque pro lubitu imprimi

 poſſit,

possit, vnde sine vllo dubio certus motus sequetur, cuius determinatio merito postulari potest.

## Casus secundus.

quo assumitur $p = \surd(\lambda x + \mu)$.

§. 10. Hinc igitur erit

$$X = \int \frac{dx}{pp} = \int \frac{dx}{\lambda x + \mu} = \frac{1}{\lambda} l \frac{\lambda x + \mu}{\mu} \text{ et } A = \frac{1}{\lambda} l \frac{\lambda a + \mu}{\mu},$$

ergo

$$\frac{X}{A} = \frac{l \frac{\lambda x + \mu}{\mu}}{l \frac{\lambda a + \mu}{\mu}}.$$

Tum vero pro crassitie inuenienda, ob

$$p = \surd(\lambda x + \mu) \text{ et } l p = \tfrac{1}{2} l (\lambda x + \mu), \text{ erit}$$

$$\frac{dp}{p\,dx} = \frac{\frac{1}{2}\lambda}{\lambda x + \mu},$$

ac denuo differentiando

$$\frac{ddp}{p\,dx^2} - \frac{dp^2}{pp\,dx^2} = \frac{-\frac{1}{2}\lambda\lambda}{(\lambda x + \mu)^2}.$$

Est vero

$$\frac{dp^2}{pp\,dx^2} = \frac{\frac{1}{4}\lambda\lambda}{(\lambda x + \mu)^2}$$

ex quo colligitur

$$\frac{ddp}{p\,dx^2} = \frac{-\frac{1}{4}\lambda\lambda}{(\lambda x + \mu)^2};$$

hinc igitur oritur crassities

$$uu = \frac{c^3 k}{AA(\lambda x + \mu)^2}(ii\pi\pi + \tfrac{1}{4}\lambda\lambda AA).$$

Pona-

Ponatur igitur

$$k(ii\pi\pi+\tfrac{1}{4}\lambda\lambda AA)=f, \text{ vt fit } k=\frac{f}{ii\pi\pi+\frac{1}{4}\lambda\lambda AA},$$

vbi $f$ eſt quantitas a numero $i$ non pendens; erit ergo $uu=\frac{c^3 f}{AA(\lambda x+\mu)^2}$, vnde pro initio A, vbi $x=0$, erit craſſities $\frac{c^3 f}{AA\,\mu\mu}$, et pro altero termino, vbi $x=a$, erit craſſities $=\frac{c^3 f}{AA(\lambda a+\mu)^2}$. Quod ſi ergo prior craſſities ponatur vt ante $=\alpha\alpha$, poſterior vero $=\beta\beta$, ponaturque $\frac{c^3 f}{AA}=ee$, erit $\alpha\alpha=\frac{ee}{\mu\mu}$ et $\beta\beta=\frac{ee}{(\lambda a+\mu)^2}$; ex priore ſtatim fit $\mu=\frac{e}{\alpha}$, ex poſteriore vero $\lambda=\frac{e-\beta\mu}{\beta a}=\frac{e(\alpha-\beta)}{\alpha\beta a}$, quibus valoribus ſubſtitutis erit $ee=\frac{c^3 f}{AA}$; quae formula ob $A=\frac{\alpha\beta a}{e(\alpha-\beta)}\,l\frac{\alpha}{\beta}$, praebet hanc aequationem:

$$1=\frac{c^3 f(\alpha-\beta)^2}{\alpha\alpha\beta\beta aa(l\frac{\alpha}{\beta})^2}, \text{ ſicque erit}$$

$$c^3 f=\frac{\alpha\alpha\beta\beta aa(l\frac{\alpha}{\beta})^2}{(\alpha-\beta)^2}$$

ita vt in genere ſit craſſities

$$uu=\frac{\alpha^2\beta^2 a^2}{((\alpha-\beta)x+\beta a)^2},$$

quae ergo etiam non a tenſione $c^3$ pendet, prorſus vti rei natura poſtulat, et craſſities erit in X reciproce vt

$$((\alpha-\beta)x+\beta a)^2.$$

§. 11. Definita igitur chorda in ſingulis punctis, in formulam pro tempore inquiramus, et quia poſuimus $ee=\frac{c^3 f}{AA}$, erit

$$f=\frac{AA\,ee}{c^3}=\frac{\alpha\alpha\beta\beta aa}{c^3(\alpha-\beta)^2}(l\frac{\alpha}{\beta})^2. \text{ Poſueramus vero}$$

 $f=$

$$f = k\left(ii\pi\pi + \tfrac{1}{4}\lambda\lambda AA\right) = k\left(ii\pi\pi + \tfrac{1}{4}\left(l\tfrac{\alpha}{\beta}\right)^2\right),$$

ſicque

$$k = \frac{f}{ii\pi\pi + \frac{1}{4}\left(l\frac{\alpha}{\beta}\right)^2} \text{ quocirca erit}$$

$$k = \frac{\alpha\alpha\beta\beta aa\left(l\frac{\alpha}{\beta}\right)^2}{c^3(\alpha-\beta)^2\left(ii\pi\pi + \frac{1}{4}\left(l\frac{\alpha}{\beta}\right)^2\right)};$$

quare cum tempus vnius vibrationis ſit $\pi\sqrt{\frac{k}{2g}}$, et ipſe ſonus editus

$$= \frac{\sqrt{2g}}{\pi\sqrt{k}} = \frac{(\alpha-\beta)\sqrt{2gc^3\left(ii\pi\pi + \frac{1}{4}\left(l\frac{\alpha}{\beta}\right)^2\right)}}{\pi\alpha\beta a\, l\frac{\alpha}{\beta}},$$

patet, ſi literae $i$ ſucceſſiue tribuamus valores 1, 2, 3, 4, etc. ſonos diuerſos progredi ſecundum hanc progreſſionem:

$$\sqrt{\left(\pi\pi + \tfrac{1}{4}\left(l\tfrac{\alpha}{\beta}\right)^2\right)},\ \sqrt{\left(4\pi\pi + \tfrac{1}{4}\left(l\tfrac{\alpha}{\beta}\right)^2\right)},\ \sqrt{\left(9\pi\pi + \tfrac{1}{4}\left(l\tfrac{\alpha}{\beta}\right)^2\right)}$$
$$\sqrt{\left(16\pi\pi + \tfrac{1}{4}\left(\tfrac{\alpha}{\beta}\right)^2\right)},\ \text{etc.}$$

vnde intelligitur hos ſonos nulla harmonia inter ſe eſſe connexos, quoniam ſinguli per formulas irrationales exprimuntur. Hic notaſſe iuuabit, ſi ſumatur $\beta = \alpha$, tum caſum prodire pro chordis vniformiter craſſis: fiet enim vbique $uu = \alpha\alpha$, et quia vbique formula $\frac{\alpha-\beta}{l\frac{\alpha}{\beta}}$ abit in $\alpha$, ſonus editus erit $\frac{i\sqrt{2gc^3}}{\alpha a}$, id quod egregie conuenit cum iis, quae de his chordis ſunt inuenta.

§. 12. Nunc denique etiam formulam $\frac{i\pi X}{A}$ expediamus, et quia inuenimus $\lambda = \frac{e(\alpha-\beta)}{\alpha\beta a}$, et $\mu = \frac{e}{a}$; erit

$$X = \frac{\alpha\beta a}{e(\alpha-\beta)}\, l\,\frac{\alpha x - \beta x + \beta a}{\beta a},$$

O 3 hinc-

hincque:

$$A = \frac{\alpha\beta a}{e(\alpha-\beta)}\, l\frac{\alpha}{\beta}, \text{ vnde fit}$$

$$\frac{X}{A} = \frac{l\frac{\alpha x - \beta x + \beta a}{\beta a}}{l\frac{\alpha}{\beta}}.$$

Denique cum fit

$$p = \sqrt{(\lambda x + \mu)} = \sqrt{\left(\frac{e(\alpha-\beta)x + \beta a}{\alpha\beta a}\right)},$$

factor constans $\sqrt{\frac{e}{\alpha\beta a}}$, commode in coëfficiente constante C comprehendi poteſt. His igitur inuentis omnes motus regulares, quos quidem haec chorda recipere poteſt, in ſequenti formula generali continentur:

$$y = C\sqrt{((\alpha-\beta)x + \beta a)}\ \text{ſin.}\ i\pi\,\frac{l\frac{(\alpha-\beta)x + \beta a}{\beta a}}{l\frac{\alpha}{\beta}} \times$$

$$\times\ \text{ſin.}\left(\zeta + t\,\frac{(\alpha-\beta)\sqrt{2gc^3\left(ii\pi\pi + \frac{1}{4}\left(l\frac{\alpha}{\beta}\right)^2\right)}}{\alpha\beta a\, l\frac{\alpha}{\beta}}\right).$$

Quocirca ſi in hac formula loco $i$ ſucceſſiue ſcribantur numeri 1, 2, 3, 4, etc. et ſingulis literis arbitrariis C et $\zeta$ alii valores quicunque tribuantur, qui ſint $C'$, $\zeta'$; $C''$, $\zeta''$; $C'''$, $\zeta'''$; etc. omnes iſtae formulae in vnam ſummam collectae dabunt aequationem generalem pro omnibus plane motibus mixtis, quibus iſta chorda contremiſcere poteſt. Neque vero propterea haec ſolutio pro perfecta haberi poteſt, quoniam neutiquam ad quamuis datam figuram initialem adcommodari poteſt.

§. 13. In reſolutione huius caſus inſigne paradoxon occurrit: quod in formulis omnes ſonos exhibentibus etiam poni poſſe videtur $i = 0$; hinc enim prodiret ſonus editus $\frac{(\alpha-\beta)\sqrt{\frac{1}{2}gc^3}}{\pi\alpha\beta a}$, cum tamen fiat hoc caſu $y = 0$.

Ad

Ad hoc autem paradoxon resoluendum ſpectemus literam $i$ vt infinite paruam, ita vt ſinus anguli euaneſcentis $\frac{i\pi X}{A}$ ipſi angulo aequalis cenſeri queat; tum vero ne ipſa applicata euaneſcat, loco C ſcribatur $\frac{C}{i}$, ita vt iam habeamus:

$$y = \pi C \sqrt{((\alpha-\beta)x+\beta a)}\,\frac{l\frac{(\alpha-\beta)x+\beta a}{\beta a}}{l\frac{\alpha}{\beta}}\,\text{ſin.}\left(\zeta + \frac{t(\alpha-\beta)\sqrt{\tfrac{1}{2}g}\,c^{x}}{\alpha\beta a}\right).$$

Neque iam vllum dubium eſſe poteſt, quin motus vibratorius huic formulae futurus eſſet conformis, ſi modo haec formula in ſtatu noſtrae chordae locum inuenire poſſet, vbi aſſuminus, poſito $x = a$ fieri $y = 0$, quod quia in hac formula non euenit, euidens eſt, talem motum penitus eſſe excludendum.

## Caſus tertius

### quo aſſumitur $p = \sqrt{(\lambda xx + 2\mu x + \nu)}$.

§. 14. Hic ergo caſus ambos praecedentes in ſe complectitur: primum ſcilicet quando $\lambda xx + 2\mu x + \nu$ eſt quadratum; ſecundum vero quando $\lambda = 0$; praeterea vero infinite latius patet. Hinc igitur erit

$$X = \int\frac{dx}{pp} = \int\frac{dx}{\lambda xx + 2\mu x + \nu},$$

cuius formulae integratio vel a logarithmis pendet, vel a quadratura circuli, prouti denominator vel habuerit factores reales, vel ſecus, id quod hic notaſſe ſufficit, quandoquidem hoc integrale ipſa litera X deſignari conueniet. Quod ſi autem in integrali ponatur $x = a$, habebitur valor literae A, quae ergo in ſe continebit literas $\lambda$, $\mu$ et $\nu$. Nunc igitur pro craſſitie $uu$ inuenienda, quoniam habemus

$$lp = \tfrac{1}{2}l(\lambda xx + 2\mu x + \nu) \text{ erit } \frac{dp}{pdx} = \frac{\lambda x + \mu}{\lambda xx + 2\mu x + \nu};$$

hinc

hincque porro differentiando

$$\frac{ddp}{pdx^2} - \frac{dp^2}{ppdx^2} = -\frac{\lambda\lambda xx - 2\lambda\mu x + \lambda\nu - 2\mu\mu}{(\lambda xx + 2\mu x + \nu)^2}.$$

Addatur

$$\frac{dp^2}{ppdx^2} = \frac{\lambda\lambda xx + 2\lambda\mu x + \mu\mu}{(\lambda xx + 2\mu x + \nu)^2}$$

et nanciſcemur

$$\frac{ddp}{pdx^2} = \frac{\lambda\nu - \mu\mu}{(\lambda xx + 2\mu x + \nu)^2},$$

quam ob rem pro craſſitie habebimus

$$uu = c^3 k \left( \frac{ii\pi\pi}{AA(\lambda xx + 2\mu x + \nu)^2} - \frac{\lambda\nu + \mu\mu}{(\lambda xx + 2\mu x + \nu)^2} \right).$$

Nunc igitur ponamus $k(ii\pi\pi + (\mu\mu - \lambda\nu)AA) = f$, et habebimus $uu = \frac{c^3 f}{AA(\lambda xx + 2\mu x + \nu)^2}$, qua aequatione lex craſſitiei per totam chordam exprimitur. Ponamus autem vt hactenus $\frac{c^3 f}{AA} = ee$, vt fiat:

$$uu = \frac{ee}{(\lambda xx + 2\mu x + \nu)^2}, \text{ ideoque } u = \frac{e}{\lambda xx + 2\mu x + \nu}.$$

§. 15. Quia hic habemus tres coefficientes $\lambda$, $\mu$ et $\nu$, eos non ſolum ita definire licebit, vt in vtroque termino craſſities datam obtineat magnitudinem; ſed etiam efficere poterimus, vt chorda quoque in alio loco, veluti puncto medio C, datam nanciſcatur craſſitiem. Statuamus igitur craſſitiem in termino $A = \alpha\alpha$, in altero termino $B = \beta\beta$ et in ipſo chordae medio $C = \gamma\gamma$; hocque modo prodibunt tres ſequentes aequationes:

$$\alpha = \frac{e}{\nu}, \quad \beta = \frac{e}{\lambda aa + 2\mu a + \nu}, \quad \gamma = \frac{e}{\frac{1}{4}\lambda aa + \mu a + \nu},$$

vnde colligimus ſequentes valores:

$$\nu = \frac{e}{\alpha}; \quad \lambda aa + 2\mu a = \frac{e(\alpha - \beta)}{\alpha\beta};$$

$$\tfrac{1}{4}\lambda aa + \mu a = \frac{e(\alpha - \gamma)}{\alpha\gamma}, \text{ hinc}$$

 $\lambda =$

$$\lambda = \frac{2e}{aa}\left(\frac{\alpha\gamma - 2\alpha\beta + \beta\gamma}{\alpha\beta\gamma}\right); \quad \mu = \frac{e}{2a} \cdot \frac{4\alpha\beta - \alpha\gamma - 3\beta\gamma}{\alpha\beta\gamma}; \quad \nu = \frac{e}{\alpha}.$$

Qui ſinguli valores factorem complectuntur $e$, quem autem hactenus vidimus ex calculo iterum excedere; ita vt in his valoribus loco $e$ vbique vnitatem ſcribere liceat: quo obſeruato in genere erit craſſities chordae

$$uu = \frac{1}{(\lambda xx + 2\mu x + \nu)^2}.$$

§. 16. Quia igitur poſuimus $\frac{c^3 f}{AA} = ee$, erit $f = \frac{AAee}{c^3}$, hincque porro

$$k = \frac{AAee}{c^3(ii\pi\pi + (\mu\mu - \lambda\nu)AA)};$$

tum vero hinc porro erit tempus vnius vibrationis

$$= \pi\sqrt{\frac{k}{2g}} = \frac{\pi A e}{\sqrt{eg c^3(ii\pi\pi + (\mu\mu - \lambda\nu)AA)}},$$

ideoque ipſe ſonus

$$= \frac{\sqrt{(2gc^3(ii\pi\pi + (\mu\mu - \lambda\nu)AA))}}{\pi A e};$$

in quibus formulis iterum tenendum eſt, vbicunque occurrat litera $e$, eius loco tuto ſcribi poſſe vnitatem. At vero eſt

$$\mu\mu - \lambda\nu = \frac{16\alpha\alpha\beta\beta - 8\alpha\beta\beta\gamma - 8\alpha\alpha\beta\gamma + \beta\beta\gamma\gamma - 2\alpha\beta\gamma\gamma + \alpha\alpha\gamma\gamma}{4\alpha\alpha\beta\beta\gamma\gamma aa}.$$

Ceterum quia hic quoque omnes ſoni ſimplices per formulas irrationales exprimuntur, nulla inter eos harmonia locum habere poteſt. Et quoniam in genere formulam X euoluere non licet, pro motu et figura chordae haec habebitur aequatio:

$$y = C\sqrt{(\lambda xx + 2\mu x + \nu)}\,\text{ſin.}\,\frac{i\pi X}{A}\,\text{ſin.}\left(\zeta + t\sqrt{\frac{2g}{f}}\right),$$

hincque ex diuerſis valoribus pro $i$ aſſumtis aequatio generalis omnes plane motus continens facile conſtruetur, vt in caſu praecedente.

§. 17.

§. 17. In his tribus casibus commode vsu venit, vt pro omnibus valoribus numeri $i$ eadem lex crassitiei definiri potuerit, vnde deinceps omnes series simplices assignare licuit. Omnibus autem reliquis casibus pro $p$ assumtis, vbi formulae $\frac{1}{p^4}$ et $\frac{ddp}{pdx^2}$ non communem nanciscuntur denominatorem, etiam longitudo penduli simplicis $k$ non amplius per formulam solum numerum $i$ inuoluentem exprimi potest; vnde etiam lex crassitiei, scilicet $uu$, a numero $i$ pendebit, ita vt pro singulis sonis simplicibus lex crassitiei continuo mutaretur. His igitur casibus tantum pro vnico sono simplici chorda construi poterit, cuius quidem inuestigatio nulla plane attentione digna est censenda, quoniam circa tales chordas vnicum tantum sonum simplicem exhibere licet, dum omnes reliqui motus nobis prorsus incogniti essent mansuri.

§. 18. His igitur circa solutionem Illustris *Bernoulli* ingeniosissimam perpensis, consideremus nunc casus, quibus solutionem perfectam exhibere licet, visuri, num casus, quos hic euoluimus, in iis contineantur, nec ne.

## Inuestigatio chordarum ratione crassitiei, pro quibus solutionem perfectam exhibere licet.

§. 19. Hic igitur totum negotium reducitur ad integrationem completam aequationis fundamentalis, differentialis secundi gradus, quam principia motus nobis suppeditarunt, quae est:

$$\frac{uu}{2g}\left(\frac{ddy}{dt^2}\right) = c^2\left(\frac{ddy}{dx^2}\right), \text{ siue } \frac{uu}{2gc^2}\left(\frac{ddy}{dt^2}\right) = \frac{ddy}{dx^2},$$



vbi

vbi quaeritur: quali functione ipsius $x$ crassities $uu$ exprimi debeat, vt huius aequationis integrale completum per functiones arbitrarias exprimi queat? At vero nulli adhuc alii casus pro integratione huius aequationis erui potuerunt, nisi qui contineantur in hac forma:

$$y = p\,\Gamma:(ft \pm X) + q\,\Gamma':(ft \pm X) + r\,\Gamma'':(ft \pm X) + s\,\Gamma''':(ft \pm X)$$

vbi characteres functionum $\Gamma$, $\Gamma'$, $\Gamma''$, $\Gamma'''$ etc. ita a se inuicem pendent, vt fit

$$d.\,\Gamma z = dz\,\Gamma' z;\quad d.\,\Gamma' z = dz\,\Gamma'' z;\quad d.\,\Gamma'' z = dz\,\Gamma''' z;\ \text{etc.}$$

Hoc scilicet modo pro applicata $y$ abscissae $x$ respondente eiusmodi elicitur functio duarum variabilium $x$ et $t$, quae omnes plane motus, quibus chorda contremiscere potest, in se complectantur; quandoquidem hinc ad quoduis tempus $t$ figura chordae definiri poterit. Quin etiam huiusmodi functiones, quoniam sunt arbitrariae, ita assumi possunt, vt pro initio motus, vbi $t = 0$, datum chordae statum producant, ex quo deinceps totus motus secuturus determinari queat. Hinc igitur casus simpliciores euoluamus.

## Casus primus

### quo ponitur $y = p\,\Gamma:(ft \pm X)$.

§. 20. Quoniam hic X denotat functionem ipsius $x$ tantum, eiusque solum differentiale in euolutionem ingreditur, eius loco scribamus $\int v\,dx$, vt habeamus hanc formulam: $y = p\,\Gamma:(ft + \int v\,dx)$, cuius differentialia, prout vel $t$ vel $x$ assumitur variabile, sequenti modo erunt ex-

expressa; dum scilicet literae $p$ et $v$ solam variabilem $x$ in se complectuntur:

$$\left(\frac{dy}{dt}\right)=fp\,\Gamma'\ldots\ldots$$
$$\left(\frac{dy}{dx}\right)=\frac{dp}{dx}\Gamma\ldots+pv\Gamma'\ldots$$
$$\left(\frac{ddy}{dt^2}\right)=ffp\,\Gamma''\ldots\ldots$$
$$\left(\frac{ddy}{dx^2}\right)=\frac{ddp}{dx^2}\Gamma\ldots+\left(\frac{vdp}{dx}+\frac{d.pv}{dx}\right)\Gamma'\ldots+pvv\,\Gamma''\ldots.$$

vbi puncta post characteres $\Gamma$, $\Gamma'$, $\Gamma''$, denotant formulam $ft+\int v\,dx$.

§. 21. His igitur valoribus inuentis aequatio proposita induet sequentem formam:

$$\frac{ffpuu}{2gc^3}\Gamma''\ldots=\frac{ddp}{dx^2}\Gamma\ldots+\frac{(vdp+d.pv)}{dx}\Gamma'\ldots+pvv\Gamma''\ldots.$$

Quare cum tantum similes functiones inter se conferri queant, tres aequationes hinc nascentur:

I. $\frac{ddp}{dx^2}=0$, II. $v\,dp+d.pv=0$, III. $\frac{ffpuu}{2gc^3}=pvv$.

Ex prima statim colligitur $p=\alpha x+\beta$; secunda vero dat $2v\,dp+p\,dv=0$, quae ducta in $p$ et integrata dat

$$pp\,v=\gamma,\ \text{ideoque}\ v=\frac{\gamma}{pp}=\frac{\gamma}{(\alpha x+\beta)^2};$$

ex tertia vero aequatione elicimus

$$uu=\frac{2gc^3vv}{ff}=\frac{2gc^3\gamma\gamma}{ff(\alpha x+\beta)^4}.$$

Sicque innotescit crassities $uu$, pro qua formula integralis assumta locum habet.

§. 22. Quoniam quantitas constans $f$ pro lubitu accipi potest, statuamus $ff=2gc^3$, seu potius loco $2gc^3$ scribamus $ff$, et habebimus $uu\frac{\gamma\gamma}{(\alpha x+\beta)^4}$; ita vt crassities $uu$ sit reciproce vt $(\alpha x+\beta)^4$; vnde patet in hoc casu contineri primum casum solutionis *Bernoullianae*; ita vt

 etiam

etiam pro isto casu solutionem perfectam exhibere liceat; quam igitur vt obtineamus, notasse iuuabit, literam $\gamma$ tam positiue quam negatiue accipi posse, ita vt sit $v = \frac{\pm\gamma}{(\alpha x + \beta)}$; hicque vterque valor seorsim cum functione arbitraria coniungi poterit, ita vt integrale completum pro hac lege crassitiei $uu = \frac{\gamma\gamma}{(\alpha x + \beta)^4}$ sequentem formam sit habiturum:

$$y = (\alpha x + \beta)\Gamma : (ft + \int \frac{\gamma dx}{(\alpha x + \beta)^2}) - (\alpha x + \beta)\Delta : (ft - \int \frac{\gamma dx}{(\alpha x + \beta)^2}).$$

§. 23. Integretur autem formula $\frac{\gamma dx}{(\alpha x + \beta)^2}$ ita, vt euanescat posito $x = 0$, eritque hoc integrale $+ \frac{\gamma x}{\beta(\alpha x + \beta)}$ sicque erit:

$$y = (\alpha x + \beta)\Gamma : (ft + \frac{\gamma x}{\beta(\alpha x + \beta)}) - (\alpha x + \beta)\Delta : (ft - \frac{\gamma x}{\beta(\alpha x + \beta)}),$$

vbi $f$ ex tensione chordae $c^s$ ita definitur, vt sit $f = \sqrt{2gc^r}$.

§. 24. Nunc igitur ante omnia hanc expressionem generalissimam ad statum nostrae chordae accommodari oportet, quo requiritur: primo vt in ipso termino A, vbi $x = 0$, semper sit $y = 0$; tum vero etiam pro altero termino B, vbi sit $x = a$, pariter perpetuo sit $y = 0$; quare pro priore conditione adimplenda, posito $x = 0$, perpetuo esse debet

$$0 = \beta\Gamma : ft - \beta\Delta : ft, \text{ siue } \Delta : ft = \Gamma : ft;$$

vnde patet, functionem $\Delta$ plane congruere debere cum functione $\Gamma$, quandoquidem pro eodem valore $ft$ vtraque praebere debet evndem valorem, ideoque loco $\Delta$ scribi oportet $\Gamma$, ita vt iam nostra aequatio sit:

$$y = (\alpha x + \beta)\Gamma : (ft + \frac{\gamma x}{\beta(\alpha x + \beta)}) - (\alpha x + \beta)\Gamma : (ft - \frac{\gamma x}{\beta(\alpha x + \beta)}),$$

ad quam formam melius intelligendam notetur, functionem quamcunque, veluti $\Gamma : z$, semper lineam curuam pro lubitu

bitu ductam repraesentari posse, cuius singulae applicatae abscissis $z$ respondentes referant functionem ipsam $\Gamma : z$.

§. 25. Pro altera conditione statuamus $x = a$, fierique debebit:

$$0 = (\alpha a + \beta)\Gamma : (ft + \tfrac{\gamma a}{\beta(\alpha a + \beta)}) - (\alpha a + \beta)\Gamma : (ft - \tfrac{\gamma a}{\beta(\alpha a + \beta)})$$

vnde prodit aequatio

$$\Gamma : (ft + \tfrac{\gamma a}{\beta(\alpha a + \beta)}) = \Gamma : (ft - \tfrac{\gamma a}{\beta(\alpha a + \beta)}),$$

cuius vim quo melius perspiciamus, ponamus

$$ft - \tfrac{\gamma a}{\beta(\alpha a + \beta)} = z, \text{ vt sit } ft = z + \tfrac{\gamma a}{\beta(\alpha a + \beta)}$$

atque fieri debebit

$$\Gamma : (z + \tfrac{2\gamma a}{\beta(\alpha a + \beta)}) = \Gamma z;$$

vnde patet, curuam seu scalam, qua functio $\gamma$ repraesentatur, ita esse debere comparatam, vt, posito breuitatis gratia $\frac{\gamma a}{\beta(\alpha a + \beta)} = k$, omnibus abscissis

$$z, \; z + 2k, \; z + 4k, \; z + 6k, \; z + 8k, \text{ etc.}$$

eadem semper respondeat applicata. Quamobrem sumto super axe EH interuallo $EF = 2k$, super eo describatur pro lubitu curua $eof$, quae eadem deinceps per aequalia interualla FG, GH super axe assumta, continuo replicetur, quod quo fieri possit tantum opus est vt applicatae $Ee$ et $Ff$ aequales statuantur. Praeterea enim descriptio istius curuae $eof$ penitus a nostro arbitrio pendet, nihilque prorsus refert, siue ea aequatione quapiam comprehendi queat, nec ne; quemadmodum fusius exposui in determinatione generali motuum, quos chordae vniformiter crassae recipere possunt. Tab. IV. Fig. 2.

§. 26.

§. 26. Tali ergo ſcala conſtituta, noſtrae chordae ad quoduis tempus expedite definiri et pro qualibet abſciſſa $x$ ei reſpondens applicata $y$ aſſignari poterit. Primo enim ſuper axe EFGH capiatur interuallum $ET = ft$, vbi applicata ſit $Tt$, tum vtrinque ab hoc puncto T abſcindantur interualla aequalia TP et $TQ = \frac{\gamma x}{\beta(\alpha x + \beta)}$, vt habeantur applicatae

$$Pp = \Gamma : \left(ft + \frac{\gamma x}{\beta(\alpha x + \beta)}\right) \text{ et } Qq = \Gamma : \left(ft - \frac{\gamma x}{\beta(\alpha x + \beta)}\right)$$

quocirca ex his duabus applicatis applicata quaeſita $y$ ita determinabitur, vt ſit $y = (\alpha x + \beta)(Pp - Qq)$. Tum vero, quia per interuallum $EF = 2k = \frac{2\gamma a}{\beta(\alpha a + \beta)}$ ſcalam pro lubitu delineare licet, ea ſemper ita conſtrui poterit, vt pro initio, vbi $t = 0$, non ſolum chorda datam obtineat figuram, ſed etiam in ſingulis punctis datum motum nanciſcatur. Hoc igitur caſu, prorſus atque in chordis vniformiter craſſis, ex ſtatu quocunque chordae initiali, totus motus quo deinceps contremiſcet, determinari poterit.

§. 27. Pro motus autem initio, vbi $t = 0$, aequatio noſtra ita ſe habebit:

$$y = (\alpha x + \beta)\,\Gamma\,\frac{\gamma x}{\beta(\alpha x + \beta)} - (\alpha x + \beta)\,\Gamma\,\bar{}\frac{\gamma x}{\beta(\alpha x + \beta)},$$

vnde manifeſtum eſt, elapſo tempore $t$ tanto, vt ſit $ft = 2k$, chordam eundem prorſus ſtatum eſſe recuperaturam; quare cum chorda interea duas vibrationes abſoluiſſe cenſeatur, tempus vnius vibrationis erit $= \frac{k}{f} = \frac{\gamma a}{\beta f(\alpha a + \beta)}$, idque in minutis ſecundis expreſſum; vnde ipſe ſonus, ſeu numerus vibrationum vno minuto ſecundo editarum, erit $= \frac{\beta f(\alpha a + \beta)}{\gamma a}$. Fieri vero etiam poteſt vt chorda eodem

tempore

tempore $\frac{k}{f}$ duas pluresue vibrationes abſoluat, quod eueniet, ſi ſcala illa $e\,o\,f$, ſuper interuallo $E\,F = 2\,k$ exſtructa, ex duabus pluribusue partibus inter ſe aequalibus componatur.

§. 28. Nunc igitur penitus euictum eſt, chordas huius ſpeciei, quarum craſſities reciproce proportionalis eſt poteſtati quartae huius formulae: $\alpha\,x + \beta$, ſimili modo ad omnes motus vibratorios eſſe accommodatas, ac chordas vniformiter craſſas, quemadmodum Illuſtris *Bernoulli* aſſeuerauit; atque adeo etiam pro iis ſolutionem aeque perfectam dari poſſe vidimus; ita vt pro quocunque ſtatu initiali, ad quem chorda fuerit perducta, ſemper totus motus, quo chorda deinceps contremiſcet, aſſignari queat; de quo quidem Celeberr. *Bernoulli*, eo ſaltem tempore, vehementer dubitaſſe videtur.

## Caſus ſecundus.

### quo ponitur $y = p\,\Gamma : (f\,t \pm X) + q\,\Gamma' : (f\,t \pm X)$.

§. 29. Ponamus iterum $X = \int v\,d\,x$, vt habeamus:

$$y = p\,\Gamma : (f\,t + \int v\,d\,x) + q\,\Gamma' (f\,t + \int v\,d\,x),$$

et quia $p$, $q$, $v$ ſunt functiones ipſius $x$ tantum, fiet bis differentiando

$$\left(\frac{d\,d\,y}{d\,t^2}\right) = f\,f\,p\,\Gamma'' \,\ldots\ldots + f\,f\,q\,\Gamma''' \,\ldots\ldots$$

Deinde poſita ſola $x$ variabili erit

$$\left(\frac{d\,d\,y}{d\,x^2}\right) = \left(\frac{d\,d\,p}{d\,x^2}\right)\Gamma \ldots + \left(\frac{2\,v\,d\,p + p\,d\,v}{d\,x}\right)\Gamma' \ldots$$
$$+ p\,v\,v\,\Gamma'' + q\,v\,v\,\Gamma''' \ldots$$
$$+ \left(\frac{d\,d\,q}{d\,x^2}\right)\Gamma' \ldots + \left(\frac{2\,v\,d\,q + q\,d\,v}{d\,x}\right)\Gamma'',$$

quae formula quia posito $f\,f = 2\,g\,c^3$ iterum aequalis esse debet huic:

$$p\,u\,u\,\Gamma'' \ldots\ldots q\,u\,u\,\Gamma''' \ldots\ldots$$

habebimus ſequentes aequationes:

$$\text{I. } \left(\frac{d\,d\,p}{d\,x^2}\right) = 0. \quad \text{II. } 2\,v\,d\,p + p\,d\,v + \left(\frac{d\,d\,q}{d\,x}\right) = 0.$$

$$\text{III. } p\,v\,v + \frac{2\,v\,d\,q + q\,d\,v}{d\,x} = p\,u\,u. \quad \text{IV. } q\,v\,v = q\,u\,u$$

ex quarum prima ſtatim colligitur $p = \alpha\,x + \beta$, ex quarta vero $u\,u = v\,v$, qui valores in binis reliquis ſubſtituti dabunt

$$\text{II. } 2\,\alpha\,v\,d\,x + (\alpha\,x + \beta)\,d\,v + \frac{d\,d\,q}{d\,x} = 0.$$

$$\text{III. } (\alpha\,x + \beta)\,v\,v + \frac{2\,v\,d\,q + q\,d\,v}{d\,x} = (\alpha\,x + \beta)\,v\,v,$$

vnde fit $2\,v\,d\,q + q\,d\,v = 0$, hincque $q\,q\,v = \gamma$, ideoque $v = \frac{\gamma}{q\,q}$, qui valor in ſecunda ſubſtitutus praebet

$$\frac{2\,\alpha\,\gamma\,d\,x}{q\,q} - \frac{2\,(\alpha\,x + \beta)\,\gamma\,d\,q}{q^3} + \frac{d\,d\,q}{d\,x} = 0,$$

ſiue

$$2\,\alpha\,\gamma\,q\,d\,x - 2\,\gamma\,(\alpha\,x + \beta)\,d\,q + \frac{q^3\,d\,d\,q}{d\,x} = 0.$$

§. 30. Quo haec aequatio ſimplicior reddatur, retineamus in calculo literam $p = \alpha\,x + \beta$, vt fit $d\,x = \frac{d\,p}{\alpha}$, et poſtrema aequatio hanc induet formam:

$$2\,\gamma\,q\,d\,p - 2\,\gamma\,p\,d\,q + \frac{\alpha\,q^3\,d\,d\,q}{d\,p} = 0,$$

cui facile intelligitur ſatisfieri poſſe, dum $q$ certae poteſtati ipſius $p$ aequatur. Ponatur igitur $q = \delta\,p^n$, erit

$d\,q$

$$d\,q = \delta\, n\, p^{n-1}\, d\,p \text{ et } d\,d\,q = \delta\, 2\,(n-1)\, p^{n-2}\, d\,p^2,$$

atque hinc orietur ista forma:

$$2\,\gamma\,\delta\, p^n - 2\,\gamma\,\delta\, n\, p^n + \alpha\,\delta^4\, n\,(n-1)\, p^{4n-2} = 0.$$

Fiat igitur $n = 4\,n - 2$, vt poteſtates ipſius $p$ egrediantur, eritque $n = \frac{2}{3}$, quo facto prodibit $\frac{2}{3}\gamma - \frac{2}{9}\,\alpha\,\delta^3 = 0$, vnde colligitur $\gamma = \frac{1}{3}\,\alpha\,\delta^3$; ſicque habetur ſolutio maxime ſpecialis, quoniam nulla noua conſtans in calculum eſt ingreſſa. Interim tamen operae pretium erit hanc ſolutionem expedire. Cum igitur ſit $\gamma = \frac{1}{3}\,\alpha\,\delta^3$, erit $q = \delta\, p^{\frac{2}{3}} = \delta\,(\alpha\, x + \beta)^{\frac{2}{3}}$, vnde porro ſit $v = \dfrac{\alpha\,\delta}{3\,(\alpha\, x + \beta)^{\frac{4}{3}}}$, ex qua expreſſione, cum ſit $u\,u = v\,v$, lex craſſitiei definitur. Erit enim $u\,u = \dfrac{\alpha\,\alpha\,\delta\,\delta}{9\,(\alpha\, x + \beta)^{\frac{8}{3}}}$, ita vt craſſities vbique ſit reciproce vt poteſtas $(\alpha\, x + \beta)^{\frac{8}{3}}$, qui ergo caſus prorſus diſcrepat ab iis, quos ex formula *Bernoulliana* deduximus.

§. 31. Conſideremus igitur chordam ita comparatam vti inuenimus, vt ſcilicet ſit eius craſſities

$$u\,u = \frac{\alpha\,\alpha\,\delta\,\delta}{9\,(\alpha\, x + \beta)^{\frac{4}{3}}},$$

et quia inuenimus $v\,v = u\,u$, erit

$$v = + \frac{\alpha\,\delta}{3\,(\alpha\, x + \beta)^{\frac{4}{3}}}, \text{ atque } q = \delta\,(\alpha\, x + \beta)^{\frac{2}{3}}.$$

In his autem formulis quantitas $\delta$ tam negatiue quam

 poſi-

positiue accipi potest, quandoquidem hoc modo crassities immutata manet; sicque hinc geminae determinationes obtinentur, prouti $\delta$ vel positiue vel negatiue accipitur. Priore scilicet casu erit

$$v = \frac{\alpha\delta}{3(\alpha x + \beta)^{\frac{4}{3}}} \text{ et } q = \delta(\alpha x + \beta)^{\frac{2}{3}},$$

posteriore vero casu erit

$$v = \frac{-\alpha\delta}{3(\alpha x + \beta)^{\frac{4}{3}}} \text{ et } q = -\delta(\alpha x + \beta)^{\frac{2}{3}}.$$

§. 32. Hinc ergo pro applicata $y$ duos nanciscimur valores, qui in genere coniuncti integrale completum exhibent, ita vt habeamus:

$$\begin{aligned} y = {} & (\alpha x + \beta)\,\Gamma:\left(ft + \int\frac{\alpha\delta\, dx}{3(\alpha x + \beta)^{\frac{4}{3}}}\right) \\ & + \delta(\alpha x + \beta)^{\frac{2}{3}}\,\Gamma':\left(ft + \int\frac{\alpha\delta\, dx}{3(\alpha x + \beta)^{\frac{4}{3}}}\right) \\ & - (\alpha x + \beta)\,\Delta:\left(ft - \int\frac{\alpha\delta\, dx}{3(\alpha x + \beta)^{\frac{4}{3}}}\right) \\ & + \delta(\alpha x + \beta)^{\frac{2}{3}}\,\Delta':\left(ft - \int\frac{\alpha\delta\, dx}{3(\alpha x + \beta)^{\frac{4}{3}}}\right), \end{aligned}$$

quae forma continet integrale completum, quoniam duas functiones arbitrarias $\Gamma$ et $\Delta$ inuoluit. Valet autem pro casu quo crassities chordae est

$\varkappa u =$

$$uu = \frac{\alpha\alpha\delta\delta}{9(\alpha x + \beta)^{\frac{8}{3}}}.$$

Tum vero eſt $f = \sqrt{2gc^{1}}$, vbi $c^{1}$ denotat tenſionem chordae. Formula autem integralis, quae hic occurrit eſt

$$\int \frac{\alpha\delta dx}{3(\alpha x + \beta)^{\frac{4}{3}}} = \frac{-\delta}{(\alpha x + \beta)^{\frac{1}{3}}} + \frac{\delta}{\beta^{\frac{1}{3}}}$$

$$= \delta \left( \frac{\sqrt[3]{(\alpha x + \beta)} - \sqrt[3]{\beta}}{\sqrt[3]{\beta}\,(\alpha x + \beta)} \right),$$

quod integrale ita ſumſimus, vt euaneſcat poſito $x = o$. Quodſi ergo hoc ipſum integrale litera X deſignemus, vt ſit $X = \frac{\delta}{\sqrt[3]{\beta}} - \frac{\delta}{\sqrt[3]{(\alpha x + \beta)}}$, formula noſtra generalis erit:

$$y = (\alpha x + \beta)\Gamma : (ft + X) + \delta(\alpha x + \beta)^{\frac{2}{3}}\Gamma' : (ft + X)$$
$$- (\alpha x + \beta)\Delta : (ft - X) + \delta(\alpha x + \beta)^{\frac{2}{3}}\Delta' : (ft - X)$$

§. 33. Applicemus igitur hanc formulam ad ſtatum noſtrae chordae, quo ea in punctis B et A fixa ſupponitur. Primo igitur, poſito $x = o$, vnde etiam ſit $X = o$, requiritur vt fiat

$$o = \beta\Gamma : ft + \delta\beta^{\frac{2}{3}}\Gamma' : ft - \beta\Delta : ft + \delta\beta^{\frac{2}{3}}\Delta' ft,$$

vnde patet, hic non amplius aequalitatem functionum $\Gamma$ et $\Delta$ locum habere, vt praecedente caſu vſu venit. Praeterea vero, poſito $x = a$, fiat $X = A$, ita vt ſit

$$A = \frac{\delta}{\sqrt[3]{\beta}} - \frac{\delta}{\sqrt[3]{(\alpha a + \beta)}},$$

atque haec inſuper conditio adimpleri debebit, vt ſit:

$$0 = (\alpha a + \beta)\,\Gamma : (ft + A) + \delta\,(\alpha a + \beta)^{\frac{2}{3}}\,\Gamma' : (ft + A)$$
$$-\,(\alpha a + \beta)\,\Delta : (ft - A) + \delta\,(\alpha a + \beta)^{\frac{2}{3}}\,\Delta' : (ft - A)$$

atque ex his duabus conditionibus tam indoles vtriusque functionis $\Gamma$ et $\Delta$, quam continuatio ſcalarum quae has functiones repraeſentant, definiri debet.

§. 34. Priore conditione, qua eſſe debet

$$\Gamma : z - \Delta : z + \frac{\delta}{\sqrt[3]{\beta}}\,\Gamma' : z + \frac{\delta}{\sqrt[3]{\beta}}\,\Delta' : z = 0, \text{ ſiue}$$

$$\Delta : z - \Gamma\,z = \frac{\delta}{\sqrt[3]{\beta}}\,(\Gamma'\,z + \Delta' : z),$$

praeſcribitur certa relatio inter ambas functiones $\Gamma$ et $\Delta$, ad quam inueſtigandam, poſito breuitatis gratia $\frac{\delta}{\sqrt[3]{\beta}} = \lambda$, ſtatuamus

$$\Delta : z - \Gamma : z = \lambda\,(\Gamma' : z + \Delta' : z) = 2\lambda\,\Theta'\,z$$

vt ſit $\Gamma' : z + \Delta'\,z = 2\,\Theta'\,z$, vnde ſit integrando

$$\Gamma : z + \Delta : z = 2\,\Theta : z.$$

Cum igitur ſit

$$\Delta\,z - \Gamma : z = 2\lambda\,\Theta' : z$$

inde colligitur:

$$\Gamma : z = \Theta : z - \lambda\,\Theta' : z \text{ et } \Delta\,z = \Theta\,z + \lambda\,\Theta' : z$$

hincque porro

$$\Gamma' : z = \Theta'\,z - \lambda\,\Theta''\,z \text{ et } \Delta' : z = \Theta' : z + \lambda\,\Theta'' : z$$

ſicque ambae functiones $\Gamma$ et $\Delta$ ad nouam functionem $\Theta$ ſunt reductae; quocirca noſtra aequatio generalis induet hanc formam:

$$y=(\alpha x+\beta)\,\Theta:(ft+X)-\frac{\delta(\alpha x+\beta)^{\frac{2}{3}}}{\sqrt[3]{\beta}}(\sqrt[3]{(\alpha x+\beta)}-\sqrt[3]{\beta})\times$$

$$\times\,\Theta':(ft+X)-\frac{\delta\delta}{\sqrt[3]{\beta}}(\alpha x+\beta)^{\frac{2}{3}}\,\Theta'':(ft+X)$$

$$-(\alpha x+\beta)\,\Theta:(ft-X)-\frac{\delta(\alpha x+\beta)^{\frac{2}{3}}}{\sqrt[3]{\beta}}\times$$

$$\times(\sqrt[3]{(\alpha x+\beta)}-\sqrt[3]{\beta})\,\Theta':(ft-x)+\frac{\delta\delta}{\sqrt[3]{\beta}}(\alpha x+\beta)^{\frac{2}{3}}\,\Theta'':(ft+X)$$

§. 35. Progrediamur igitur ad alteram conditionem, qua posito $x=a$ etiam fieri debet $y=0$; tum autem fiat $X=A$: ac statuamus breuitatis gratia

$$\frac{\delta(\alpha a+\beta)^{\frac{2}{3}}}{\sqrt[3]{\beta}}(\sqrt[3]{(\alpha a+\beta)}-\sqrt[3]{\beta})=B(\alpha a+\beta)\ \text{et}$$

$$\frac{\delta\delta}{\sqrt[3]{\beta}}(\alpha a+\beta^{\frac{2}{3}})=C(\alpha a+\beta),$$

atque haec conditio postulat vt sit

$$0=\Theta:(ft+A)-B\,\Theta':(ft+A)-C\,\Theta'':(ft+A)$$
$$-\Theta:(ft-A)-B\,\Theta':(ft-A)+C\,\Theta'':(ft-A).$$

Ponamus hic $ft-A=z$, eritque $ft+A=z+2A$, ita vt esse debeat:

$$\Theta:(z+2A)-B\,\Theta':(z+2A)-C\,\Theta'':(z+2A)$$
$$=\Theta:z+B\,\Theta':z-C\,\Theta'':z,$$

ex qua conditione continuatio scalarum pro functionibus

$\Theta$,

$\Theta$, $\Theta'$, $\Theta''$ peti debet. Ex iis enim, quae de chordis aequaliter craſſis ſunt tradita, intelligitur, ſuper axe per interuallam $= 2A$ curuam quamcunque pro lubitu deſcribi poſſe, cuius applicatae referant functionem $\Theta : z$, quae ſcilicet abſciſſis $z$ reſpondeant; tum enim ex hac curua facile conſtruentur eae, quibus functiones $\Theta' z$ et $\Theta'' : z$ referuntur. Sicque iſtae functiones $\Theta : z$, $\Theta' : z$ et $\Theta'' : z$, repraeſentabuntur pro omnibus abſciſſis $z$, minoribus quam $2A$. At vero pro maioribus abſciſſis continuatio harum ſcalarum ita debet inſtitui, vt illi conditioni poſteriori ſatisfiat.

§. 36. Hic igitur ſtatim perſpicitur, non amplius fore $\Theta : (z + 2A) = \Theta z$, vti in chordis ordinariis vſu venit, ideoque continuatas portiones harum ſcalarum a portione principali eſſe diſcrepaturas; quamobrem ad hanc diſcrepantiam explorandam introducamus nouam functionem $\Lambda$, ac ſtatuamus

$$\Theta : (z + 2A) = \Theta : z + \Lambda : z, \text{ eritque}$$

$$\Theta' : (z + 2A) = \Theta' : z + \Lambda' : z \text{ et}$$

$$\Theta'' : (z + 2A) = \Theta'' : z + \Lambda'' : z;$$

quibus valoribus introductis habebimus

$$\Lambda : z - B\Lambda' : z - C\Lambda'' : z = 2B\Theta' : z,$$

cui aequationi quo facilius ſatisfieri poſſit, ſtatuamus

$$\Lambda : z = 2B\Pi' : z, \text{ vt fiat}$$

$$\Theta' : z = \Pi' : z - B\Pi'' : z - C\Pi''' : z,$$

vnde fit integrando

$$\Theta : z = \Pi : z - B\Pi' : z - C\Pi'' : z, \text{ et}$$

$\Theta''$

$$\Theta'' : z = \Pi'' : z - B\,\Pi''' : z - C\,\Pi'''' : z.$$

Deinde vero pro continuatione erit

$$\Theta : (z + 2A) = \Theta : z + 2B\,\Pi' : z, \text{ siue}$$

$$\Theta : (z + 2A) = \Pi z + B\,\Pi' : z - C\,\Pi'' : z,$$

hincque porro

$$\Theta' (z + 2A) = \Pi' : z + B\,\Pi'' : z - C\,\Pi''' : z \text{ et}$$

$$\Theta'' : (z + 2A) = \Pi'' : z + B\,\Pi''' : z - C\,\Pi'''' : z.$$

Sicque totum negotium iam huc eſt perductum: vt, conſtituta pro lubitu ſcala, functionem $\Pi : z$ referente, pro abſciſſis $z$, terminum $2A$ non ſuperantibus, hincque conſtructis ſcalis pro functionibus deriuatis $\Pi' : z$, $\Pi'' : z$, $\Pi''' : z$, $\Pi'''' : z$, inde ſcalae ante memoratae pro functionibus $\Theta : z$, $\Theta' : z$, $\Theta'' : z$ exſtrui queant, vsque ad abſciſſam $z = 2A$, quae deinceps per formulas modo datas

$$\Theta : (z + 2A), \; \Theta' (z + 2A), \; \Theta'' : (z + 2A), \text{ etc.}$$

continuo vsque ad maiores abſciſſas continuari poſſunt. Hocque modo omnes plane motus, quos chorda talis recipere poteſt, ita definiri poterunt, vt etiam ad quemuis ſtatum initialem accommodari queant. Caeterum hic non eſt diffitendum, hanc conſtructionem maxime eſſe operoſam: verum pro inſtituto praeſente ſufficiet poſſibilitatem ſolutionis perfectae oſtendiſſe.

§. 37. Quae autem hic ſunt tradita tantum ad caſum maxime particularem pertinent, quandoquidem pro aequatione differentiali ſecundi gradus ad quam fuimus perducti, quae erat

$$q\,dp - p\,dq = \frac{\alpha q^3\,d\,dq}{2\gamma d\,p},$$

tantum eius integrale ſpecialiſſimum $q = \delta p^{\frac{2}{3}}$ aſſumſimus; quamobrem operae pretium erit inueſtigare, num iſtius aequationis integrale completum reperire liceat, nec ne. Quoniam igitur huic aequationi ſatisfacit valor $p^{\frac{2}{3}}$ pro $q$ aſſumtus, vnde $\frac{dq}{dp}$ tali formulae: $\frac{1}{p^{\frac{1}{3}}}$ proportionalis eſt: ſtatuamus pro integrali completo inueſtigando

$$q = p^{\frac{2}{3}} r, \text{ et } \frac{dq}{dp} = \frac{s}{p^{\frac{1}{3}}}.$$

Ex priore aequatione fiet $dq = p^{\frac{2}{3}} dr + p^{-\frac{1}{3}} r\, dp$, quae ergo expreſſio aequari debet huic: $s p^{-\frac{1}{3}} dp$, vnde ſequitur fore $\frac{dp}{p} = \frac{3\, dr}{3 s - 2 r}$.

§. 38. Quodſi autem hos valores in ipſa noſtra aequatione ſubſtituamus, pro qua ob $dp$ conſtans aſſumtum ſit

$$\frac{ddq}{dp} = \frac{d.dq}{dp} = \frac{ds}{p^{\frac{1}{3}}} - \frac{1}{3}\frac{s\, dp}{p^{\frac{4}{3}}},$$

loco $\frac{\alpha}{2\gamma}$ ſcribendo literam $n$, illa aequatio hanc induet formam:

$$p^{\frac{2}{3}} r\, dp - p^{\frac{2}{3}} s\, dp = n p p r^3 \left(\frac{ds}{p^{\frac{1}{3}}} - \frac{s\, dp}{3 p^{\frac{4}{3}}}\right)$$

$$= n r^3 (p^{\frac{5}{3}} ds - \tfrac{1}{3} p^{\frac{2}{3}} s\, dp),$$

quae aequatio per $p^{\frac{2}{3}}$ diuiſa dat

$r\,dp$

$$r\,d\,p - s\,d\,p = n\,r^3\,p\,d\,s - \tfrac{1}{3}\,n\,r^3\,s\,d\,p,$$

vnde colligimus

$$\frac{d\,p}{p} = \frac{3\,n\,r^3\,d\,s}{3\,r - 3\,s + n\,r^3\,s}.$$

Hi igitur valores pro $\frac{d\,p}{p}$ inuenti nobis ſuppeditant hanc aequationem differentialem primi gradus inter $r$ et $s$:

$$3\,n\,r^3\,s\,d\,s - 2\,n\,r^4\,d\,s - n\,r^3\,s\,d\,r = 3\,r\,d\,r - 3\,s\,d\,r,$$

ſiue

$$3\,s\,d\,s - 2\,r\,d\,s - s\,d\,r = \frac{3\,(r-s)\,d\,r}{n\,r^3},$$

quae, poſito $r - s = t$, vt ſit $s = r - t$, reducitur ad hanc formam ſimpliciorem:

$$-\,r\,d\,t - 2\,t\,d\,r + 3\,t\,d\,t = \frac{3\,t\,d\,r}{n\,r^3},$$

cuius quidem membrum prius integrabile redditur multiplicatum per $r - t = s$, ita vt prior forma hoc modo poſſit exhiberi:

$$d.\,s^3 - r\,s\,s = \frac{3\,s\,(r-s)\,d\,r}{n\,r^3},$$

ſiue etiam

$$\frac{d.\,s^3 - r\,s\,s}{s^3 - r\,s\,s} = \frac{3\,s\,(r-s)\,d\,r}{n\,r^3\,(s^3 - r\,s\,s)};\; = \frac{-\,3\,d\,r}{n\,r^3\,s}.$$

Quomodocunque autem haec aequatio tractetur, eius integratio omnes vires Analyſeos fruſtrari videtur.

§. 39. Multo minus integratio generalis ſuccedet, ſi pro valore ipſius $y$ adhuc plures terminos aſſumere vellemus, veluti

$$y = p\,\Gamma:(f\,t \pm X) + q\,\Gamma'(f\,t \pm X) + r\,\Gamma'':(f\,t \pm X).$$



Inte-

Integralia autem particularia pro determinatione craſſitiei ſequerentur caſus integrabiles aequationis *Riccatianae*, quemadmodum iam dudum eſt obſeruatum, vnde perſpicitur: binos caſus poſteriores, quos ex ſolutione *Bernoulliana* deduximus neutiquam ad iſtos caſus referri poſſe; quamobrem illi tanto maiore attentione digni ſunt cenſendi.

DE