NOVA METHODVS
INTEGRANDI FORMVLAS DIFFERENTIALES RATIONALES
SINE
SVBSIDIO QVANTITATVM IMAGINARIARVM.

Auctore
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Theorema I.

§. 1. *Si fuerit* $xx - 2x \cos. \omega + 1 = 0$, *tum omnes potestates ipsius* x *reduci poterunt ad formam simplicem* $\alpha x + \beta$.

Demonstratio.

Cum sit $xx - 2x \cos. \omega + 1 = 0$, erit

$$x^{\lambda+2} = 2x^{\lambda+1} \cos. \omega - x^{\lambda},$$

vnde si potestates x^{λ} et $x^{\lambda+1}$ ad formam praescriptam

$\alpha x + \beta$ redigi queant, tum etiam poteſtas $x^{\lambda+1}$ per eandem formam exprimi poterit. Incipiamus igitur a poteſtatibus infimis, quas ita exhibeamus:

$$x = \frac{x\,\text{ſin.}\,\omega}{\text{ſin.}\,\omega} \text{ et } x\,x = \frac{2\,x\,\text{ſin.}\,\omega\,\text{coſ.}\,\omega - \text{ſin.}\,\omega}{\text{ſin.}\,\omega} = \frac{x\,\text{ſin.}\,2\,\omega - \text{ſin.}\,\omega}{\text{ſin.}\,\omega}.$$

His igitur conſtitutis, cum ſit

$$2\,\text{coſ.}\,\omega\,\text{ſin.}\,\lambda\,\omega = \text{ſin.}\,(\lambda + 1)\,\omega + \text{ſin.}\,(\lambda - 1)\,\omega,$$

ex his duabus formulis facile eliciemus ſequentes:

$$x^3 = 2\,x\,x\,\text{coſ.}\,\omega - x = \frac{x\,\text{ſin.}\,3\,\omega - \text{ſin.}\,2\,\omega}{\text{ſin.}\,\omega},$$

$$x^4 = 2\,x^3\,\text{coſ.}\,\omega - x\,x = \frac{x\,\text{ſin.}\,4\,\omega - \text{ſin.}\,3\,\omega}{\text{ſin.}\,\omega},$$

hocque modo quousque libuerit progredi licet; atque hinc in genere concludimus fore

$$x^n = \frac{x\,\text{ſin.}\,n\,\omega - \text{ſin.}\,(n-1)\,\omega}{\text{ſin.}\,\omega},$$

quae ergo expreſſio formam habet $\alpha x + \beta$.

Corollarium 1.

§. 2. Cum ſit

$$\text{ſin.}\,(n-1)\,\omega = \text{ſin.}\,n\,\omega\,\text{coſ.}\,\omega - \text{coſ.}\,n\,\omega\,\text{ſin.}\,\omega,$$

hoc valore ſubſtituto fiet

$$x^n = \frac{x - \text{coſ.}\,\omega}{\text{ſin.}\,\omega}\,\text{ſin.}\,n\,\omega + \text{coſ.}\,n\,\omega;$$

quamobrem ſi fuerit $x\,x - 2\,x\,\text{coſ.}\,\omega + 1 = 0$, pro omnibus poteſtatibus ipſius x habebimus hanc reductionem:

$$x^n = \frac{x - \text{coſ.}\,\omega}{\text{ſin.}\,\omega}\,\text{ſin.}\,n\,\omega + \text{coſ.}\,n\,\omega,$$

qua forma deinceps potiſſimum vtemur.

Corollarium 2.

§. 3. Hinc igitur erit:

x

$$x^{k+n} = \frac{x - \cos.\omega}{\sin.\omega} \sin.(k+n)\omega + \cos.(k+n)\omega \quad \text{et}$$
$$x^{k-n} = \frac{x - \cos.\omega}{\sin.\omega} \sin.(k-n)\omega + \cos.(k-n)\omega,$$

quare his formulis addendis, ob

$$\sin.(k+n)\omega + \sin.(k-n)\omega = 2\sin.k\omega \cos.n\omega \quad \text{et}$$
$$\cos.(k+n)\omega + \cos.(k-n)\omega = 2\cos.k\omega \cos.n\omega$$

fiet

$$x^{k+n} + x^{k-n} = \frac{2(x - \cos.\omega)}{\sin.\omega} \sin.k\omega \cos.n\omega + 2\cos.k\omega \cos.n\omega$$

siue

$$x^{k+n} + x^{k-n} = 2\cos.n\omega \left(\left(\frac{x - \cos.\omega}{\sin.\omega} \right) \sin.k\omega + \cos.k\omega \right).$$

Corollarium 3.

§. 4. Sin autem potestatem posteriorem a priore subtrahamus, ob

$$\sin.(k+n)\omega - \sin.(k-n)\omega = 2\cos.k\omega \sin.n\omega \quad \text{et}$$
$$\cos.(k+n)\omega - \cos.(k-n)\omega = -2\sin.k\omega \sin.n\omega,$$

habebimus

$$x^{k+n} - x^{k-n} = \frac{2(x - \cos.\omega)}{\sin.\omega} \cos.k\omega \sin.n\omega - 2\sin.k\omega \sin.n\omega. \text{ h. e.}$$
$$x^{k+n} - x^{k-n} = 2\sin.n\omega \left(\left(\frac{x - \cos.\omega}{\sin.\omega} \right) \cos.k\omega - \sin.k\omega \right).$$

Corollarium 4.

§. 5. Etiamsi nostra demonstratio tantum ad potestates integras ipsius x perduxit, tamen ex indole harum formarum facile intelligitur, eas etiam pro exponentibus fractis, vel adeo irrationalibus, locum habere, quandoquidem in ipsis his formulis nihil inest, quod tantum ad valores integros exponentis n restringatur; tum vero etiam nihil impedit, quominus exponenti n valores nega-

 tiui

tiui tribuantur. Si enim v. g. ſumamus $n = \frac{1}{2}$, per formulam Coroll. I. eſſe debet

$$\sqrt{x} = \frac{x - \text{coſ.}\,\omega}{\text{ſin.}\,\omega}\,\text{ſin.}\,\tfrac{1}{2}\,\omega + \text{coſ.}\,\tfrac{1}{2}\,\omega,$$

vnde ſumtis quadratis, ob

$$(x - \text{coſ.}\,\omega)^2 = x\,x - 2\,x\,\text{coſ.}\,\omega + \text{coſ.}\,\omega^2 = -\,\text{ſin.}\,\omega^2,$$

habebimus

$$x = -\,\text{ſin.}\,\tfrac{1}{2}\,\omega^2 + \frac{2\,(x - \text{coſ.}\,\omega)}{\text{ſin.}\,\omega}\,\text{ſin.}\,\tfrac{1}{2}\,\omega\,\text{coſ.}\,\tfrac{1}{2}\,\omega + \text{coſ.}\,\tfrac{1}{2}\,\omega^2,$$

quae forma, ob

$$2\,\text{ſin.}\,\tfrac{1}{2}\,\omega\,\text{coſ.}\,\tfrac{1}{2}\,\omega = \text{ſin.}\,\omega \quad \text{et} \quad \text{coſ.}\,\tfrac{1}{2}\,\omega^2 - \text{ſin.}\,\tfrac{1}{2}\,\omega^2 = \text{coſ.}\,\omega,$$

abit in $x = x$, h. e. aequationem identicam.

Scholion.

§. 6. Formulae quas hic ſumus adepti egregie conueniunt cum iis, quas calculus imaginariorum ſuppeditat. Cum enim aequatio $x\,x - 2\,x\,\text{coſ.}\,\omega + 1 = 0$ contineat has radices: $x = \text{coſ.}\,\omega \pm \text{ſin.}\,\omega.\sqrt{-1}$, erit, vti in analyſi eſt oſtenſum, $x^n = \text{coſ.}\,n\,\omega \pm \text{ſin.}\,n\,\omega.\sqrt{-1}$; quare cum ſit

$$x^n - \text{coſ.}\,n\,\omega = \pm\,\text{ſin.}\,n\,\omega.\sqrt{-1} \quad \text{et}$$

$$x - \text{coſ.}\,\omega = \pm\,\text{ſin.}\,\omega\sqrt{-1},$$

illa forma per hanc diuiſa dabit

$$\frac{x^n - \text{coſ.}\,n\,\omega}{x - \text{coſ.}\,\omega} = \frac{\text{ſin.}\,n\,\omega}{\text{ſin.}\,\omega},$$

vnde ſequitur fore

$$x^n - \text{coſ.}\,n\,\omega = \frac{x - \text{coſ.}\,\omega}{\text{ſin.}\,\omega}\,\text{ſin.}\,n\,\omega,$$

prorſus vti in Coroll. I. inuenimus. Ceterum noſtrum Theorema generalius proponi et ad aequationem

$x\,x$

$$x x - 2 a x \operatorname{cof.} \omega + a a$$

extendi potuisset; tum enim prodiisset

$$x^n = \frac{a^{n-1} x \operatorname{fin.} n \omega - a^n \operatorname{fin.} (n-1) \omega}{\operatorname{fin.} \omega},$$

deinde etiam

$$x^n = \frac{a^{n-1}(x - a \operatorname{cof.} \omega)}{\operatorname{fin.} \omega} \operatorname{fin.} n \omega + a^n \operatorname{cof.} n \omega,$$

quae formulae a prioribus non discrepant, nisi quod hic littera a homogeneitatem dimensionum expleat. Hae scilicet formulae ex illis immediate sequentur, si ibi loco x scribatur $\frac{x}{a}$; sed breuitati et concinnitati consulentes eiusmodi tantum casus euoluemus, in quibus pro a commode vnitatem scribere liceat.

Theorema II.

§. 7. *Si fuerit* $x x - 2 x \operatorname{cof.} \omega + 1 = 0$, *omnes functiones rationales integrae, quaecunque potestates ipsius* x *in iis occurrant, semper reduci possunt ad hanc formam simplicem* $\alpha x + \beta$.

Demonstratio.

Si functio proposita iam penitus fuerit euoluta, ita vt nullos factores complectatur, tum ea ope reductionis $x^n = \frac{x - \operatorname{cof.} \omega}{\operatorname{fin.} \omega} \operatorname{fin.} n \omega + \operatorname{cof.} n \omega$ sponte redigitur ad talem formam: $\frac{F(x - \operatorname{cof.} \omega)}{\operatorname{fin.} \omega} + G$. Verum si functio proposita duobus constet factoribus, veluti $P p$, ac per istam reductionem prodierit

$$P = \frac{F(x - \operatorname{cof.} \omega)}{\operatorname{fin.} \omega} + G \text{ et } p = \frac{f(x - \operatorname{cof.} \omega)}{\operatorname{fin.} \omega} + g,$$

tum

tum facta multiplicatione, ob $(x - \text{cos}.\ \omega)^2 = - \text{sin}.\ \omega^2$ colligitur fore

$$Pp = -Ff + Gg + \frac{(Fg + fG)(x - \text{cos}.\ \omega)}{\text{sin}.\ \omega},$$

quod ergo productum eiusdem est formae; vnde simul patet, quotcunque eiusmodi dentur factores, eorum productum semper ad eandem formam reduci posse.

Corollarium 1.

§. 8. Quodsi hoc modo prodierit

$$P = \frac{F(x - \text{cos}.\ \omega)}{\text{sin}.\ \omega} + G,$$

tum erit

$$P(x - \text{cos}.\ \omega) = -F\ \text{sin}.\ \omega + G(x - \text{cos}.\ \omega),$$

quae expressio ideo est notatu digna, quod in sequentibus integrationibus vbique occurret.

Corollarium 2.

§. 9. Si functio P factorem habuerit

$$xx - 2x\ \text{cos}.\ \omega + 1,$$

tum posito, vti assumimus,

$$xx - 2x\ \text{cos}.\ \omega + 1 = 0,$$

valor ipsius P etiam euanescere debet. Hoc ergo casu formula $\frac{F(x - \text{cos}.\ \omega)}{\text{sin}.\ \omega} + G$ fiet $= 0$, id quod, ob x quantitatem indefinitam, aliter euenire nequit, nisi fuerit et $F = 0$ et $G = 0$. Atque hinc vicissim, si facta reductione prodeat $P = 0$, hoc certum erit signum, ipsam functionem inuolvere factorem $xx - 2x\ \text{cos}.\ \omega + 1$.

Theo-

Theorema III.

§. 10. *Si fuerit* $xx - 2x \cos. \omega + 1 = 0$, *tum etiam omnes functiones fractae rationales semper ad formam simplicem* $\alpha x + \beta$ *reduci possunt.*

Demonstratio.

Sit enim proposita functio quaecunque fractio $\frac{P}{p}$, atque adhibita nostra reductione prodierit

$$P = \frac{F(x - \cos.\omega)}{\sin.\omega} + G \text{ et } p = \frac{f(x - \cos.\omega)}{\sin.\omega} + g,$$

ita vt peruenerimus ad hanc fractionem:

$$\frac{P}{p} = \frac{\frac{F(x - \cos.\omega)}{\sin.\omega} + G}{\frac{f(x - \cos.\omega)}{\sin.\omega} + g}.$$

Iam vt ipsam litteram x ex denominatore expellamus, multiplicemus tam numeratorem quam denominatorem per formulam $\frac{f(x - \cos.\omega)}{\sin.\omega} - g$; sic enim, ob

$$(x - \cos.\omega)^2 = - \sin.\omega^2,$$

pro denominatore reperiemus: $-ff - gg$; at vero pro numeratore:

$$- Ff + \frac{(fG - Fg)(x - \cos.\omega)}{\sin.\omega} - Gg,$$

vnde mutatis signis forma nostrae fractionis erit

$$\frac{P}{p} = \frac{\frac{(Fg - fG)(x - \cos.\omega)}{\sin.\omega} + Ff + Gg}{ff + gg},$$

siue concinnius:

$$\frac{P}{p} = \frac{Fg - fG}{ff + gg} \cdot \frac{x - \cos.\omega}{\sin.\omega} + \frac{Ff + Gg}{ff + gg}.$$

Problema.

§. 11. *Propoſita formula differentiali rationali quacunque, eam in ſuas fractiones partiales reſoluere, ac deinceps eius integrale inueſtigare.*

Solutio.

Repraeſentetur formula differentialis ſub hac ſpecie: $\frac{P}{Q}\cdot\frac{dx}{x} = \frac{P\,dx}{Qx}$, ita tamen, vt $\frac{P}{x}$ maneat functio integra, ne x ſit factor denominatoris. Ante omnia quaerantur igitur ipſius Q omnes factores tam ſimplices quam duplices reales; et quia ſimplices nulla laborant difficultate, hic tantum duplices ſum contemplaturus, quorum forma ſit $xx - 2x\cos\omega + 1$, ita vt, poſito $xx - 2x\cos\omega + 1 = 0$, quantitas Q ſimul in nihilum abeat, ex qua conditione omnes valores anguli ω elici poterunt, ita vt hoc modo omnes factores denominatoris Q obtineantur. Nunc igitur fractionem $\frac{P}{Qx}$ in totidem fractiones partiales reſolui oportet, quot inuenti fuerint factores formae

$$xx - 2x\cos\omega + 1.$$

Sit igitur in genere fractio partialis ex iſto factore nata $= \frac{\alpha x + \beta}{xx - 2x\cos\omega + 1}$, quandoquidem nouimus, eius numeratorem talem formam: $\alpha x + \beta$ habere debere; pro reliquis autem fractionibus partialibus omnibus ſcribamus litteram R, ita vt eſſe debeat

$$\frac{P}{Qx} = \frac{\alpha x + \beta}{xx - 2x\cos\omega + 1} + R,$$

et multiplicando per $xx - 2x\cos\omega + 1$ habebimus:

$$\frac{P(xx - 2x\cos\omega + 1)}{Qx} = \alpha x + \beta + R(xx - 2x\cos\omega + 1).$$

Quodſi ergo iam faciamus $xx - 2x\cos\omega + 1 = 0$, erit

αx

$$\alpha x + \beta = \frac{P(xx - 2x\,\text{cof.}\,\omega + 1)}{Qx} = \frac{P}{x}\cdot\frac{xx - 2x\,\text{cof.}\,\omega + 1}{Q},$$

vbi in priore factore $\frac{P}{x}$ ista substitutio nullam habet difficultatem; verum in altera fractione $\frac{xx - 2x\,\text{cof.}\,\omega + 1}{Q}$, quia, posito $xx - 2x\,\text{cof.}\,\omega + 1 = 0$, non solum numerator sed etiam denominator Q euanescit, secundum praecepta cognita vtriusque loco eius differentiale scribamus, siquidem hoc casu fieri debet

$$\frac{xx - 2x\,\text{cof.}\,\omega + 1}{Q} = \frac{2dx(x - \text{cof.}\,\omega)}{dQ},$$

sicque obtinebitur numerator quaesitus

$$\alpha x + \beta = \frac{2Pdx(x - \text{cof.}\,\omega)}{x\,dQ}.$$

Ponamus igitur per hanc substitutionem fieri

$$P = \frac{F(x - \text{cof.}\,\omega)}{\text{fin.}\,\omega} + G \text{ et } \frac{x\,dQ}{dx} = \frac{f(x - \text{cof.}\,\omega)}{\text{fin.}\,\omega} + g,$$

ita vt sit

$$\frac{P\,dx}{x\,dQ} = \frac{\frac{F(x - \text{cof.}\,\omega)}{\text{fin.}\,\omega} + G}{\frac{f(x - \text{cof.}\,\omega)}{\text{fin.}\,\omega} + g},$$

quae forma per Theorema tertium reducitur ad hanc:

$$\frac{P\,dx}{x\,dQ} = \frac{Fg - fG}{ff + gg}\cdot\frac{x - \text{cof.}\,\omega}{\text{fin.}\,\omega} + \frac{Ff + Gg}{ff + gg},$$

quae ergo insuper per $2(x - \text{cof.}\,\omega)$ multiplicata, ob

$$(x - \text{cof.}\,\omega)^2 = -\,\text{fin.}\,\omega^2,$$

praebet numeratorem quaesitum:

$$\alpha x + \beta = \frac{2(Ff + Gg)(x - \text{cof.}\,\omega)}{ff + gg} + \frac{2(fG - Fg)}{ff + gg}\,\text{fin.}\,\omega.$$

Multiplicetur igitur ista forma per $\frac{dx}{xx - 2x\,\text{cof.}\,\omega + 1}$, atque obtinebitur pars integralis ex hac fractione partiali oriunda:

$$\frac{2(Ff + Gg)}{ff + gg}\int\frac{dx(x - \text{cof.}\,\omega)}{xx - 2x\,\text{cof.}\,\omega + 1} + \frac{fG - Fg}{ff + gg}\,\text{fin.}\,\omega\int\frac{dx}{xx - 2x\,\text{cof.}\,\omega + 1}.$$

Hic igitur pro priore parte manifesto est

$$\int \frac{dx\,(x - \cos.\omega)}{xx - 2x\cos.\omega + 1} = l\sqrt{(xx - 2x\cos.\omega + 1)},$$

quod integrale iam ita est sumtum, vt euanescat posito $x = 0$; pro altero autem membro facile reperitur:

$$\int \frac{dx\,\sin.\omega}{1 - 2x\cos.\omega + xx} = \text{A tang.}\,\frac{x\sin.\omega}{1 - x\cos.\omega},$$

quod itidem euanescit posito $x = 0$, quocirca pars integralis, ex denominatoris Q factore $xx - 2x\cos.\omega + 1$ orta erit

$$\frac{2(Ff + Gg)}{ff + gg}\,l\sqrt{(xx - 2x\cos.\omega + 1)}$$
$$+ \frac{2(fG - Fg)}{ff + gg}\,\text{A tang.}\,\frac{x\sin.\omega}{1 - \cos.\omega}.$$

Corollarium 1.

§. 12. Duo autem casus hic singularem euolutionem postulant: alter; quo $\omega = 0$, alter vero quo $\omega = 180°$; priore enim casu denominator $xx - 2x\cos.\omega + 1$ abit in $(x - 1)^2$, posteriore vero in $(x + 1)^2$. Cum autem hinc plus concludere non liceat, quam vel $1 - x$ vel $1 + x$ esse factorem denominatoris, his casibus pars integralis in genere inuenta tantum ad semissem redigi debet, quemadmodum in principiis Calculi integralis fusius est ostensum. Ceterum his casibus posterior pars a circulo pendens semper euanescet.

Corollarium 2.

§. 13. Praeter hos autem binos casus portio integralis ex formula $xx - 2x\cos.\omega + 1$ semper constabit duabus partibus, altera logarithmica altera circulari, nisi forte fuerit vel $Ff + Gg = 0$, vel $fG - gF = 0$. Priore enim casu haec portio tantum arcum circularem inuoluet, posteriore vero tantum logarithmum.

Scho-

Scholion.

§. 14. Quoniam assumimus denominatoris Q factorem esse $xx - 2x\cos.\omega + 1$, alias denominatoris formas hic non contemplabimur, nisi quarum omnes factores tali formula exprimi queant. Tales autem formulae simpliciores occurrunt tres sequentes:

$$Q = 1 + x^{2k},\ Q = 1 - x^{2k},\ Q = 1 + 2x^k\cos.\eta + x^{2k},$$

vbi quidem in prioribus potestati ipsius x exponentem parem tribuimus, quoniam casus, quibus esset impar, facile ad hanc formam reduci possunt. Si enim denominator esset $1 \pm x^i$, denotante i numerum imparem, tantum loco x scribamus y^2, prodibitque talis forma: $1 \pm y^{2i}$; at tali substitutione natura formulae differentialis neutiquam mutatur. Hos ergo tres casus in sequentibus tribus Problematibus particularibus omni cura percurramus, quo magis praestantia istius nouae methodi prae aliis, quae adhuc in vsu fuerunt, eluceat.

Problema particulare I.

§. 15. *Si fuerit* $Q = 1 + x^{2k}$, *inuestigare integrale huius formulae differentialis:* $\frac{P\,dx}{(1 + x^{2k})\,x}$; *vbi quidem* $\frac{P}{x}$ *sit functio integra, in qua nullae potestates altiores occurrant, quam exponentis* $2k$, *ne scilicet ista fractio euadat spuria.*

Solutio.

Cum sit $Q = 1 + x^{2k}$, sit eius factor trinomialis quicunque $= xx - 2x\cos.\omega + 1$, ita vt numerus talium factorum sit $= k$; quare cum, posito

$$x x - 2 x \operatorname{cos.} \omega + 1 = 0,$$

etiam ipsa formula $1 + x^{2k}$ euanescere debeat, facta substitutione debita secundum Theorema II. fiet

$$Q = 1 + \frac{(x - \operatorname{cos.} \omega)}{\operatorname{sin.} \omega} \operatorname{sin.} 2 k \omega + \operatorname{cos.} 2 k \omega,$$

qui valor cum debeat euanescere, erit tam $\operatorname{sin.} 2 k \omega = 0$, quam $1 + \operatorname{cos.} 2 k \omega = 0$. Conditio ergo posterior praebet $\operatorname{cos.} 2 k \omega = - 1$; vnde intelligitur, angulum $2 k \omega$ esse debere vel π, vel 3π, vel 5π, vel in genere $(2 i - 1) \pi$, denotante $2 i - 1$ numerum imparem quemcunque. Valores igitur anguli ω erunt sequentes:

$$1^{\circ}.\ \omega = \frac{\pi}{2 k},\ 2^{\circ}.\ \omega = \frac{3 \pi}{2 k},\ 3^{\circ}.\ \omega = \frac{5 \pi}{2 k},$$

et generatim $\omega = \frac{(2 i - 1) \pi}{2 k}$, quorum numerus cum esse debeat $= k$, vltimus valor erit $\omega = \frac{(2 k - 1)}{2 k} \pi$; singulis autem istis valoribus simul prior conditio adimpletur, quae esse debet $\operatorname{sin.} 2 k \omega = 0$. Quodsi iam pro ω vnusquisque horum valorum accipiatur, atque ponatur

$$x x - 2 x \operatorname{cos.} \omega + 1 = 0,$$

quicunque fuerit numerator P, sumamus facta hac substitutione fieri

$$P = \frac{F (x - \operatorname{cos.} \omega)}{\operatorname{sin.} \omega} + G;$$

tum vero erit $\frac{x d Q}{d x} = 2 k x^{2 k}$, vnde, cum nostro casu fieri debeat $Q = 0$, erit vtique $x^{2 k} = - 1$, sicque fiet

$$\frac{x d Q}{d x} = - 2 k.$$

Cum igitur haec formula in genere posita sit $\frac{f (x - \operatorname{cos.} \omega)}{\operatorname{sin.} \omega} + g$, erit nunc $f = 0$ et $g = - 2 k$, quo inuento secundum praecepta ante tradita pars integralis ex hoc factore denominatoris $x x - 2 x \operatorname{cos.} \omega + 1$, oriunda erit

$-\frac{G}{k}$

$$-\frac{G}{k}\, l\sqrt{(xx - 2x\cos.\omega + 1)} + \frac{F}{k}\, A\, tang.\, \frac{x \sin.\omega}{1 - x\cos.\omega};$$

consequenter si ex singulis valoribus anguli ω istae partes integralis formentur, et in vnam summam colligantur, impetrabitur totum integrale formulae differentialis propositae; et quia hoc casu nunquam fieri potest vel $\omega = 0$, vel $\omega = \pi$, cautione supra indicata non erit opus.

Corollarium 1.

§. 16. Quodsi numerator P fuerit potestas simplex ipsius x, puta x^m, existente $m > 0$, at $m < 2k$, vt formula integranda sit $\int \frac{x^{m-1}\, dx}{1 + x^{2k}}$, posito

$$xx - 2x\cos.\omega + 1 = 0, \text{ erit formula}$$
$$P = x^m = \frac{(x - \cos.\omega)}{\sin.\omega}\sin.m\omega + \cos.m\omega, \text{ ideoque}$$
$$F = \sin.m\omega \text{ et } G = \cos.m\omega,$$

vnde quaelibet portio integralis induet hanc formam:

$$-\frac{\cos.m\omega}{k}\, l\sqrt{(xx - 2x\cos.\omega + 1)}\, \frac{\sin.m\omega}{k}\, A\, tang.\, \frac{x\sin.\omega}{1 - x\cos.\omega}$$

et aggregatum omnium harum partium, siquidem loco ω successiue singuli eius valores substituantur, dabit totum integrale formulae huius propositae, ita sumtum, vt euanescat posito $x = 0$.

Corollarium 2.

§. 17. Si numerator P ex pluribus huiusmodi terminis constet, vt sit $P = a x^\alpha + b x^\beta + c x^\gamma +$ etc. integratio maiore difficultate non laborat; erit enim

$$F = a\sin.\alpha\omega + b\sin.\beta\omega + c\sin.\gamma\omega \text{ etc. et}$$
$$G = a\cos.\alpha\omega + b\cos.\beta\omega + c\cos.\gamma\omega \text{ etc.}$$

hincque totum integrale facile expedietur.

Scho-

Scholion.

§. 18. Hic autem occurrit casus imprimis memorabilis, quo sumitur $P = x^{k-n} + x^{k+n}$, quem in sequente Problemate speciali seorsim euoluamus.

Problema speciale.

§. 19. *Proposita formula differentiali* $\frac{x^{k-n} + x^{k+n}}{1 + x^{2k}} \cdot \frac{dx}{x}$ *eius totum integrale euoluere.*

Solutio.

Cum hic sit $P = x^{k-n} + x^{k+n}$, si statuamus
$xx - 2x \cos.\omega + 1 = 0$, fiet
$$P = \frac{x - \cos.\omega}{\sin.\omega}(\sin.(k-n)\omega) + \sin.(k-n)$$
$$+ \cos.(k-n)\omega + \cos.(k-n)\omega,$$
vnde sponte se produnt litterae F et G; cum autem in genere sit
$$\sin.p + \sin.q = 2\sin.\tfrac{p+q}{2} \cdot \cos.\tfrac{p-q}{2} \text{ et}$$
$$\cos.p + \cos.q = 2\cos.\tfrac{p+q}{2} \cdot \cos.\tfrac{p-q}{2},$$
facta hac reductione reperietur:
$$F = 2\sin.k\omega \cos.n\omega \text{ et } G = 2\cos.k\omega \cos.n\omega.$$
Cum autem in genere sit
$$\omega = \frac{(2i-1)\pi}{2k}, \text{ erit } \sin.k\omega = \sin.\frac{(2i-1)\pi}{2},$$
cuius valor est vel $+1$, vel -1; vtrumuis autem locum habeat, semper erit $\cos.k\omega = 0$, ita vt sit
$$F = 2\sin.\frac{(2i-1)\pi}{2}\cos.n\frac{(2i-1)\pi}{2k} \text{ et } G = 0 \cos.;$$
quibus

quibus valoribus inuentis pars integralis ex hoc factore generali oriunda erit

$$\frac{2}{k}\,\text{fin.}\,\frac{(2i-1)\pi}{2}\,\text{cof.}\,\frac{(2i-1)n\pi}{2k}\,\text{A tag.}\,\frac{x\,\text{fin.}\,(\frac{(2i-1)\pi}{2k})}{1-x\,\text{cof.}\,\frac{(2i-1)\pi}{2k}}.$$

Hinc ergo, fi loco i fucceffiue fcribamus valores 1, 2, 3, 4, etc. vsque ad k, totum integrale quaefitum fequenti forma exprimetur:

$$\frac{2}{k}\,\text{cof.}\,\frac{n\pi}{2k}\,\text{A tang.}\,\frac{x\,\text{fin.}\,\frac{\pi}{2k}}{1-x\,\text{cof.}\,\frac{\pi}{2k}}$$

$$-\frac{2}{k}\,\text{cof.}\,\frac{3n\pi}{2k}\,\text{A tang.}\,\frac{x\,\text{fin.}\,\frac{3\pi}{2k}}{1-x\,\text{cof.}\,\frac{3\pi}{2k}}$$

$$+\frac{2}{k}\,\text{cof.}\,\frac{5nk}{2k}\,\text{A tang.}\,\frac{x\,\text{fin.}\,\frac{5\pi}{2k}}{1-x\,\text{cof.}\,\frac{5\pi}{2k}}$$

$$-\frac{2}{k}\,\text{cof.}\,\frac{7n\pi}{2k}\,\text{A tang.}\,\frac{x\,\text{fin.}\,\frac{7\pi}{2k}}{1-x\,\text{cof.}\,\frac{7\pi}{2k}}$$

$$+\ -\ -\ -\ -\ -\ -\ -$$

$$-\ -\ -\ -\ -\ -\ -\ -$$

$$+\frac{2}{k}\,\text{fin.}\,\frac{(2k-1)\pi}{2k}\,\text{cof.}\,\frac{(2k-1)n\pi}{2k}\,\text{A tg.}\,\frac{x\,\text{fin.}\,\frac{(2k-1)\pi}{2k}}{1-x\,\text{cof.}\,\frac{(2k-1)\pi}{2k}},$$

vbi imprimis notatu dignum vfu venit, vt omnes partes logarithmicae fe mutuo deftruxerint.

Corollarium 1.

§. 20. Quod fi ergo fumamus $n=0$, ita vt formula integranda fit $\int\frac{2x^k\,dx}{1+x^{2k}}$, eius integrale hoc modo exprimetur:

$$\frac{2}{k} \text{A. tang.} \frac{x \,\text{fin.}\, \frac{\pi}{2k}}{1 - x \,\text{cof.}\, \frac{\pi}{2k}} - \frac{2}{k} \text{A tang.} \frac{x \,\text{fin.}\, \frac{3\pi}{2k}}{1 - x \,\text{cof.}\, \frac{3\pi}{2k}}$$

$$+ \frac{2}{k} \text{A tang.} \frac{x \,\text{fin.}\, \frac{5\pi}{2k}}{1 - x \,\text{cof.}\, \frac{5\pi}{2k}} + \cdots\cdots$$

$$+ \frac{2}{k} \,\text{fin.}\, \frac{(2k-1)\pi}{2k} \text{A tang.} \frac{x \,\text{fin.}\, \frac{(2k-1)\pi}{2k}}{1 - x \,\text{cof.}\, \frac{(2k-1)\pi}{2k}}.$$

At pofito $x^k = z$, ob $\frac{dx}{x} = \frac{dz}{kz}$, formula integralis induet hanc formam: $\int \frac{2\,dz}{k(1+zz)}$, cuius integrale manifefto eft

$$\frac{2}{k} \text{A tang.}\, z = \frac{2}{k} \text{A tang.}\, x^k,$$

vnde fequitur fore

$$\text{A tang.}\, x^k = \text{A tang.} \frac{x \,\text{fin.}\, \frac{\pi}{2k}}{1 - x \,\text{cof.}\, \frac{\pi}{2k}} - \text{A tang.} \frac{x \,\text{fin.}\, \frac{3\pi}{2k}}{1 - x \,\text{cof}\, \frac{3\pi}{2k}}$$

$$+ \text{A tang.} \frac{x \,\text{fin.}\, \frac{5\pi}{2k}}{1 - x \,\text{cof.}\, \frac{5\pi}{2k}} - \cdots\cdots$$

$$\pm \,\text{fin.}\, \frac{(2k-1)\pi}{2k} \text{A tang.} \frac{x \,\text{fin.}\, \frac{(2k-1)\pi}{2k}}{1 - x \,\text{cof.}\, \frac{(2k-1)\pi}{2k}},$$

quod fane eft Theorema maxima attentione dignum.

Corollarium 2.

§. 21. Ad hoc Theorema illuftrandum fumamus $k = 1$, et ob $\text{fin.}\, \frac{\pi}{2} = 1$ et $\text{cof.}\, \frac{\pi}{2} = 0$ prodit manifefto

$$\text{A tang.}\, x = \text{A tang.}\, x.$$

At fumto $k = 2$, ob $\text{fin.}\, \frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $\text{cof.}\, \frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $\text{fin.}\, \frac{3\pi}{4} = \frac{1}{\sqrt{2}}$ et $\text{cof.}\, \frac{3\pi}{4} = \frac{-1}{\sqrt{2}}$, fiet

$$\text{A tang.}\, xx = \text{A tang.} \frac{x}{\sqrt{2} - x} - \text{A tang.} \frac{x}{\sqrt{2} + x}.$$

Cum

Cum autem in genere ſit

$$\text{A tang.}\, p - \text{A tang.}\, q = \text{A tang.}\, \tfrac{p - q}{1 + pq},$$

hoc caſu erit

$$p = \tfrac{x}{\sqrt{2} - x} \text{ et } q = \tfrac{x}{\sqrt{2} + x}, \text{ ideoque}$$

$$p - q = \tfrac{2xx}{2 - xx} \text{ et } 1 + pq = \tfrac{2}{2 - xx},$$

vnde manifeſto prodit

$$\text{A tang.}\, xx = \text{A tang.}\, xx.$$

Sumamus porro $k = 3$, et ob

$$\text{ſin.}\, \tfrac{\pi}{6} = \tfrac{1}{2},\ \text{coſ.}\, \tfrac{\pi}{6} = \tfrac{\sqrt{3}}{2},\ \text{ſin.}\, \tfrac{3\pi}{6} = 1,\ \text{coſ.}\, \tfrac{3\pi}{6} = 0,$$

$$\text{ſin.}\, \tfrac{5\pi}{6} = \tfrac{1}{2} \text{ et coſ.}\, \tfrac{5\pi}{6} = -\tfrac{\sqrt{3}}{2}, \text{ reperietur}$$

$$\text{A tang.}\, x^3 = \text{A tang.}\, \tfrac{x}{2 - x\sqrt{3}} - \text{tang.}\, x + \text{A tang.}\, \tfrac{x}{2 + x\sqrt{3}},$$

vbi per reductionem ſuperiorem Arcus primus et tertius iunctim ſumti, ob

$$p = \tfrac{x}{2 - x\sqrt{3}} \text{ et } q = \tfrac{-x}{2 + x\sqrt{3}},$$

praebent A tang. $\frac{x}{1 - xx}$, a quo ſi ſubtrahatur A tang. x, remanebit A tang. x^3.

Scholion.

§. 22. Ceterum veritas huius theorematis in genere comodiſſime ſumendis differentialibus oſtendi poteſt. Cum enim ſit

$$d.\, \text{A tang.}\, x^k = \frac{k x^{k-1}\, dx}{1 + x^{2k}} \text{ et}$$

$$d.\, \text{A tang.}\, \tfrac{x\, \text{ſin.}\, \omega}{1 - x\, \text{coſ.}\, \omega} = \tfrac{dx\, \text{ſin.}\, \omega}{1 - 2x\, \text{coſ.}\, \omega + xx},$$

ſi loco ω valores debiti ſucceſſiue ſubſtituantur et per dx diuidatur, reſultabit ſequens aequatio:

$$\frac{k\,x^{k-1}}{1+x^{2k}} = \frac{\text{fin.}\,\frac{\pi}{2k}}{1-2x\,\text{cof.}\,\frac{\pi}{2k}+xx} - \frac{\text{fin.}\,\frac{3\pi}{2k}}{1-2x\,\text{cof.}\,\frac{3\pi}{2k}+xx},$$

$$+ \;\ldots\ldots,\; \pm \frac{\text{fin.}\,\frac{(2k-1)\pi}{2k}}{1-2x\,\text{cof.}\,\frac{(2k-1)\pi}{2k}+xx},$$

quae funt eae ipfae fractiones partiales, in quas functio fracta $\frac{k\,x^{k-1}}{1+x^{2k}}$ refoluitur. Ceterum cum in hac integratione omnes logarithmi excefferint, duplex quaeftio circa integrale inuentum inftitui poteft, altera, qua quaeritur eius valor cafu $x=\infty$, altera vero cafu quo fumitur $x=1$.

Quaeftio prior.

§. 23. *Propofita formula differentiali* $\frac{x^{k-n}+x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}$, *eius integralis valorem inueftigere, qui oritur fi poft integrationem ponitur* $x=\infty$.

Solutio.

Cum quilibet arcus in expreffione integralis inventi §. 19 in genere fit huiusmodi: A tang. $\frac{x\,\text{fin.}\,\omega}{1-x\,\text{cof.}\,\omega}$, fi ftatuatur $x=\infty$, is hanc induet formam: A tang. $(-\text{tang.}\,\omega)$. Quia autem $-\text{tang.}\,\omega = +\text{tang.}\,(\pi-\omega)$, ifte arcus fiet $=\pi-\omega$; quare fi loco ω fucceffiue valores debitos fubftituamus, integrale quaefitum fequenti ferie exprimetur:

$$\frac{2}{k}\left(\pi-\frac{\pi}{2k}\right)\text{cof.}\,\frac{n\pi}{2k} - \frac{2}{k}\left(\pi-\frac{3\pi}{2k}\right)\text{cof.}\,\frac{3n\pi}{2k}$$
$$+\frac{2}{k}\left(\pi-\frac{5\pi}{2k}\right)\text{cof.}\,\frac{5n\pi}{2k} - \frac{2}{k}\left(\pi-\frac{7\pi}{2k}\right)\text{cof.}\,\frac{7n\pi}{2k}$$
$$+\;\ldots\ldots\;\pm\frac{2}{k}\left(\pi-\frac{(2k-1)\pi}{2k}\right)\text{cof.}\,\frac{(2k-1)n\pi}{2k},$$

cuius

cuius vltimum membrum habebit signum $+$, quoties fuerit $2k-1$ numerus formae $4\alpha+1$, siue $k=2\alpha+1$, ideoque k numerus impar; at vero signum $-$ valebit, si $2k-1$ fuerit formae $4\alpha-1$, siue $k=2\alpha$, ideoque numerus par. Ad valorem huius seriei inueniendum ponamus:

$$S=\left(1-\tfrac{1}{2k}\right)\cos.\tfrac{n\pi}{2k}-\left(1-\tfrac{3}{2k}\right)\cos.\tfrac{3n\pi}{2k}+\left(1-\tfrac{5}{2k}\right)\cos.\tfrac{5n\pi}{2k}$$
$$-\left(1-\tfrac{7}{2k}\right)\cos.\tfrac{7n\pi}{2k}+\left(1-\tfrac{9}{2k}\right)\cos.\tfrac{9n\pi}{2k}$$
$$-\ldots\pm\left(1-\tfrac{(2k-1)}{2k}\right)\cos.\tfrac{(2k-1)n\pi}{2k},$$

ita vt valor noster quaesitus sit $\frac{2\pi S}{k}$. Quo nunc valorem ipsius S inuestigemus, multiplicemus vtrinque per $2\cos.\frac{n\pi}{2k}$, et cum in genere sit

$$2\cos.\tfrac{n\pi}{2k}\cos.\tfrac{(2i-1)n\pi}{2k}=\cos.\tfrac{in\pi}{k}+\cos.\tfrac{(i-1)n\pi}{k},$$

adhibita ista reductione reperietur

$$\left\{\begin{array}{l} \qquad\qquad +\left(1-\frac{1}{2k}\right)\cos.\frac{n\pi}{k}-\left(1-\frac{3}{2k}\right)\cos.\frac{2n\pi}{k} \\ +\left(1-\frac{1}{2k}\right)-\left(1-\frac{3}{2k}\right)\cos.\frac{n\pi}{2k}+\left(1-\frac{5}{2k}\right)\cos.\frac{2n\pi}{k} \\ \qquad\qquad +\left(1-\frac{5}{2k}\right)\cos.\frac{3n\pi}{k}-\left(1-\frac{7}{2k}\right)\cos.\frac{4n\pi}{k}+\text{etc.} \\ \qquad\qquad -\left(1-\frac{7}{2k}\right)\cos.\frac{3n\pi}{k}+\left(1-\frac{9}{2k}\right)\cos.\frac{4n\pi}{k}-\text{etc.} \end{array}\right\}$$

vbi patet, quemlibet terminum superiorem cum sequente inferiori in vnicum coalescere, ita vt tantum primus inferior, qui est $\left(1-\frac{1}{2k}\right)$, et vltimus superior, qui est $\pm\frac{1}{2k}\cos.n\pi$, solitarii relinquantur; facta ergo hac contractione reperietur:

$$2S\cos.\tfrac{n\pi}{2k}=1-\tfrac{1}{2k}\pm\tfrac{1}{2k}\cos.n\pi+\tfrac{1}{k}\cos.\tfrac{n\pi}{k}-\tfrac{1}{k}\cos.\tfrac{2n\pi}{k}$$
$$+\tfrac{1}{k}\cos.\tfrac{3n\pi}{n}-\ldots\mp\tfrac{1}{k}\cos.\tfrac{(k-1)n\pi}{k},$$

vbi signum superius valet, si k fuerit numerus impar, inferius autem si k fuerit numerus par. Ponamus porro ad hanc seriem summandam

$$T = \cos.\frac{n\pi}{k} - \cos.\frac{2n\pi}{k} + \cos.\frac{3n\pi}{k} - \cos.\frac{4n\pi}{k} + \ldots \pm \cos.\frac{(k-1)n\pi}{k},$$

ita vt hoc valore T inuento futurum sit

$$2\,S\cos.\frac{n\pi}{2k} = 1 - \frac{1}{2k} \pm \frac{1}{2k}\cos.n\pi + \frac{T}{k}.$$

Multiplicemus simili modo vtrinque per $2\cos.\frac{n\pi}{2k}$, et in subsidium vocata eadem reductione reperietur:

$$2\,T\cos.\frac{n\pi}{2k} = \left\{\begin{array}{l} \quad\quad\quad\quad\quad + \cos.\frac{3n\pi}{2k} - \cos.\frac{5n\pi}{2k} \\ + \cos.\frac{n\pi}{2k} - \cos.\frac{3n\pi}{2k} + \cos.\frac{5n\pi}{2k} \\ + \cos.\frac{7n\pi}{2k} \ldots \ldots \pm \cos.\frac{(2k-1)n\pi}{2k} \\ - \cos.\frac{7n\pi}{2k} \quad \ldots \ldots \end{array}\right\}$$

vbi omnes termini se mutuo destruunt, praeter primum inferiorem et vltimum superiorem, ita vt obtineamus:

$$2\,T\cos.\frac{n\pi}{2k} = \cos.\frac{n\pi}{2k} \mp \cos.\frac{(2k-1)n\pi}{2k}.$$

Quia autem

$$\frac{(2k-1)}{2k}n\pi = n\pi - \frac{n\pi}{2k}, \text{ erit}$$

$$\cos.\frac{(2k-1)n\pi}{2k} = \cos.n\pi\cos.\frac{n\pi}{2k} + \sin.n\pi\sin.\frac{n\pi}{2k},$$

quoniam vero n supponitur numerus integer; erit

$$\sin.n\pi = 0, \text{ ideoque}$$

$$2\,T\cos.\frac{n\pi}{2k} = \cos.\frac{n\pi}{2k} \mp \cos.n\pi\cos\frac{n\pi}{2k},$$

vnde fit

$$T = \tfrac{1}{2} \mp \tfrac{1}{2}\cos.n\pi, \text{ quo valore substituto fiet}$$

$$2\,S\cos.\frac{n\pi}{2k} = 1, \text{ consequenter}$$

$S = \frac{1}{2 \operatorname{cof.} \frac{n\pi}{2k}}$, ideoque valor noſter quaeſitus erit

$\frac{\pi}{k \operatorname{cof.} \frac{n\pi}{2k}}$, vnde naſcitur ſequens

Theorema 1.

§. 24. *Iſta formula integralis:*

$$\int \frac{x^{k-n} + x^{k+n}}{1 + x^{2k}} \cdot \frac{dx}{x}$$

a termino $x = 0$ *vsque ad terminum* $x = \infty$ *extenſa, producit hunc valorem:* $\frac{\pi}{k \operatorname{cof.} \frac{n\pi}{2k}}$, cuius demonſtratio ex praecedente paragrapho liquet. Huic adiungi poteſt ſequens *Theorema*, quod prorſus ſingulari demonſtratione ex iſto deriuare licet.

Theorema 2.

§. 25. *Si tam iſta formula integralis:* $\int \frac{x^{k-n}}{1 + x^{2k}} \cdot \frac{dx}{x}$, *quam haec:* $\int \frac{x^{k+n}}{1 + x^{2k}} \cdot \frac{dx}{x}$, *a termino* $x = 0$ *vsque ad* $x = \infty$ *extendatur, vtraque producet eandem ſummam, quae eſt*

$$\frac{\pi}{2k \operatorname{cof.} \frac{n\pi}{2k}}.$$

Demonſtratio.

Ponatur $S = \int \frac{x^{k-n}}{1 + x^{2k}} \cdot \frac{dx}{x}$, ſiquidem integratio a termino $x = 0$ vsque ad terminum $x = \infty$ extendatur, ac pona-

ponatur $x = \frac{1}{z}$, ita vt iam integratio abſolui debeat a termino ∞ vsque ad o, et ob $\frac{dx}{x} = -\frac{dz}{z}$ habebitur nunc $-\int \frac{z^{-k+n}}{1+z^{-2k}} \cdot \frac{dz}{z}$, quae, ſi numerator ac denominator multiplicetur per z^{2k}, abit in hanc:

$$S = -\int \frac{z^{k+n}}{1+z^{2k}} \cdot \frac{dz}{z}$$

integratione a termino $z = \infty$ vsque ad $z = 0$ extenſa. Hinc permutatis terminis integrationis erit

$$S = \int \frac{z^{k+n}}{1+z^{2k}} \cdot \frac{dz}{z}$$

a termino $z = 0$ vsque ad $z = \infty$, vnde ſi loco z ſcribatur x, manifeſtum eſt, vtramque formulam, a termino $x = 0$ vsque ad $x = \infty$ extenſam, eandem habere ſummam S. Cum igitur ambae hae formulae iunctae praebeant ſummam $2S = \frac{\pi}{k} \operatorname{cof.} \frac{n\pi}{2k}$, erit vtique vtriusque formulae valor ſeorſim $S = \frac{\pi}{2k \operatorname{cof.} \frac{n\pi}{2k}}$.

Quaeſtio altera.

§. 26. *Propoſita formula differentiali*

$$\frac{x^{k-n} + x^{k+n}}{1+x^{2k}} \cdot \frac{dx}{x}$$

eius integralis valorem inueſtigare, qui oritur, ſi poſt integrationem ponatur $x = 1$.

Solutio.

Cum in forma integralis generali quilibet terminus inuentus ſit $\frac{2}{n} \operatorname{cof.} n\omega \operatorname{A tang.} \frac{x \operatorname{ſin.} \omega}{1 - x \operatorname{cof.} \omega}$, fiat hic $x = 1$

ac

ac prodibit $\frac{2}{k}$ cof. $n\,\omega$ A tang. $\frac{\text{fin.}\,\omega}{1-\text{cof.}\,\omega}$, quae forma ob

$$\text{fin.}\,\omega = 2\,\text{fin.}\,\tfrac{1}{2}\omega\,\text{cof.}\,\tfrac{1}{2}\omega \text{ et } 1-\text{cof.}\,\omega = 2\,\text{fin.}\,\tfrac{1}{2}\omega^2$$

abit in hanc:

$$\tfrac{2}{k}\,\text{cof.}\,n\,\omega \text{ A tang. } \frac{\text{cof.}\,\frac{1}{2}\omega}{\text{fin.}\,\frac{1}{2}\omega}$$

quae, cum fit

$$\frac{\text{cof.}\,\frac{1}{2}\omega}{\text{fin.}\,\frac{1}{2}\omega} = \text{cot.}\,\tfrac{1}{2}\omega = \text{tang.}\,(\tfrac{\pi}{2}-\tfrac{1}{2}\omega),$$

porro transformatur in hanc:

$$\tfrac{2}{k}\,\text{cof.}\,n\,\omega\,(\tfrac{\pi}{2}-\tfrac{1}{2}\omega) = \tfrac{1}{k}(\pi-\omega)\,\text{cof.}\,n\,\omega.$$

Quod fi igitur hic loco ω fucceffiue fcribamus eius valores, qui funt $\frac{\pi}{2k}$, $\frac{3\pi}{2k}$, $\frac{5\pi}{2k}$, vsque ad $\frac{(2k-1)\pi}{2k}$, valor integralis quaefitus exprimetur per hanc progreffionem:

$$\tfrac{1}{k}(\pi-\tfrac{\pi}{2k})\,\text{cof.}\,\tfrac{n\pi}{2k} - \tfrac{1}{k}(\pi-\tfrac{3\pi}{2k})\,\text{cof.}\,\tfrac{3n\pi}{2k} + \tfrac{1}{k}(\pi-\tfrac{5\pi}{2k})\,\text{cof.}\,\tfrac{5n\pi}{2k}$$
$$-\tfrac{1}{k}(\pi-\tfrac{7\pi}{2k})\,\text{cof.}\,\tfrac{7n\pi}{2k} - - - - - \pm \tfrac{1}{k}(\pi-\tfrac{(2k-1)}{2k}\pi)\,\text{cof.}\,\tfrac{(2k-1)n\pi}{2k}$$

vbi fignorum ambiguorum fuperius valet fi k fuerit numerus impar, inferius vero fi par. Comparemus hanc expreffionem cum ea, ad quam in quaeftione praecedente eft peruentum, ac reperiemus, hanc illius praecife effe femiffem, vnde eius valor erit $\frac{\pi}{2k\,\text{cof.}\,\frac{n\pi}{2k}}$, ficque habebitur fequens

Theorema.

§. 27. *Ifta formula integralis:* $\int \frac{x^{k-n}+x^{k+n}}{1+x^{2k}} \cdot \frac{dx}{x}$, *a termino* $x=0$ *vsque ad terminum* $x=1$ *extenfa, producet hunc valorem:* $\frac{\pi}{2k\,\text{cof.}\,\frac{n\pi}{2k}}$.

Corollarium.

§. 28. Cum igitur huius formulae integrale, a termino $x = 0$ vsque ad $x = 1$ extensum, sit dimidium eius, quod a termino $x = 0$ vsque ad $x = \infty$ extenditur, sequitur, si eadem formula integralis a termino $x = 1$ vsque ad $x = \infty$ extendatur, eius valorem quoque fore $\frac{\pi}{2k \operatorname{cof.} \frac{n\pi}{2k}}$, praeterea vero vtriusque valor aequabitur huic formulae integrali: $\int \frac{x^{k \pm n}}{1 + x^{2k}} \cdot \frac{dx}{x}$, siquidem ab $x = 0$ vsque ad $x = \infty$ extendatur.

Problema particulare II.

§. 29. *Si sumatur* $Q = 1 - x^{2k}$, *inuestigare integrale huius formulae differentialis*: $\frac{P\,dx}{(1 - x)^{2k}}$, *vbi quidem* $\frac{P}{x}$ *sit functio integra, in qua nullae potestates altiores occurrant quam exponentis* $2k$, *ne scilicet ista fractio euadat spuria.*

Solutio.

Cum sit $Q = 1 - x^{2k}$, statim duo eius habentur factores simplices reales, qui sunt $1 - x$ et $1 + x$, quare partes integrales ex iis oriundas primum inuestigemus. Ponamus igitur pro factore $1 - x$ fractionem

$$\frac{P}{x(1 - x^{2k})} = \frac{a}{1 - x} + R,$$

vbi R complectitur omnes reliquas partes, vnde per $1 - x$ multiplicando habebimus

$$\frac{P(1-x)}{x(1-x^{2k})}=\alpha+R(1-x);$$

quare ſi faciamus $x=1$, nanciſcemur

$$\alpha=\frac{P}{x}\cdot\frac{1-x}{1-x^{2k}},$$

cuius poſterioris fractionis, poſito $x=1$, tam numerator quam denominator euaneſcit, hinc eorum loco ſcribamus eorum differentialia, fietque

$$\alpha=\frac{P}{x}\cdot\frac{1}{2kx^{2k-1}}.$$

Fiat igitur nunc $x=1$, quo caſu abeat P in B, ac prodibit $\alpha=\frac{B}{2k}$; ex fractione autem partiali $\frac{\alpha}{1-x}$ porro reperitur pars integralis inde nata

$$=-\alpha\, l(1-x)=\frac{-B}{2k}l(1-x).$$

Pro altero factore $1+x$ faciamus ſimili modo

$$\frac{P}{x(1-x^{2k})}=\frac{\beta}{1+x}+R,$$

vnde per $1+x$ multiplicando fit

$$\frac{P(1+x)}{x(1-x^{2k})}=\beta+R(1+x),$$

quare ſi faciamus $x=-1$, erit

$$\beta=\frac{P}{x}\cdot\frac{1+x}{1-x^{2k}},$$

vbi in poſteriore fractione differentialia tam ſupra quam infra ſcribantur, vt prodeat

$$\beta=\frac{P}{x}\cdot\frac{1}{-2kx^{2k-1}}=\frac{-P}{2kx^{2k}}=\frac{-P}{2k},$$

 poſito

posito scilicet $x = -1$. Ponamus ergo, facto $x = -1$, functionem P abire in functionem C, fietque $\beta = -\frac{C}{2k}$, et ex fractione partiali $\frac{\beta}{1+x}$, obtinebitur pars inde nata integralis $\beta l(1+x) = -\frac{C}{2k} l(1+x)$, sicque ex ambobus factoribus $(1+x)$ et $(1-x)$ nascuntur hae duae partes integrales: $-\frac{B}{2k} l(1-x) - \frac{C}{2k} l(1+x)$.

His expeditis sit formulae $1 - x^{2k}$ factor trinominalis quicunque $1 - 2x \cos. \omega + xx$, quo facto $= 0$, ista formula $1 - x^{2k}$ induet hanc formam:

$$1 - \frac{(x - \cos. \omega)}{\sin. \omega} \sin. 2k\omega - \cos. 2k\omega,$$

quae formula, cum debeat euanescere, has suppeditat conditiones:

$$1^{\circ}.\ \sin. 2k\omega = 0 \text{ et } 2^{\circ}.\ \cos. 2k\omega = 1;$$

ex posteriore statim intelligitur angulum ω sequentes valores accipere posse:

$$1^{\circ}.\ \omega = 0,\ 2^{\circ}.\ \omega = \frac{2\pi}{2k} = \frac{\pi}{k},\ 3^{\circ}.\ \omega = \frac{4\pi}{2k} = \frac{2\pi}{2k},$$

et in genere $\omega = \frac{i\pi}{k}$. Quia igitur numerus horum valorum debet esse $= k$, primus autem $\omega = 0$ tantum factori simplici respondet, numerus valorum debet sumi $k+1$, ita vt iam vltimus futurus sit $\frac{k\pi}{k} = \pi$, vnde alter factor simplex $1+x$ nascitur; hoc autem modo simul primae conditioni satisfit, qua esse debet $\sin. 2k\omega = 0$. Nunc consideremus factorem generalem $xx - 2x \cos. \omega + 1$, quo posito $= 0$ fiat

$$P = \frac{F(x - \cos. \omega)}{\sin. \omega} + G, \text{ eritque } \frac{x\,dQ}{dx} = -2k\,x^{2k};$$

at vero iam vidimus, tum fieri $x^{2k} = 1$ sicque $\frac{x\,dQ}{dx} = -2k$,

pro

pro qua forma in genere posuimus $\frac{f(x - \text{cos.}\,\omega)}{\text{sin.}\,\omega} + g$, quo circa pro nostro casu erit $f = 0$ et $g = -2k$. Cum igitur pro hoc factore in genere inuenta sit ista pars integralis:

$$\frac{2(Ff + Gg)}{ff + gg}\, l \sqrt{(xx - 2x\,\text{cos.}\,\omega + 1)}$$
$$+ \frac{2(fG - Fg)}{ff + gg}\, \text{A tang.}\, \frac{x\,\text{sin.}\,\omega}{1 - x\,\text{cos.}\,\omega},$$

erit ista pars nostro casu

$$-\frac{G}{k}\, l \sqrt{(xx - 2x\,\text{cos.}\,\omega + 1)} + \frac{F}{k}\, \text{A tang.}\, \frac{x\,\text{sin.}\,\omega}{1 - x\,\text{cos.}\,\omega},$$

consequenter si loco ω successiue scribantur valores indicati, scilicet $\omega = 0$, $\omega = \frac{\pi}{k}$, $\omega = \frac{2\pi}{k}$, vsque ad $\omega = (k - i)\frac{\pi}{k}$, et omnes istae partes in vnam summam colligantur, obtinebitur totum integrale formulae propositae. Hic autem probe obseruandum est, ex parte prima et vltima eas ipsas partes oriri, quas iam pro valoribus $(1 - x)$ et $(1 + x)$ assignauimus; quare eas penitus omitti conueniet.

Corollarium 1.

§. 30. Quodsi numerator P fuerit potestas simplex ipsius x, puta x^m, existente $m > 1$ et $m < 2k$, vt formula integranda sit $\int \frac{x^{m-1}\,dx}{1 + x^{2k}}$, pro factoribus simplicibus $1 - x$ et $1 + x$ erit $B = +1$ et $C = (-1)^m$, vnde partes integralis hinc natae erunt

$$\frac{-1}{2k}\, l(1 - x) - \frac{(-1)^m}{2k}\, l(1 + x).$$

Pro reliquis vero partibus erit $F = \text{sin.}\, m\omega$ et $G = \text{cos.}\, m\omega$, vnde quaelibet portio integralis induet hanc formam:

$$-\frac{\operatorname{cof.} m\omega}{k}\, l \sqrt{(xx - 2x \operatorname{cof.} \omega + 1)} + \frac{\operatorname{fin.} m\omega}{k} \operatorname{A tang.} \frac{x \operatorname{fin.} \omega}{1 - x \operatorname{cof.} \omega},$$

vbi valores pro ω ſubſtituendi ſunt

$$\frac{\pi}{k},\ \frac{2\pi}{k},\ \frac{3\pi}{k}\ .\ .\ .\ .\ \frac{(k-1)\pi}{k}.$$

Corollarium 2.

§. 31. Si numerator P pluribus huiusmodi terminis conſtet, vt ſit $P = a x^{\alpha} + b x^{\beta} + c x^{\gamma} +$ etc. integratio maiore difficultate non laborat; erit enim

$$F = a \operatorname{fin.} \alpha\omega + b \operatorname{fin.} \beta\omega + c \operatorname{fin.} \gamma\omega + \text{etc.} \quad \text{et}$$

$$G = a \operatorname{cof.} \alpha\omega + b \operatorname{cof.} \beta\omega + c \operatorname{cof.} \gamma\omega + \text{etc.}$$

hincque totum integrale facile expedietur.

Scholion.

§. 32. Caſus prae ceteris memoratu dignus, qui hic occurrit eſt, quo ſtatuitur $P = x^{k-n} - x^{k+n}$, quippe quo omnes partes logarithmicae ſe mutuo tollere reperiuntur, vnde eum in ſequente Problemate euoluamus.

Problema ſpeciale.

§. 33. *Propoſita formula differentiali* $\frac{x^{k-n} - x^{k+n}}{1 + x^{2k}} \cdot \frac{dx}{x}$ *eius totum integrale inueſtigare.*

Solutio.

Quia hic eſt $P = x^{k-n} - x^{k+n} = x^{k-n}(1 - x^{2n})$, ob n numerum integrum poſito tam $x = 1$ quam $x = -1$ iſte valor euaneſcet, vnde fiet tam $B = 0$ quam $C = 0$, ſicque partes integrales ex factoribus ſimplicibus natae ſponte

ſponte euaneſcunt. Pro factore autem duplici quocunque

$$1 - 2x\,\text{coſ.}\,\omega + xx,$$

eo facto $= 0$ reperietur:

$$P = \frac{x - \text{coſ.}\,\omega}{\text{ſin.}\,\omega}\left(\text{ſin.}\,(k-n)\,\omega - \text{ſin.}\,(k+n)\,\omega\right)$$
$$+ \text{coſ.}\,(k-n)\,\omega - \text{coſ.}\,(k+n)\,\omega,$$

hincque colligitur:

$$F = \text{ſin.}\,(k-n)\,\omega - \text{ſin.}\,(k+n)\,\omega \text{ et}$$
$$G = \text{coſ.}\,(k-n)\,\omega - \text{coſ.}\,(k+n)\,\omega.$$

Cum autem in genere ſit

$$\text{ſin.}\,p - \text{ſin.}\,q = 2\,\text{ſin.}\,\tfrac{p-q}{2}\,\text{coſ.}\,\tfrac{p+q}{2} \text{ et}$$
$$\text{coſ.}\,p - \text{coſ.}\,q = 2\,\text{ſin.}\,\tfrac{q-p}{2}\,\text{ſin.}\,\tfrac{p+q}{2}, \text{ ob}$$
$$p = (k-n)\,\omega \text{ et } q = (k+n)\,\omega \text{ erit}$$
$$F = -2\,\text{ſin.}\,n\omega\,\text{coſ.}\,k\omega \text{ et } G = 2\,\text{ſin.}\,n\omega\,\text{ſin.}\,k\omega.$$

Eſt vero vti vidimus in genere $\omega = \frac{i\pi}{k}$, vnde fit

$$\text{ſin.}\,k\omega = \text{ſin.}\,i\pi = 0 \text{ et } \text{coſ.}\,k\omega = \pm 1,$$

ſcilicet valebit $+1$, ſi i eſt numerus par, et -1 ſi i impar. Ad hanc autem ambiguitatem euitandam retineamus coſ. $k\omega$, atque habebimus

$$F = -2\,\text{ſin.}\,n\omega\,\text{coſ.}\,k\omega \text{ et } G = 0.$$

Ex his igitur pars integralis quaecunque in genere erit:

$$-\frac{2\,\text{ſin.}\,n\omega\,\text{coſ.}\,k\omega}{k}\,\text{A tang.}\,\frac{x\,\text{ſin.}\,\omega}{1 - x\,\text{coſ.}\,\omega},$$

vbi tantum opus eſt loco ω valores indicatos ſucceſſiue ſubſtitui; et quia pro primo $\omega = 0$ et vltimo $\omega = \pi$ partes integrales ſponte euaneſcunt, perinde eſt ſiue valores primus et vltimus reiiciantur ſiue retineantur, quamobrem totum integrale quaeſitum ſequenti modo exprimetur:

+ 2

$$+\frac{2}{k}\,\text{fin.}\,\frac{n\pi}{k}\,\text{A tang.}\,\frac{x\,\text{fin.}\,\frac{\pi}{k}}{1-x\,\text{cof.}\,\frac{\pi}{k}}$$

$$-\frac{2}{k}\,\text{fin.}\,\frac{2n\pi}{k}\,\text{A tang.}\,\frac{x\,\text{fin.}\,\frac{2\pi}{k}}{1-x\,\text{cof.}\,\frac{2}{k}}$$

$$+\frac{2}{k}\,\text{fin.}\,\frac{3n\pi}{k}\,\text{A tag.}\,\frac{x\,\text{fin.}\,\frac{3\pi}{k}}{1-x\,\text{cof.}\,\frac{3\pi}{k}}$$

$$- \quad - \quad - \quad - \quad - \quad - \quad - \quad -$$

$$+ \quad - \quad - \quad - \quad - \quad - \quad - \quad -$$

$$\pm\frac{2}{k}\,\text{fin.}\,\frac{(k-1)n\pi}{k}\,\text{A tang.}\,\frac{x\,\text{fin.}\,\frac{(k-1)\pi}{k}}{1-x\,\text{cof.}\,\frac{(k-1)\pi}{k}},$$

vbi fignum fuperius valet fi n fuerit numerus par, inferius vero fi impar.

Corollarium.

§. 34. Si hic fumere velimus $n=0$, formula integranda fponte euanefcit, ita vt hic nihil memoratu dignum refultet, vnde in valores huius integralis pro cafibus $x=\infty$ et $x=1$ inquiramus.

Quaeftio prior.

§. 35. *Propofita formula differentiali*

$$\frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x},$$

eius valorem integralem inueftigare, qui oritur fi poft integrationem ponitur $x=\infty$.

Solutio.

Cum arcuum qui hic occurrunt forma generalis fit A tang. $\frac{x\,\text{fin.}\,\omega}{1-x\,\text{cof.}\,\omega}$, ea, pofito $x=\infty$, vti ante vidimus, abit in $\pi-\omega$, vnde litterae ω valores fuos ordine tribuendo

valor

valor quaesitus nostrae formulae integralis sequenti progressione exprimetur:

$$\tfrac{2}{k}(\pi-\tfrac{\pi}{k})\,\text{fin.}\tfrac{n\pi}{k}-\tfrac{2}{k}(\pi-\tfrac{2\pi}{k})\,\text{fin.}\tfrac{2n\pi}{k}+\tfrac{2}{k}(\pi-\tfrac{3\pi}{k})\,\text{fin.}\tfrac{3n\pi}{k}$$
$$-\tfrac{2}{k}(\pi-\tfrac{4\pi}{k})\,\text{fin.}\tfrac{4n\pi}{k}+\ldots\ldots\pm\tfrac{2}{k}(\pi-(\tfrac{(k-1)\pi}{k})\,\text{fin.}\tfrac{(k-1)n\pi}{k}.$$

Ad huius valorem inuestigandum ponamus:

$$S=(1-\tfrac{1}{k})\,\text{fin.}\tfrac{n\pi}{k}-(1-\tfrac{2}{k})\,\text{fin.}\tfrac{2n\pi}{k}+(1-\tfrac{3}{k})\,\text{fin.}\tfrac{3n\pi}{k}$$
$$-(1-\tfrac{4}{k})\,\text{fin.}\tfrac{4n\pi}{k}\ldots\ldots\pm(1-\tfrac{(k-1)}{k})\,\text{fin.}\tfrac{(k-1)n\pi}{k}$$

ita vt valor quem quaerimus sit $\frac{2\pi S}{k}$. Multiplicemus igitur vt hactenus vtrinque per 2 cof. $\frac{n\pi}{2k}$, et cum in genere sit

$$2\,\text{fin.}\tfrac{in\pi}{k}\,\text{cof.}\tfrac{n\pi}{2k}=\text{fin.}\tfrac{(2i+1)n\pi}{2k}+\text{fin.}\tfrac{(2i-1)n\pi}{2k},$$

facta hac reductione perueniemus ad sequentem expressionem:

$$2S\,\text{cof.}\tfrac{n\pi}{2k}=\begin{cases}\qquad\qquad+(1-\tfrac{1}{k})\,\text{fin.}\tfrac{3n\pi}{2k}-(1-\tfrac{2}{k})\,\text{fin.}\tfrac{5n\pi}{2k}\\(1-\tfrac{1}{k})\,\text{fin.}\tfrac{n\pi}{2k}-(1-\tfrac{2}{k})\,\text{fin.}\tfrac{3n\pi}{2k}+(1-\tfrac{3}{k})\,\text{fin.}\tfrac{5n\pi}{2k}\end{cases}$$
$$\begin{matrix}+(1-\tfrac{3}{k})\,\text{fin.}\tfrac{7n\pi}{2k}\ldots\pm(1-\tfrac{(k-3)}{k})\,\text{fin.}\tfrac{(2k-3)n\pi}{2k}\\-(1-\tfrac{5}{k})\,\text{fin.}\tfrac{7n\pi}{2k}\ldots\mp(1-\tfrac{(k-1)}{k})\,\text{fin.}\tfrac{(2k-3)n\pi}{2k}\end{matrix}\Bigg\}$$

vbi quilibet terminus superior cum sequente inferiori in vnum contrahi potest, vnde primum inferiorem cum vltimo superiori seorsim exhibeamus hoc modo:

$$2S\,\text{cof.}\tfrac{n\pi}{2k}=(1-\tfrac{1}{k})\,\text{fin.}\tfrac{n\pi}{2k}\pm\tfrac{1}{k}\,\text{fin.}\tfrac{(2k-1)n\pi}{2k}$$
$$+\tfrac{1}{k}\,\text{fin.}\tfrac{3n\pi}{2k}-\tfrac{1}{k}\,\text{fin.}\tfrac{5n\pi}{2k}+\tfrac{1}{k}\,\text{fin.}\tfrac{7n\pi}{2k}\mp\tfrac{1}{k}\,\text{fin.}\tfrac{(2k-3)n\pi}{2k}.$$

Hoc igitur modo vltimus superior cum reliquis eandem legem sequitur, ita vt ponere liceat:

$$2S\,\text{cof.}\tfrac{n\pi}{2k}=(1-\tfrac{1}{k})\,\text{fin.}\tfrac{n\pi}{2k},$$
$$+\tfrac{1}{k}\,\text{fin.}\tfrac{3n\pi}{2k}-\tfrac{1}{k}\,\text{fin.}\tfrac{5n\pi}{2k}+\tfrac{1}{k}\,\text{fin.}\tfrac{7n\pi}{2k}\ldots\pm\tfrac{1}{k}\,\text{fin.}\tfrac{(2k-1)n\pi}{2k}.$$

Statuamus porro

$$T = \text{fin.}\tfrac{3n\pi}{2k} - \text{fin.}\tfrac{5n\pi}{2k} + \text{fin.}\tfrac{7n\pi}{2k} - \ldots \pm \text{fin.}\tfrac{(2k-1)n\pi}{2k},$$

vt fit

$$2\,S\,\text{cof.}\tfrac{n\pi}{2k} = (1 - \tfrac{1}{k})\,\text{fin.}\tfrac{n\pi}{2k} + \tfrac{T}{k}.$$

Iam iterum multiplicemus per $2\,\text{cof.}\frac{n\pi}{2k}$, et adhibita eadem reductione reperiemus:

$$2\,T\,\text{cof.}\tfrac{n\pi}{2k} = \text{fin.}\tfrac{2n\pi}{k} - \text{fin.}\tfrac{3n\pi}{k} + \text{fin.}\tfrac{4n\pi}{k} \ldots \pm \text{fin.}\,n\pi$$
$$+ \text{fin.}\tfrac{n\pi}{k} - \text{fin.}\tfrac{2n\pi}{k} + \text{fin.}\tfrac{3n\pi}{k} - \text{fin.}\tfrac{4n\pi}{k} \qquad \pm \text{fin.}\tfrac{(k-1)n\pi}{k},$$

vbi, destructis terminis qui se mutuo tollunt, obtinebitur:

$$2\,T\,\text{cof.}\tfrac{n\pi}{2k} = \text{fin.}\tfrac{n\pi}{k} \pm \text{fin.}\,n\pi = \text{fin.}\tfrac{n\pi}{k}, \text{ ob } n\pi = 0.$$

Quia igitur est

$$\text{fin.}\tfrac{n\pi}{k} = 2\,\text{fin.}\tfrac{n\pi}{2k}\,\text{cof.}\tfrac{n\pi}{2k}, \text{ erit } T = \text{fin.}\tfrac{n\pi}{2k},$$

quo valore fubstituto fiet

$$2\,S\,\text{cof.}\tfrac{n\pi}{2k} = \text{fin.}\tfrac{n\pi}{2k},$$

ideoque $S = \frac{1}{2}\,\text{tang.}\frac{n\pi}{2k}$, consequenter nostrae formulae integralis, casu $x = \infty$, valor erit $\frac{\pi}{k}\,\text{tang.}\frac{n\pi}{2k}$, vnde nascitur sequens

Theorema.

§. 36. *Ista formula integralis:* $\int \frac{x^{k-n} + x^{k+n}}{1 - x^{2k}} \cdot \frac{dx}{x}$, *a termino* $x = 0$ *vsque ad terminum* $x = \infty$ *extensa, producit hunc valorem:* $\frac{\pi}{k}\,\text{tang.}\frac{n\pi}{2k}$.

Scholion.

§. 37. Cum haec formula duabus constet partibus, si simili modo, vt supra factum est, in valorem vtriusque seorsim inquirere velimus, vtriusque valor adeo imaginaginarius esse deprehendetur, id quod facile inde percipitur, quod posito $x = 1$ ipsa fractio iam in infinitum excrescat. Tractemus autem vt supra partem priorem, ponendo $S = \int \frac{x^{k-n}}{1-x^{2k}} \cdot \frac{dx}{x}$, a termino $x = 0$ vsque ad $x = \infty$, ac faciendo $x = \frac{1}{z}$, fiet

$$S = -\int \frac{z^{-k+n}}{1 - z^{-2k}} \cdot \frac{dz}{z} = -\int \frac{z^{k+n}}{z^{2k} - 1} \cdot \frac{dz}{z},$$

a termino $z = \infty$ vsque ad $z = 0$, ergo mutatis terminis integrationis erit

$$S = -\int \frac{z^{k+n}}{1 - z^{2k}} \cdot \frac{dz}{z}$$

a termino $z = 0$ vsque ad $z = \infty$; vnde si loco z scribamus x et has formulas iungamus, erit

$$2S = \int \frac{x^{k-n} - x^{k+n}}{1 - x^{2k}} \cdot \frac{dx}{x} = \frac{\pi}{k} \text{ tang. } \frac{n\pi}{2k}.$$

vnde prodiret $S = \frac{\pi}{2k}$ tang. $\frac{n\pi}{2k}$. Haec autem conclusio admitti nequit, quoniam nostra formula integralis eatenus tantum ad arcum circularem reduci poterit, quatenus numerator $x^{k-n} - x^{k+n}$ cum denominatore $1 - x^{2k}$ factorem communem habet $1 - xx$, qui ergo semper diuisionem tolli posset. Verum sumta tantum alterutra parte, iste factor $1 - xx$ ex denominatore non tollitur, ex eoque igitur necessario nasceretur pars integralis vel huius formae: $\alpha\, l \frac{1+x}{1-x}$, vel huius: $\alpha\, l(1-xx)$, quae vtraque forma, sumto $x = \infty$, fit imaginaria.

 Quae-

Quaestio altera.

§. 38. *Proposita formula differentiali*

$$\frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x}$$

eius integrale inuestigare, quod oritur, si post integrationem ponitur $x=1$.

Solutio.

Si in forma generali arcuum, quibus integrale exprimitur, quae est A tang. $\frac{x\,\sin.\omega}{1-x\,\cos.\omega}$, ponatur $x=1$, prodit vt ante vidimus $\frac{\pi}{2}-\frac{\omega}{2}$, qui valor cum sit dimidius eius quem casu praecedente habuimus, statim patet, valorem nostrum fore $\frac{\pi}{2k}$ tang. $\frac{n\pi}{2k}$, vnde nascitur istud

Theorema.

§. 39. *Ista formula integralis:* $\int\frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x}$, *a termino* $x=0$ *vsque ad terminum* $x=1$ *extensa, producet hunc valorem:* $\frac{\pi}{2k}$ tang. $\frac{n\pi}{2k}$.

Corollarium.

§. 40. Hinc si eiusdem formulae integrale a termino $x=1$ vsque ad $x=\infty$ extendatur, eius valor quoque erit $\frac{\pi}{2k}$ tang. $\frac{n\pi}{2k}$, quandoquidem hi duo valores iunctim sumti valorem casus praecedentis producere debent.

Problema particulare III.

§. 41. *Si sumatur* $Q=1+2x^k\cos.\eta+x^{2k}$, *investi-*

vestigare integrale huius formulae differentialis:

$$\frac{P\,dx}{x(1 + 2x^k \operatorname{cos.} \eta + x^{2k})},$$

vbi quidem $\frac{P}{x}$ *sit functio integra, in qua nullae potestates altiores occurrant quam exponentis* $2k$.

Solutio.

Quia denominator Q alios factores simplices, praeter imaginarios, non admittit, nisi casu quo $\eta = 180°$, sit eius factor trinomialis in genere $1 - 2x \operatorname{cos.} \omega + xx$, quo posito $= 0$ fiet

$$Q = \frac{x - \operatorname{cos.} \omega}{\operatorname{sin.} \omega}(\operatorname{sin.} 2k\omega + 2 \operatorname{cos.} \eta \operatorname{sin.} k\omega)$$
$$+ \operatorname{cos.} 2k\omega + 2 \operatorname{cos.} \eta \operatorname{cos.} k\omega + 1,$$

quae forma, quia debet esse nihilo aequalis, postulat has duas conditiones:

I. $\operatorname{sin.} 2k\omega + 2 \operatorname{cos.} \eta \operatorname{sin.} k\omega = 0$ et

II. $\operatorname{cos.} 2k\omega + 2 \operatorname{cos.} \eta \operatorname{cos.} k\omega + 1 = 0$.

Cum igitur sit

$\operatorname{sin.} 2k\omega = 2 \operatorname{sin.} k\omega \operatorname{cos.} k\omega$ et $\operatorname{cos.} 2k\omega + 1 = 2 \operatorname{cos.} k\omega^2$,

prior conditio dat

$$2 \operatorname{sin.} k\omega (\operatorname{cos.} k\omega + \operatorname{cos.} \eta) = 0,$$

et secunda conditio

$$2 \operatorname{cos.} k\omega (\operatorname{cos.} k\omega + \operatorname{cos.} \eta) = 0;$$

vtrique igitur conditioni satisfit simul, si fuerit

$$\operatorname{cos.} k\omega + \operatorname{cos.} \eta = 0,$$

quod quo facilius fieri possit sumamus $\eta = \pi - \theta$, vt habeatur $\operatorname{cos.} k\omega = \operatorname{cos.} \theta$. Omnes autem anguli cum η communem cofinum habentes funt: θ, $2\pi + \theta$, $4\pi + \theta$, $6\pi + \theta$, et in genere $2i\pi + \theta$, quamobrem ftatuamus pro ω fequentes valores:

$$\omega = \frac{\theta}{k}, \quad \omega = \frac{2\pi + \theta}{k}, \quad \omega = \frac{4\pi + \theta}{k}, \text{ etc.}$$

et in genere $\omega = \frac{2i\pi + \theta}{k}$, quorum valorum numerus cum debeat effe $= k$, vltimus erit

$$\omega = \frac{2(k-1)\pi + \theta}{k}, \text{ fiue } \omega = \frac{-2\pi + \theta}{k}.$$

His conftitutis confideremus formulam $\frac{x\,dQ}{dx}$, quae erit

$$= 2k(x^k \operatorname{cof.} \eta + x^{2k}),$$

quae per conditionem

$$xx - 2x \operatorname{cof.} \omega + 1 = 0, \text{ ob } \operatorname{cof.} \eta = -\operatorname{cof.} \theta,$$

reducitur ad hanc formam:

$$2k \frac{(x - \operatorname{cof.} \omega)}{\operatorname{fin.} \omega} (\operatorname{fin.} 2k\omega - \operatorname{cof.} \theta \operatorname{fin.} k\omega)$$
$$+ 2k(\operatorname{cof.} 2k\omega - \operatorname{cof.} \theta \operatorname{cof.} k\omega),$$

pro qua in genere fumfimus

$$= \frac{f(x - \operatorname{cof.} \omega)}{\operatorname{fin.} \omega} + g, \text{ ficque erit}$$
$$f = 2k(\operatorname{fin.} 2k\omega - \operatorname{cof.} \theta \operatorname{fin.} k\omega) \text{ et}$$
$$g = 2k(\operatorname{cof.} 2k\omega - \operatorname{cof.} \theta \operatorname{cof.} k\omega).$$

Loco $\operatorname{fin.} 2k\omega$ et $\operatorname{cof.} 2k\omega$ fcribamus valores ante indicatos, prodibitque

$$f = 2k \operatorname{fin.} k\omega (2 \operatorname{cof.} k\omega - \operatorname{cof.} \theta) \text{ et}$$
$$g = 2k(2 \operatorname{cof.} k\omega^2 - 1 - \operatorname{cof.} \theta \operatorname{cof.} k\omega),$$

cum autem effe debeat $\operatorname{cof.} k\omega = \operatorname{cof.} \theta$, fiet

$$f = 2k \operatorname{fin.} k\omega \operatorname{cof.} \theta = k \operatorname{fin.} 2k\omega \text{ et}$$
$$g = 2k(\operatorname{cof.} \theta^2 - 1) = -2k \operatorname{fin.} \theta^2.$$

Quia

Quia igitur in genere eſt $\omega = \frac{2i\pi + \theta}{k}$, erit

$$2k\omega = 4i\pi + 2\theta = 2\theta,$$

ita vt iam habeamus $f = k$ ſin. $2\theta = 2k$ ſin. θ coſ. θ, ita vt ſit $ff + gg = 4kk$ ſin. θ^2, quocirca ſi, poſito

$$1 - 2x \operatorname{cof.} \omega + xx = 0,$$

functio P transformetur in hanc formam: $\frac{F(x - \operatorname{cof.} \omega)}{\operatorname{fin.} \omega} + G$, ex denominatoris Q factore $1 - 2x$ coſ. $\omega + xx$ orietur iſta pars integralis:

$$\frac{F \operatorname{cof.} \theta - G \operatorname{fin.} \theta}{k \operatorname{fin.} \theta} l \surd (xx - 2x \operatorname{cof.} \omega + 1)$$
$$+ \frac{G \operatorname{cof.} \theta + F \operatorname{fin.} \theta}{k \operatorname{fin.} \theta} \text{A tang.} \frac{x \operatorname{fin.} \omega}{1 - x \operatorname{cof.} \omega}.$$

Tantum igitur ſupereſt vt loco ω ordine ſubſtituantur omnes eius valores, qui ſunt: $\frac{\theta}{k}$, $\frac{2\pi + \theta}{k}$, $\frac{4\pi + \theta}{k}$ - - - $\frac{2(k-1)\pi + \theta}{k}$, et ſumma omnium harum formularum praebebit totum integrale quaeſitum.

Corollarium.

§. 42. Si fuerit numerator P ſimplex poteſtas ipſius x, ſcilicet $P = x^m$, tum fiet $F = \operatorname{fin.} m\omega$ et $G = \operatorname{cof.} m\omega$, vnde pars integralis ex denominatoris factore indefinito: $1 - 2x$ coſ. $\omega + xx$, oriunda erit

$$\frac{\operatorname{cof.} \theta \operatorname{fin.} m\omega - \operatorname{fin.} \theta \operatorname{cof.} m\omega}{k \operatorname{fin.} \theta} l \surd (xx - 2x \operatorname{cof.} \omega + 1)$$
$$+ \frac{\operatorname{cof.} \theta \operatorname{cof.} m\omega + \operatorname{fin.} \theta \operatorname{fin.} m\omega}{k \operatorname{fin.} \theta} \text{A tang.} \frac{x \operatorname{fin.} \omega}{1 - x \operatorname{cof.} \omega},$$

vnde ſimul patet, ſi functio P ex pluribus huiusmodi poteſtatibus fuerit compoſita, quemadmodum integrationem abſolui oporteat.

Problema ſpeciale.

§. 43. *Propoſita formula differentiali* $\frac{x^{k-n} + x^{k+n}}{1 + 2x^k \operatorname{cof.} \eta + x^{2k}} \cdot \frac{dx}{x}$ *eius totum integrale inueſtigare.*

Solu-

Solutio.

Cum hic ſit $P = x^{k-n} + x^{k+n}$, erit
$F = 2\,\text{ſin.}\,k\,\omega\,\text{coſ.}\,n\,\omega$ et $G = 2\,\text{coſ.}\,k\,\omega\,\text{coſ.}\,n\,\omega$,
vbi $\text{coſ.}\,k\,\omega = \text{coſ.}\,\theta$, quibus valoribus ſubſtitutis pro parte integralis logarithmica erit:

$$\frac{F\,\text{coſ.}\,\theta - G\,\text{ſin.}\,\theta}{k\,\text{ſin.}\,\theta} = \frac{2\,\text{coſ.}\,\theta\,\text{coſ.}\,n\,\omega\,(\text{ſin.}\,k\,\omega - \text{ſin.}\,\theta)}{k\,\text{ſin.}\,\theta}.$$

Cum autem in genere ſit $\omega = \frac{2i\pi + \theta}{k}$, erit $\text{ſin.}\,k\omega = \text{ſin.}\,\theta$, vnde patet, hanc formulam euaneſcere, ita vt omnes partes logarithmicae ex integrali excedant. Pro partibus autem circularibus euadet coefficiens $\frac{G\,\text{coſ.}\,\theta + F\,\text{ſin.}\,\theta}{k\,\text{ſin.}\,\theta} = \frac{2\,\text{coſ.}\,n\,\omega}{k\,\text{ſin.}\,\theta}$, ſicque ex factore denominatoris indefinito $1 - 2x\,\text{coſ.}\,\omega + xx$ oritur iſta pars integralis:

$$\frac{2\,\text{coſ.}\,n\,\omega}{k\,\text{ſin.}\,\theta}\ \text{A tang.}\ \frac{x\,\text{ſin.}\,\omega}{1 - x\,\text{coſ.}\,\omega}.$$

In hac ergo formula pro ω ordine ſcribamus eius valores, qui ſunt $\frac{\theta}{k}$, $\frac{2\pi+\theta}{k}$, $\frac{4\pi+\theta}{k}$ etc. vsque ad $\frac{2(k-1)\pi+\theta}{k}$; vbi meminiſſe oportet eſſe $\theta = \pi - \eta$, et quo formulae non nimis fiant perplexae vtamur ſequentibus valoribus:

$$\frac{2\pi}{k} = \alpha,\quad \frac{\theta}{k} = \beta,\quad \frac{2n\pi}{k} = \gamma,\ \text{et}\ \frac{n\pi}{k} = \delta,$$

vt valores ipſius ω fiant

$$\beta,\ \alpha+\beta,\ 2\alpha+\beta,\ 3\alpha+\beta,\ \text{- - - -}\ (k-1)\alpha+\beta.$$

At vero omnes valores anguli $n\,\omega$ erunt ordine

$$\delta,\ \gamma+\delta,\ 2\gamma+\delta,\ 3\gamma+\delta,\ \text{- - - -}\ (k-1)\gamma+\delta.$$

His igitur valoribus adhibitis totum integrale quod quaerimus erit

$$\frac{2\,\text{coſ.}\,\delta}{k\,\text{ſin.}\,\theta}\ \text{A tang.}\ \frac{x\,\text{ſin}\,\beta}{1 - x\,\text{coſ.}\,\beta} + \frac{2\,\text{coſ.}\,(\gamma+\delta)}{k\,\text{ſin.}\,\theta}\ \text{A tang.}\ \frac{x\,\text{ſin.}\,(\alpha+\beta)}{1 - x\,\text{coſ.}\,(\alpha+\beta)}$$

$$+ \frac{2\,\text{coſ.}\,(2\gamma+\delta)}{k\,\text{ſin.}\,\theta}\ \text{A tang.}\ \frac{x\,\text{ſin.}\,(2\alpha+\beta)}{1 - x\,\text{coſ.}\,(2\alpha+\beta)} + \frac{2\,\text{coſ.}\,(3\gamma+\delta)}{k\,\text{ſin.}\,\theta}\ \text{A tang.}\ \frac{x\,\text{ſin.}\,(3\alpha+\beta)}{1 - x\,\text{coſ.}\,(3\alpha+\beta)}$$

$$+ \text{- - - - - - - -} + \frac{2\,\text{coſ.}\,((k-1)\gamma+\delta)}{k\,\text{ſin.}\,\theta}\ \text{A tang.}\ \frac{x\,\text{ſin.}\,((k-1)\alpha+\beta)}{1 - x\,\text{coſ.}\,((k-1)\alpha+\beta)}$$

Corol-

Corollarium 1.

§. 44. Euoluamus casum quo $n = 0$, ideoque etiam $\gamma = 0$ et $\delta = 0$, et quia hoc casu formula integralis euadet $2\int \frac{x^{k-1}\,dx}{1 + 2x^k \operatorname{cos.}\eta + x^{2k}}$, pro ea statuamus $x^k = z$, atque ob $x^{k-1} = \frac{dz}{k}$, habebitur $\frac{2}{k}\int \frac{dz}{1 + 2z\operatorname{cos.}\eta + zz}$, cuius integrale erit:

$$= \frac{2}{k\operatorname{sin.}\eta}\ \text{A tang.}\ \frac{z\operatorname{sin.}\eta}{1 - z\operatorname{cos.}\eta} = \frac{2}{k\operatorname{sin.}\theta}\ \text{A tang.}\ \frac{z\operatorname{sin.}\theta}{1 - z\operatorname{cos.}\theta},$$

vnde cum in serie inuenta omnes coefficientes arcuum fiant $\frac{2}{k\operatorname{sin.}\theta}$, per hunc coefficientem diuidendo habebimus sequentem aequationem:

$$\text{A tang.}\ \frac{x^k \operatorname{sin.}\theta}{1 - x^k\operatorname{cos.}\theta} = \text{A tang.}\ \frac{x\operatorname{sin.}\beta}{1 - x\operatorname{cos.}\beta} + \text{A tg.}\ \frac{x\operatorname{sin.}(\alpha+\beta)}{1 - x\operatorname{cos.}(\alpha+\beta)}$$
$$+ \text{A tg.}\ \frac{x\operatorname{sin.}(2\alpha+\beta)}{1 - x\operatorname{cos.}(2\alpha+\beta)} + \text{A tg.}\ \frac{x\operatorname{sin.}((k-1)\alpha+\beta)}{1 - x\operatorname{cos.}((k-1)\alpha+\beta)},$$

vbi recordandum est esse $\alpha = \frac{2\pi}{k}$, $\beta = \frac{\theta}{k}$.

Corollarium 2.

§. 45. Ponamus esse $\theta = 90° = \frac{\pi}{2}$, et aequatio modo inuenta hanc induet formam:

$$\text{A tang.}\,x = \text{A tg.}\ \frac{x\operatorname{sin.}\frac{\pi}{2k}}{1 - x\operatorname{cos.}\frac{\pi}{2k}} + \text{A tg.}\ \frac{x\operatorname{sin.}\frac{5\pi}{2k}}{1 - x\operatorname{cos.}\frac{5\pi}{2k}} + \text{A tg.}\ \frac{x\operatorname{sin.}\frac{9\pi}{2k}}{1 - x\operatorname{cos.}\frac{9\pi}{2k}}$$
$$+ \text{A tg.}\ \frac{x\operatorname{sin.}\frac{13\pi}{2k}}{1 - x\operatorname{cos.}\frac{13\pi}{2k}} + \ \cdot\ \cdot\ \cdot\ + \text{A tg.}\ \frac{x\operatorname{sin.}\left(\frac{4k-3)\pi}{2k}\right.}{1 - x\operatorname{cos.}\frac{(4k-3)\pi}{2k}}.$$

Sit $k = 1$, eritque A tang. $x =$ A tang. x.

Sit $k = 2$, eritque $\text{A tang.}\ xx = \text{A tang.}\ \frac{x}{\sqrt{2} - x} + \text{A tag.}\ \frac{-x}{\sqrt{2} + x}$

$$= \text{A tang.}\ \frac{x}{\sqrt{2} - x} - \text{A tang.}\ \frac{x}{\sqrt{2} + x}.$$

Sit

Sit $k = 3$, eritque $\text{A tang.}\, x^3 = \text{A tang.}\, \frac{x}{2 - x\sqrt{3}} + \text{A tg.}\, \frac{x}{2 + x\sqrt{3}} - \text{A tg}\, x$

etc. etc.

Haec igitur feries ab ea quam fupra §. 20 inuenimus, prorfus difcrepat, etiamfi vtriusque valor fit idem, fcil. $\text{A tang.}\, x^k$.

Quaeftio prior.

§. 46. *Propofita formula differentialis*

$$\frac{x^{k-n} + x^{k+n}}{1 + x^k \operatorname{cof.} \eta + x^{2k}} \cdot \frac{dx}{x}$$

eius integralis valorem inueftigare, qui oritur, fi poft integrationem ponatur $x = 1$.

Solutio.

Cum pofito $x = \infty$ in genere fit

$$\text{A tang.}\, \frac{x \operatorname{fin.} \omega}{1 - x \operatorname{cof.} \omega} = \pi - \omega,$$

valor integralis quem quaerimus hac ferie exprimetur:

$$\frac{2 \operatorname{cof.} \delta}{k \operatorname{fin.} \theta} \cdot (\pi - \beta) + \frac{2 \operatorname{cof.} (\gamma + \delta)}{k \operatorname{fin.} \theta} (\pi - \alpha - \beta) + \frac{2 \operatorname{cof.} (2\gamma + \delta)}{k \operatorname{fin.} \theta} (\pi - 2\alpha - \beta)$$
$$+ \frac{2 \operatorname{cof.} (3\gamma + \delta)}{k \operatorname{fin.} \theta} (\pi - 3\alpha - \beta) \ldots + \frac{2 \operatorname{cof.} ((k-1)\gamma + \delta)}{k \operatorname{fin.} \theta} (\pi - (k-1)\alpha - \beta)$$

Statuamus igitur

$$S = (\pi - \beta) \operatorname{cof.} \delta + (\pi - \alpha - \beta) \operatorname{cof.} (\gamma + \delta) + (\pi - 2\alpha - \beta) \operatorname{cof.} (2\gamma + \delta)$$
$$\ldots\ldots + (\pi - (k-1)\alpha - \beta) \operatorname{cof.} ((k-1)\gamma + \delta),$$

vt fit valor quaefitus $\frac{2S}{k \operatorname{fin.} \theta}$. Multiplicemus vtrinque per $2 \operatorname{fin.} \frac{1}{2}\gamma$, et cum fit

$$2 \operatorname{fin.} \tfrac{1}{2}\gamma \operatorname{cof.} q = \operatorname{fin.} (\tfrac{1}{2}\gamma + q) - \operatorname{fin.} (q - \tfrac{1}{2}\gamma),$$

hac reductione adhibita fiet:

$$2\,S\,\text{fin.}\,\tfrac{1}{2}\gamma = \begin{cases} -(\pi-\beta)\,\text{fin.}(\delta-\tfrac{1}{2}\gamma) + (\pi-\beta)\,\text{fin.}(\tfrac{1}{2}\gamma+\delta) \\ \qquad\qquad\qquad\qquad -(\pi-\alpha-\beta)\,\text{fin.}(\tfrac{1}{2}\gamma+\delta) \end{cases}$$

$$\left.\begin{aligned} &+(\pi-\alpha-\beta)\,\text{fin.}(\tfrac{3}{2}\gamma+\delta)+(\pi-2\alpha-\beta)\,\text{fin.}(\tfrac{5}{2}\gamma+\delta) \\ &-(\pi-2\alpha-\beta)\,\text{fin.}(\tfrac{3}{2}\gamma+\delta)+(\pi-3\alpha-\beta)\,\text{fin.}(\tfrac{5}{2}\gamma+\delta) \end{aligned}\right\}\text{etc.}$$

quae feries contractis terminis fimilibus tranfit in hanc:

$$\begin{aligned} 2\,S\,\text{fin.}\,\tfrac{1}{2}\gamma = &-(\pi-\beta)\,\text{fin.}\,(\delta-\tfrac{1}{2}\gamma)+\alpha\,\text{fin.}\,(\tfrac{1}{2}\gamma+\delta) \\ &+\alpha\,\text{fin.}\,(\tfrac{3}{2}\gamma+\delta)+\alpha\,\text{fin.}\,(\tfrac{5}{2}\gamma+\delta) \\ &+\ldots\ldots+\alpha\,\text{fin.}\,(\tfrac{(2k-3)}{2}\gamma+\delta) \\ &+(\pi-(k-1)\alpha-\beta)\,\text{fin.}\,(\tfrac{(2k-1)}{2}\gamma+\delta), \end{aligned}$$

vbi cum fit $\alpha = \frac{2\pi}{k}$ et $\beta = \frac{\theta}{k}$, erit

$$\pi-(k-1)\,\alpha-\beta = \alpha-\pi-\beta.$$

Ponatur

$$\begin{aligned} T = &\,\text{fin.}\,(\tfrac{1}{2}\gamma+\delta)+\text{fin.}\,(\tfrac{3}{2}\gamma+\delta)+\text{fin.}\,(\tfrac{5}{2}\gamma+\delta) \\ &+\text{fin.}\,(\tfrac{7}{2}\gamma+\delta)+\ldots\ldots+\text{fin.}\,((k-\tfrac{1}{2})\gamma+\delta), \end{aligned}$$

vt nancifcamur:

$$\begin{aligned} 2\,S\,\text{fin.}\,\tfrac{1}{2}\gamma = &-(\pi-\beta)\,\text{fin.}\,(\delta-\tfrac{1}{2}\gamma) \\ &-(\pi+\beta)\,\text{fin.}\,((k-\tfrac{1}{2})\gamma+\delta)+\alpha\,T, \end{aligned}$$

quae expreffio ob $k\gamma = 2n\pi$ reducitur ad $-2\pi\,\text{fin.}(\delta-\tfrac{1}{2}\gamma)+\alpha\,T$. Nunc igitur ad quantitatem T inueniendam multiplicemus vtrinque per $2\,\text{fin.}\,\tfrac{1}{2}\gamma$, et cum in genere fit:

$$2\,\text{fin.}\,\tfrac{1}{2}\gamma\,\text{fin.}\,q = \text{cof.}\,(q-\tfrac{1}{2}\gamma)-\text{cof.}\,(q+\tfrac{1}{2}\gamma),$$

obtinebimus:

$$2\,T\,\text{fin.}\,\tfrac{1}{2}\gamma = \begin{cases} \text{cof.}\,\delta-\text{cof.}\,(\gamma+\delta)-\text{cof.}\,(2\gamma+\delta)-\text{cof.}\,(3\gamma+\delta) \\ \qquad +\text{cof.}\,(\gamma+\delta)+\text{cof.}\,(2\gamma+\delta)+\text{cof.}\,(3\gamma+\delta) \end{cases}$$

$$\left.\begin{aligned} &-\text{cof.}\,(4\gamma+\delta)\ldots\ldots-\text{cof.}\,(k\gamma+\delta) \\ &+\text{cof.}\,(4\gamma+\delta)\ldots\ldots\ldots \end{aligned}\right\}$$

quae

quae forma contrahitur in istam:

$$2\,T\,\text{sin.}\,\tfrac{1}{2}\gamma = \text{cos.}\,\delta - \text{cos.}\,(\gamma\,k + \delta).$$

Cum autem sit $\gamma = \frac{2n\pi}{k}$, erit $k\gamma = 2n\pi$, ideoque

$\text{cos.}\,(k\gamma + \delta) = \text{cos.}\,\delta$, vnde fit

$2\,T\,\text{sin.}\,\tfrac{1}{2}\gamma = 0$, ita vt nunc fit

$2\,S\,\text{sin.}\,\tfrac{1}{2}\gamma = 2\pi\,\text{sin.}\,(\tfrac{1}{2}\gamma - \delta)$, ideoque,

$$S = \frac{\pi\,\text{sin.}\,(\frac{1}{2}\gamma - \delta)}{\text{sin.}\,\frac{1}{2}\gamma}.$$

Est vero $\frac{1}{2}\gamma = \frac{n\pi}{k}$ et $\delta = \frac{n\theta}{k}$, ideoque

$\frac{1}{2}\gamma - \delta = \frac{n(\pi - \theta)}{n} = \frac{n\eta}{k}$, ob $\theta = \pi - \eta$,

hocque modo habebimus $S = \frac{\pi\,\text{sin.}\,\frac{n\eta}{k}}{\text{sin.}\,\frac{n\pi}{k}}$, consequenter valor integralis quaesiti concluditur fore

$$\frac{2\pi\,\text{sin.}\,\frac{n\eta}{k}}{k\,\text{sin.}\,\theta\,\text{sin.}\,\frac{n\pi}{k}} = \frac{2\pi\,\text{sin.}\,\frac{n\eta}{k}}{k\,\text{sin.}\,\eta\,\text{sin.}\,\frac{n\pi}{k}},$$

vnde formetur sequens Theorema.

Theorema I.

§. 47. *Haec formula integralis:*

$$\int \frac{x^{k-n} + x^{k+n}}{1 + 2x^k\,\text{cos.}\,\eta + x^{2k}} \cdot \frac{dx}{x},$$

a termino $x = 0$ *vsque ad* $x = \infty$ *extensa, producit hunc valorem:* $\frac{2\pi\,\text{sin.}\,\frac{n\eta}{k}}{k\,\text{sin.}\,\eta\,\text{sin.}\,\frac{n\pi}{k}}$. Cui adiungatur adhuc sequens.

Theo-

Theorema 2.

§. 48. *Ista vero formula integralis:*

$$\int \frac{x^{k \pm n - 1} \, dx}{1 + 2x^k \cos. \eta + x^{2k}}$$

pariter a termino $x = 0$ *vsque ad terminum* $x = \infty$ *extensa, valorem habet dimidium praecedentis, qui ergo erit*

$$\frac{\pi \sin. \frac{n\eta}{k}}{k \sin. \eta \sin. \frac{n\pi}{k}},$$

cuius demonstratio perinde succedet ac supra §. 25.

Quaestio altera.

§. 49. *Proposita formula differentiali:*

$$\int \frac{x^{k-n} + x^{k+n}}{1 + 2x^k \cos. \eta + x^{2k}} \cdot \frac{dx}{x},$$

eius integralis valorem inuestigare, qui oritur si post integrationem ponitur $x = 1$.

Solutio.

Posito $x = 1$ formula generalis A tang. $\frac{x \sin. \omega}{1 - x \cos. \omega}$, vt supra vidimus, reducitur ad $\frac{\pi - \omega}{2}$; vnde patet, singulas partes integralis duplo minores esse quam casu praecedente, vnde valor quaesitus etiam erit duplo minor

$$= \frac{\pi \sin. \frac{n\eta}{k}}{k \sin. \eta \sin. \frac{n\pi}{k}},$$ vnde nascitur sequens

Theorema.

§. 50. *Ista formula integralis:*

$$\int \frac{x^{k-n} + x^{k+n}}{1 + 2x^k \cos.\eta + x^{2k}} \cdot \frac{dx}{x},$$

a termino $x = 0$ *vsque ad* $x = 1$ *extensa, producet hunc valorem:* $\frac{\pi \sin. \frac{n\eta}{k}}{k \sin. \eta \sin. \frac{n\pi}{k}}$.

Scholion.

§. 51. In his valoribus integralibus ii casus praecipue sunt notatu digni, quibus post integrationem statuitur $x = 1$, quandoquidem tum ista integralia commode per seriem infinitam exprimere licet. Ita pro casu §. 26, quoniam est

$$\frac{1}{1 + x^{2k}} = 1 - x^{2k} + x^{4k} - x^{6k} + \text{etc.}$$

si hanc seriem multiplicemus per $(x^{k-n} + x^{k+n}) \frac{dx}{x}$, et integremus, tum vero ponamus $x = 1$, prodibit ista series infinita:

$$\frac{1}{k-n} - \frac{1}{3k-n} + \frac{1}{5k-n} - \frac{1}{7k-n} + \text{etc.}$$
$$+ \frac{1}{k+n} - \frac{1}{3k+n} + \frac{1}{5k+n} - \frac{1}{7k+n} + \text{etc.}$$

cuius ergo seriei in infinitum continuatae summa est

$\frac{\pi}{2k \cos. \frac{n\pi}{2k}}$. At pro casu §. 38. ob

$$\frac{1}{1 - x^{2k}} = 1 + x^{2k} + x^{4k} + x^{6k} + \text{etc.}$$

eodem modo operando peruenitur ad hanc seriem:

$$\frac{1}{k-n} + \frac{1}{3k-n} + \frac{1}{5k-n} + \frac{1}{7k-n} + \text{etc.}$$
$$- \frac{1}{k+n} - \frac{1}{3k+n} - \frac{1}{5k+n} - \frac{1}{7k+n} - \text{etc.}$$

cuius ergo summa erit $\frac{\pi}{2k}$ tang. $\frac{n\pi}{2k}$. Denique pro casu, quem extremo loco tractauimus, cum sit vt alibi ostendimus:

$$\frac{\sin.\eta}{1+2x^k\cos.\eta+x^{2k}}=\sin.\eta-x^k\sin.2\eta+x^{2k}\sin.3\eta-\text{etc.}$$

haec series ducta in $(x^{k-n}+x^{k+n})\frac{dx}{x}$ et integrata, sumendo $x=1$, producet hanc seriem:

$$\frac{\sin.\eta}{k-n}-\frac{\sin.2\eta}{2k-n}+\frac{\sin.3\eta}{3k-n}-\frac{\sin.4\eta}{4k-n}+\text{etc.}$$
$$+\frac{\sin.\eta}{k+n}-\frac{\sin.2\eta}{2k+n}+\frac{\sin.3\eta}{3k+n}-\frac{\sin.4\eta}{4k+n}+\text{etc.}$$

cuius ergo valor aequabitur illi quem inuenimus valori ducto in sin. η, ita vt summa huius seriei sit $=\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\frac{n\pi}{k}}$, quae series eo magis sunt memorabiles, quod alio modo earum summa vix elici potest.

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