DE INSIGNIBVS PROPRIETATIBVS VNCIARVM BINOMII AD VNCIAS QVORVMVIS POLYNOMIORVM EXTENSIS.

Auctore
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Quae non ita pridem demonstraui circa insignes proprietates, quibus vnciae Binomii ad dignitatem quamcunque euecti sunt praeditae, etiam simili modo ad vncias Trinomii et Quadrinomii atque adeo in genere Polynomii cuiusque ad dignitatem quamcunque euecti extendi possunt, id quod in hac dissertatione dilucide ostendere constitui; quae quo facilius intelligi queant, denuo ab vnciis Binomii incipiam, quarum proprietates cum iam satis perspicue a me sint demonstratae, eas hic sine demonstratione succincte ante oculos ponam, quo hoc pacto progressio ad Polynomia clarius perspiciatur: neque enim o-

pus

pus erit, omnia quae sum allaturus, demonstrationibus corroborare, quandoquidem eae omnino similes sunt iis, quae de vnciis Binomii sunt traditae.

I. De vnciis Binomii $(1+z)^n$.

§. 1. Denotet character $[\frac{n}{p}]$ quamlibet harum vnciarum, hac ratione vt sit

$$(1+z)^n = [\tfrac{n}{0}] + [\tfrac{n}{1}]z + [\tfrac{n}{2}]z^2 + [\tfrac{n}{3}]z^3 + [\tfrac{n}{4}]z^4 + \text{etc.}$$

atque euidens est fore $[\frac{n}{0}] = 1$ et $[\frac{n}{1}] = n$; tum vero quilibet sequentium terminorum ex praecedente ita definitur, vt sit

$$[\tfrac{n}{p+1}] = \tfrac{n-p}{p+1}[\tfrac{n}{p}].$$

Praeterea vero notasse iuuabit, semper fore $[\frac{n}{p}] = 0$, tam quando p est numerus negatiuus quam quando p est numerus maior quam n; id quod intelligendum est, si pro p numeri integri accipiantur. Deinde quia vnciae hae finem versus eodem ordine regrediuntur, quo ab initio progrediuntur, erit $[\frac{n}{n}] = 1$, atque in genere erit $[\frac{n}{n-p}] = [\frac{n}{p}]$. Praeterea vero istae vnciae potestatis $(1+z)^n$ cum vnciis sequentis potestatis $(1+z)^{n+1}$ ita cohaerent, vt sit $[\frac{n+1}{p+1}] = [\frac{n}{p}] + [\frac{n}{p+1}]$, vnde sequitur fore

$$[\tfrac{n+1}{p+1}] = \tfrac{n+1}{p+1}[\tfrac{n}{p}].$$

§. 2. Quo clarius appareat quales numeri his characteribus quouis casu designentur, sequentem schematismum adiungamus, in quo vnciae potestatum simpliciorum exhibeantur, simulque pro singulis valores tam ipsius n quam ipsius p indicentur:

Valores pro p

n	0,	1,	2,	3,	4,	5,	6
0	1						
1	1 +	1					
2	1 +	2 +	1				
3	1 +	3 +	3 +	1			
4	1 +	4 +	6 +	4 +	1		
5	1 +	5 +	10 +	10 +	5 +	1	
6	1 +	6 +	15 +	20 +	15 +	6 +	1

vnde patet fore exempli caussa $[\frac{6}{4}] = 15$ et $[\frac{5}{3}] = 10$. Constat autem in genere esse

$$[\frac{n}{p}] = \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-p+1}{p}.$$

§. 3. His circa significationem istorum characterum expositis omnes proprietates, quas non ita pridem demonstraui, sequenti formula succincte repraesentari possunt: $\int[\frac{m}{x}][\frac{n}{p+x}] = [\frac{m+n}{n-p}]$, vbi membrum sinistrum denotat summam progressionis, cuius singuli termini sunt producta ex binis vnciis $[\frac{m}{x}]$ et $[\frac{n}{p+x}]$, dum scilicet litterae x successiue omnes valores integri tribuuntur, quamdiu nimirum neuter factorum in nihilum abit; vnde patet incipiendum esse ab $x = 0$, hincque procedendum, donec fiat vel $x = m$ vel $p + x = n$, hoc est vsque ad $x = n - p$, si quidem fuerit $n - p < m$. Huius igitur progressionis summa demonstrata est fore $= [\frac{m+n}{n-p}]$, quae etiam ita exhiberi potest: $[\frac{m+n}{m+p}]$. Hinc igitur sequitur, sumto $p = 0$ fore $\int[\frac{m}{x}] \cdot [\frac{n}{x}] = [\frac{m+n}{n}] = [\frac{m+n}{m}]$ etc. Sumto igitur insuper

$m = n$

$m = n$, erit $\int [\frac{n}{x}]^2 = [\frac{2n}{n}]$, quod eſt illud eximium theorema, quo oſtendi eſſe

$$1 + [\tfrac{n}{1}]^2 + [\tfrac{n}{2}]^2 + [\tfrac{n}{3}]^2 + [\tfrac{n}{4}]^2 + \text{etc.} = [\tfrac{2n}{n}],$$

vbi obſeruaui iſtam ſummam etiam hoc producto exprimi poſſe:

$$\frac{2 . 6 . 10 . - - - - - 4n - 2}{1 . 2 . 3 . - - - - - n}.$$

Ita exempli gratia ſi ſumatur $n = 5$, ipſa progreſſio hinc reſultans erit

$$1 + 5^2 + 10^2 + 10^2 + 5^2 + 1 = 252,$$

prior vero expreſſio euadit

$$[\tfrac{2n}{n}] = [\tfrac{10}{5}] = \frac{10 . 9 . 8 . 7 . 6}{1 . 2 . 3 . 4 . 5} = 252,$$

alterum autem productum praebet

$$\frac{2 . 6 . 10 . 14 . 18}{1 . 2 . 3 . 4 . 5} = 252.$$

II. De vnciis Trinomii $(1 + z + zz)^n$.

§. 4. Vtamur iterum eodem charactere $(\frac{n}{p})$ ad vncias huius poteſtatis trinomialis $(1 + z + zz)^n$ deſignandas; vnde probe cauendum erit, ne cum praecedente ſignificatu $[\frac{n}{p}]$ confundantur. Statuamus ſcilicet:

$$(1 + z + zz)^n = 1 + (\tfrac{n}{1}) z + (\tfrac{n}{2}) zz + (\tfrac{n}{3}) z^3 + \text{etc.}$$

ita vt hic quoque ſit $(\frac{n}{0}) = 1$; at quo ſignificatus ſequentium characterum facilius intelligatur, adiungamus ſimilem ſchematiſmum continentem poteſtates ſimpliciores:

Valores

Valores pro p

n	0	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12,
1	1+	1+	1										
2	1+	2+	3+	2+	1								
3	1+	3+	6+	7+	6+	3+	1						
4	1+	4+	10+	16+	19+	16+	10+	4+	1				
5	1+	5+	15+	30+	45+	51+	45+	30+	15+	5+	1		
6	1+	6+	21+	50+	90+	126+	141+	126+	90+	50+	21+	6+	1

Hinc igitur patet, si fuerit exempli gratia $n = 6$ et $p = 7$, fore $(\frac{6}{7}) = 126$; similique modo si $n = 5$ et $p = 5$, erit $(\frac{5}{5}) = 51$.

§. 5. Quoniam hae vnciae pariter ordinem retrogradum seruant, earumque vltima est $= 1$; euidens est fore $(\frac{1}{2}) = 1$; $(\frac{2}{4}) = 1$; $(\frac{3}{6}) = 1$; sicque porro, ita vt in genere sit $(\frac{n}{2n}) = 1$; ergo quia eodem ordine a fine regrediuntur, quo ab initio progrediuntur, erit

$$(\frac{n}{2n-1}) = (\frac{n}{1}) \text{ et } (\frac{n}{2n-2}) = (\frac{n}{2}),$$

atque in genere $(\frac{n}{2n-p}) = (\frac{n}{p})$. Deinde quia omnes vnciae tam primam antecedentes quam vltimam sequentes sunt nullae, valor characteris $(\frac{n}{p})$ primo euanescet, quoties p fuerit numerus negatiuus, deinde pariter euanescet, si fuerit $p > 2n$.

§. 6. Porro ex hac tabula perspicuum est semper esse $(\frac{n}{1}) = n$: quemadmodum autem sequentes se habeant, non tam facile perspicitur; at vero ostendi potest, pro qualibet potestate singulas vncias per binas praecedentes

ope

ope huius formulae determinari:

$$\left(\frac{n}{p+2}\right) = \frac{n-p-1}{p+2}\left(\frac{n}{p+1}\right) + \frac{2p-p}{p+2}\left(\frac{n}{p}\right).$$

Ita si fuerit verbi gratia $n = 5$ et $p = 2$, erit ex tabula modo data

$$\left(\frac{n}{p}\right) = \left(\frac{5}{2}\right) = 15;\ \left(\frac{n}{p+1}\right) = \left(\frac{5}{3}\right) = 30;\ \left(\frac{n}{p+2}\right) = \left(\frac{5}{4}\right) = 45;$$

quia porro est

$$\frac{n-p-1}{p+2} = \frac{2}{4} \text{ et } \frac{2n-p}{p+2} = \frac{8}{4} = 2,$$

vtique erit $45 = \frac{1}{2} \cdot 30 + 2 \cdot 15$. Simili modo si fuerit $n = 6$ et $p = 5$, erit

$$\left(\frac{n}{p}\right) = \left(\frac{6}{5}\right) = 126,\ \left(\frac{n}{p+1}\right) = \left(\frac{6}{6}\right) = 141, \text{ et } \left(\frac{n}{p+2}\right) = \left(\frac{6}{7}\right) = 126;$$

euidens autem est esse

$$126 = \frac{0}{7} \cdot 141 + \frac{7}{7} \cdot 126 = 126.$$

§. 7. Quod nunc porro ad vncias sequentis potestatis $(1 + z + zz)^{n+1}$ attinet, facile ex ipsa formatione apparet, quamlibet earum aequari aggregato ex tribus vnciis potestatis antecedentis, scilicet fore

$$\left(\frac{n+1}{p+2}\right) = \left(\frac{n}{p+2}\right) + \left(\frac{n}{p+1}\right) + \left(\frac{n}{p}\right);$$

vnde si loco $\left(\frac{n}{p+2}\right)$ valor ante inuentus substituatur, reperietur

$$\left(\frac{n+1}{p+2}\right) = \frac{n+1}{p+2}\left(\frac{n}{p+1}\right) + \frac{2n+2}{p+2}\left(\frac{n}{p}\right).$$

Hinc pro potestate septima, ad quam superior tabula non est extensa, singulae vnciae sequenti modo reperientur, statuendo $n = 6$

 Si $p =$

Si $p = -2$	$\left(\frac{7}{0}\right) = \frac{7}{0} \cdot \left(\frac{6}{-1}\right) + \frac{14}{0} \cdot \left(\frac{6}{-2}\right) = 1$ vti per se constat.
Si $p = -1$	$\left(\frac{7}{1}\right) = \frac{7}{1} \cdot \left(\frac{6}{0}\right) + \frac{14}{1} \cdot \left(\frac{6}{-1}\right) = 7$
Si $p = 0$	$\left(\frac{7}{2}\right) = \frac{7}{2} \cdot \left(\frac{6}{1}\right) + \frac{14}{2} \cdot \left(\frac{6}{0}\right) = 28$
Si $p = 1$	$\left(\frac{7}{3}\right) = \frac{7}{3} \cdot \left(\frac{6}{2}\right) + \frac{14}{3} \cdot \left(\frac{6}{1}\right) = 77$
Si $p = 2$	$\left(\frac{7}{4}\right) = \frac{7}{4} \cdot \left(\frac{6}{3}\right) + \frac{14}{4} \cdot \left(\frac{6}{2}\right) = 161$
Si $p = 3$	$\left(\frac{7}{5}\right) = \frac{7}{5} \cdot \left(\frac{6}{4}\right) + \frac{14}{5} \cdot \left(\frac{6}{3}\right) = 266$
Si $p = 4$	$\left(\frac{7}{6}\right) = \frac{7}{6} \cdot \left(\frac{6}{5}\right) + \frac{14}{6} \cdot \left(\frac{6}{4}\right) = 357$

id quod egregie conuenit cum naturali harum progressionum continuatione.

§. 8. Reliquae proprietates, quarum demonstratio simili modo expediri potest, quo pro Binomio est factum, respiciunt seriem, cuius terminus generalis ita exprimitur: $\left(\frac{m}{x}\right)\left(\frac{n}{p+x}\right)$, qui scilicet est productum ex duobus quibusuis characteribus huius generis, vnde singuli termini formantur, si loco x ordine scribantur valores 0, 1, 2, 3, 4, etc. vsque ad $x = 2m$, vel vsque ad $x = 2n - p$, quorum duorum terminorum minor valet. Huius autem seriei demonstrari potest summam esse $\int\left(\frac{m}{x}\right)\left(\frac{n}{p+x}\right) = \left(\frac{m+n}{2n-p}\right)$, quae summa etiam ita exprimitur: $\left(\frac{m+n}{2m+p}\right)$. Ita si sumamus $m = 3$, series ex factore $\left(\frac{m}{x}\right)$ orta erit

$$1 + 3 + 6 + 7 + 6 + 3 + 1.$$

Deinde si sumamus $n = 3$ et $p = 2$, alter factor $\left(\frac{3}{2+x}\right)$ dat hanc seriem: $6 + 7 + 6 + 3 + 1$, cuius termini in superiores sigillatim multiplicati praebent hanc progressionem:

$$6 + 21 + 36 + 21 + 6 = 90;$$

at

at vero formula $\left(\frac{m+n}{2n-p}\right)$ fit $= \left(\frac{8}{4}\right)$, cuius valor ex tabula ſupra allata reperietur fore $= 90$.

§. 9. Hinc igitur ſi ſumamus $p = 0$ prodibit iſta ſummatio: $\int\left(\frac{m}{x}\right)\left(\frac{n}{x}\right) = \left(\frac{m+n}{2m}\right)$, vel etiam $= \left(\frac{m+n}{2n}\right)$. Quare ſi porro capiamus $m = n$, erit $\int\left(\frac{n}{x}\right)^2 = \left(\frac{2n}{2n}\right)$, cuius veritatem per ſequentia exempla exploremus:

Si	erit
$n = 0$	$1^2 = 1 = \left(\frac{0}{0}\right) = 1$
$n = 1$	$1^2 + 1^2 + 1^2 = 3 = \left(\frac{2}{2}\right) = 3$
$n = 2$	$1^2 + 2^2 + 3^2 + 2^2 + 1^2 = 19 = \left(\frac{4}{4}\right) = 19$
$n = 3$	$1 + 3^2 + 6^2 + 7^2 + 6^2 + 3^2 + 1^2 = 141 = \left(\frac{6}{6}\right) = 141$.

Hoc igitur modo, quoniam tabula noſtra non vlterius eſt extenſa, valorem aſſignare poterimus characteris $\left(\frac{8}{8}\right)$, ſumto ſcilicet $n = 4$; erit enim:

$$\left(\frac{8}{8}\right) = 1^2 + 4^2 + 10^2 + 16^2 + 19^2 + 16^2 + 10^2 + 4^2 + 1^2$$

ſiue

$$\left(\frac{8}{8}\right) = 19^2 + 2\,(1^2 + 4^2 + 10^2 + 16^2) = 1107.$$

III. De vnciis quadrinomii $(1 + z + zz + z^3)^n$.

§. 10. Vtamur hic etiam eodem charactere $\left(\frac{n}{p}\right)$ ad ſingulas vncias huius poteſtatis euolutae exprimendas, ita vt ſit

$$(1 + z + zz + z^3)^n = 1 + \left(\frac{n}{1}\right)z + \left(\frac{n}{2}\right)zz + \left(\frac{n}{3}\right)z^3 \ldots \left(\frac{n}{p}\right)z^p$$

vbi iterum manifeſto eſt $\left(\frac{n}{0}\right) = 1$; at quo ſignificatus ſequentium characterum facilius intelligatur, adiungamus ſimilem ſchematismum continentem poteſtates ſimpliciores:

 Valores

Valores pro p

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1													
1	1+	1+	1+	1										
2	1+	2+	3+	4+	3+	2+	1							
3	1+	3+	6+	10+	12+	12+	10+	6+	3+	1				
4	1+	4+	10+	20+	31+	40+	44+	40+	31+	20+	10+	4+	1	
5	1+	5+	15+	35+	65+	101+	135+	155+	155+	135+	101+	65+	35+	15
6	1+	6+	21+	56+	120+	216+	336+	456+	546+	580+	546 etc.			

Hinc igitur patet, si fuerit verbi gratia $n = 6$ et $p = 7$, fore $(\frac{6}{7}) = 456$. Similique modo si fuerit $n = 5$ et $p = 10$ erit $(\frac{5}{10}) = 101$.

§. 11. Nunc ex ordine retrogrado pro vltimis cuiusque ordinis terminis erit $(\frac{1}{3}) = 1$; $(\frac{6}{2}) = 1$; $(\frac{3}{9}) = 1$, atque adeo in genere $(\frac{n}{3n}) = 1$. Ex eodem porro principio sequitur fore $(\frac{n}{3n-p}) = (\frac{n}{p})$; hincque intelligitur, valorem formulae $(\frac{n}{p})$ in nihilum esse abiturum, tam si fuerit $p < 0$ quam si fuerit $p > 3n$, quod quidem de numeris integris est intelligendum.

§. 12. Caeterum hic itidem euidens est esse $(\frac{n}{1}) = n$; pro sequentibus autem notandum est, quemlibet terminum per ternos praecedentes ita determinari vt sit

$$\left(\frac{n}{p+3}\right) = \frac{n-p-2}{p+3}\left(\frac{n}{p+2}\right) + \frac{2n-p-1}{p+3}\left(\frac{n}{p+1}\right) + \frac{3n-p}{p+3}\left(\frac{n}{p}\right).$$

Ita si fuerit $n = 5$ et $p = 2$, erit

$$\left(\frac{5}{5}\right) = \frac{1}{5}\left(\frac{5}{4}\right) + \frac{7}{5}\left(\frac{5}{3}\right) + \frac{13}{5}\left(\frac{5}{2}\right);$$

euidens autem est ob $(\frac{5}{2}) = 15$; $(\frac{5}{3}) = 35$; $(\frac{5}{4}) = 65$ et

$(\frac{5}{5})$

$(\frac{5}{3}) = 101$ fore

$$101 = \tfrac{1}{5}(75 + 7.35 + 13.15) = 101.$$

§. 13. Sequentis autem potestatis, cuius exponens $= n + 1$, singulae vnciae per superiores ita determinantur, vt sit

$$(\tfrac{n+1}{p+3}) = (\tfrac{n}{p+3}) + (\tfrac{n}{p+2}) + (\tfrac{n}{p+1}) + (\tfrac{n}{p}),$$

vbi si loco $(\frac{n}{p+3})$ valor ante inuentus substituatur prodibit;

$$(\tfrac{n+1}{p+3}) = \tfrac{n+1}{p+3}(\tfrac{n}{p+2}) + \tfrac{2n+2}{p+3}(\tfrac{n}{p+1}) + \tfrac{3n+3}{p+3}(\tfrac{n}{p}),$$

siue erit

$$(\tfrac{n+1}{p+3}) = \tfrac{n+1}{p+3}[(\tfrac{n}{p+2}) + 2(\tfrac{n}{p+1}) + 3(\tfrac{n}{p})];$$

hinc si loco $p + 3$ scribamus p fiet

$$(\tfrac{n+1}{p}) = \tfrac{n+1}{p}[(\tfrac{n}{p-1}) + 2(\tfrac{n}{p-2}) + 3(\tfrac{n}{p-3})].$$

Ita si sumamus $n = 5$ et $p = 6$ habebimus:

$$(\tfrac{6}{6}) = \tfrac{6}{6}[(\tfrac{5}{5}) + 2(\tfrac{5}{4}) + 3(\tfrac{5}{3})],$$

ideoque substitutis valoribus ex tabula superiore erit

$$(\tfrac{6}{6}) = 101 + 2.65 + 3.35 = 336,$$

est vero vtique $(\frac{6}{6}) = 336$.

§. 14. Reliquae proprietates redeunt ad summationem seriei, cuius terminus generalis est productum $(\frac{m}{x})(\frac{n}{p+x})$. Si enim loco x ordine scribantur numeri $0, 1, 2, 3, 4$, etc. donec ad terminos euanescentes perveniatur, erit $\int(\frac{m}{x})(\frac{n}{p+x}) = (\frac{m+n}{2 3 n - p})$, quae summa etiam est $= (\frac{m+n}{3m+p})$. Hinc si $p = 0$ orietur haec summatio:

$$(\tfrac{m}{0})(\tfrac{n}{0}) + (\tfrac{m}{1})(\tfrac{n}{1}) + (\tfrac{m}{2})(\tfrac{n}{2}) + (\tfrac{m}{3})(\tfrac{n}{3}) + \text{etc.} = (\tfrac{m+n}{3n})$$

 siue

siue etiam $= (\frac{m+n}{2m})$. Quare si insuper fuerit $m = n$, orietur haec summatio:

$$(\tfrac{n}{0})^2 + (\tfrac{n}{1})^2 + (\tfrac{n}{2})^2 + (\tfrac{n}{3})^2 + (\tfrac{n}{4})^2 + \text{etc.} = (\tfrac{2n}{n}).$$

Ita si fuerit $n = 1$ erit

$$1^2 + 1^2 + 1^2 + 1^2 = (\tfrac{2}{1}) = 4.$$

Deinde si $n = 2$ erit

$$1^2 + 2^2 + 3^2 + 4^2 + 3^2 + 2^2 + 1^2 = (\tfrac{4}{2}) = 44.$$

Porro si $n = 3$ erit

$$1^2 + 3^2 + 6^2 + 10^2 + 12^2 + 12^2 + 10^2 + 6^2 + 3^2 + 1 = (\tfrac{6}{3}) = 580.$$

IV. De vnciis Polynomii cuiuscunque $(1 + z + zz + z^3 \ldots\ldots + z^\lambda)^n$.

§. 16. Facta euolutione huius potestatis designet iste character $[\frac{p}{n}]$ coefficientem potestatis z^p, ita vt sit

$$(1 + z + zz + \ldots\ldots + z^\lambda)^n$$
$$= 1 + [\tfrac{n}{1}]z + [\tfrac{n}{2}]zz + [\tfrac{n}{3}]z^3 + [\tfrac{n}{4}]z^4 \ldots + [\tfrac{n}{p}]z^p$$

vnde quia vltimus terminus huius potestatis est $z^{\lambda n}$, erit $[\frac{n}{\lambda n}] = 1$; quamobrem manifestum est, si fuerit vel p numerus negatiuus vel numerus maior quam λn, perpetuo fore $[\frac{n}{p}] = 0$; deinde quia hi coefficientes pariter ordine retrogrado gaudent, erit $[\frac{n}{\lambda n - p}] = [\frac{n}{p}]$. Porro perspicuum est fore vt supra $[\frac{n}{0}] = 1$ et $[\frac{n}{1}] = n$. Caeterum in genere nullam formulam pro singulis potestatibus adiicere licet.

§. 16. Lex autem, qua termini huius seriei:

$$[\tfrac{n}{0}] + [\tfrac{n}{1}] + [\tfrac{n}{2}] + [\tfrac{n}{3}] + [\tfrac{n}{4}] + \text{etc.}$$

ex

ex praecedentibus determinantur, ſequenti formula includi poteſt:

$$\left[\frac{n}{p+\lambda}\right] = \frac{n+1-p-\lambda}{p+\lambda}\left[\frac{n}{p+\lambda-1}\right] + \frac{2(n+1)-p-\lambda}{p+\lambda}\left[\frac{n}{p+\lambda-2}\right]$$
$$+ \frac{3(n+1)-p-\lambda}{p+\lambda}\left[\frac{n}{p+\lambda-3}\right] \ldots\ldots + \frac{\lambda(n+1)-p-\lambda}{p+\lambda}\left[\frac{n}{p}\right],$$

quae expreſſio, ſi loco $p+\lambda$ ſcribamus ſimpliciter p, induet hanc formam:

$$\left[\frac{n}{p}\right] = \frac{n+1-p}{p}\left[\frac{n}{p-1}\right] + \frac{2(n+1)-p}{p}\left[\frac{n}{p-2}\right] + \frac{3(n+1)-p}{p}\left[\frac{n}{p-3}\right]$$
$$+ \frac{4(\pi+1)-p}{p}\left[\frac{n}{p-4}\right] + \ldots\ldots \frac{\lambda(n+1)-p}{p}\left[\frac{n}{p-\lambda}\right].$$

Huius formulae ope ſinguli ſeriei termini facile formari poterunt: quia enim nouimus eſſe $\left[\frac{n}{0}\right] = 1$, antecedentes vero omnes $= 0$, pro formatione ſingulorum terminorum habebimus:

$$\left[\frac{n}{1}\right] = \frac{n}{1}\left[\frac{n}{0}\right]$$
$$\left[\frac{n}{2}\right] = \frac{n-1}{2}\left[\frac{n}{1}\right] + \frac{2n}{2}\left[\frac{n}{0}\right]$$
$$\left[\frac{n}{3}\right] = \frac{n-2}{3}\left[\frac{n}{2}\right] + \frac{2n-1}{3}\left[\frac{n}{1}\right] + \frac{3n}{3}\left[\frac{n}{0}\right]$$
$$\left[\frac{n}{4}\right] = \frac{n-3}{4}\left[\frac{n}{3}\right] + \frac{2n-2}{4}\left[\frac{n}{2}\right] + \frac{3n-1}{4}\left[\frac{n}{1}\right] + \frac{4n}{4}\left[\frac{n}{0}\right]$$
$$\left[\frac{n}{5}\right] = \frac{n-4}{5}\left[\frac{n}{4}\right] + \frac{2n-3}{5}\left[\frac{n}{3}\right] + \frac{3n-2}{5}\left[\frac{n}{2}\right] + \frac{4n-1}{5}\left[\frac{n}{1}\right] + \frac{5n}{5}\left[\frac{n}{0}\right].$$

Hic autem probe obſeruandum eſt, has formulas non vltra λ terminos continuari debere, quandoquidem pro formula generali $\left[\frac{n}{p}\right]$ vltimum membrum vidimus eſſe

$$= \frac{\lambda(n+1)-p}{p}\left[\frac{n}{p-\lambda}\right].$$

§. 17. Quodſi porro hinc ad poteſtatem ſequentem, cuius exponens eſt $n+1$, progredi velimus, vbi poteſtatis z^p coefficiens eſt $\left[\frac{n+1}{p}\right]$ ex ipſa formatione harum poteſtatum manifeſtum eſt fore

$\left[\frac{n+1}{p}\right]$

$$\left[\frac{n+1}{p}\right] = \left[\frac{n}{p}\right] + \left[\frac{n}{p-1}\right] + \left[\frac{n}{p-2}\right] + \left[\frac{n}{p-3}\right] \ldots + \left[\frac{n}{p-\lambda}\right].$$

Quodsi iam hic loco $\left[\frac{n}{p}\right]$ scribamus valorem ante exhibitum, prodibit sequens aequatio:

$$\left[\frac{n+1}{p}\right] = \frac{n+1}{p}\left[\frac{n}{p+1}\right] + \frac{2(n+1)}{p}\left[\frac{n}{p-2}\right] + \frac{3(n+1)}{p}\left[\frac{n}{p-3}\right] \ldots + \frac{\lambda(n+1)}{p}\left[\frac{n}{p-\lambda}\right],$$

vbi terminorum numerus itidem est $= \lambda$.

§. 18. Quodsi iam hinc pariter seriem formemus, cuius terminus generalis sit productum $\left[\frac{m}{x}\right]\left[\frac{n}{p+x}\right]$, cuius ergo ipsi termini singuli reperiuntur, si loco x ordine scribantur numeri 0, 1, 2, 3, 4, etc. donec perueniatur ad terminos euanescentes, id quod eueniet, quando x vel vltra λm vel vltra $\lambda n - p$ augetur; tum totius seriei summa erit $= \left[\frac{m+n}{\lambda n - p}\right]$, vel etiam $\left[\frac{m+n}{\lambda m + p}\right]$, quae summatio ita repraesentari poterit:

$$\int \left[\frac{m}{x}\right]\left[\frac{n+1}{p+x}\right] = \left[\frac{m+n}{\lambda n - p}\right] = \left[\frac{m+n}{\lambda m + p}\right],$$

ipsa autem progressio his constabit terminis:

$$\left[\frac{m}{0}\right]\left[\frac{n}{p}\right] + \left[\frac{m}{1}\right]\left[\frac{n}{p+1}\right] + \left[\frac{m}{2}\right]\left[\frac{u}{p+2}\right] + \left[\frac{m}{3}\right]\left[\frac{n}{p+3}\right] + \text{etc.}$$

Quodsi ergo fuerit $p = 0$ et $m = n$, singuli seriei termini fiunt quadrata, scilicet habebitur ista series:

$$\left[\frac{n}{0}\right]^2 + \left[\frac{n}{1}\right]^2 + \left[\frac{n}{2}\right]^2 + \left[\frac{n}{3}\right]^2 + \left[\frac{n}{4}\right]^2 + \text{etc.}$$

cuius summa erit $= \left[\frac{2n}{\lambda n}\right]$. Hoc scilicet casu ambae formulae in vnam coalescunt.

§. 19. Hoc igitur modo insignes illae proprietates, quas non ita pridem super vnciis Binomii demonstraui, extendi possunt tam ad trinomia et quadrinomia quam ad Polynomia cuiuscunque ordinis; et quemadmodum easdem

dem has proprietates pro Binomio ostendi etiam locum habere, quando exponentes m et n sunt vel numeri negatiui, vel adeo numeri fracti, eadem proprietas pro Polynomiis quibusque locum habere deprehenditur; quod quidem non tam facile perspicitur, propterea quod euolutio potestatis

$$(1 + z + z^2 + z^3 \ldots \ldots + z^\lambda)^n,$$

quando exponens n est vel numerus negatiuus vel fractus, multo magis euadit perplexa, neque etiam interpolationem admittit, quemadmodum id pro casu Binomii praestare licuit, vbi si $\left[\frac{n}{p}\right]$ designet vnciam potestatis z^p, quae ex euolutione dignitatis $(1 + z)^n$ oritur, ostendi si breuitatis gratia statuatur $l\frac{1}{x} = u$, semper fore

$$\left[\frac{n}{p}\right] = \frac{\int u^n dx}{\int u^p dx \int u^{n-p} dx},$$

si scilicet haec integralia ab $x = 0$ vsque ad $x = 1$ extendantur. Pro interpolatione enim monstraui esse

$$\int\frac{dx}{\sqrt{u}} = \sqrt{\pi} \text{ et } \int dx \sqrt{u} = \tfrac{1}{2}\sqrt{\pi},$$

denotante scilicet π peripheriam circuli, cuius diameter $= 1$.