METHODVS FACILIS OMNIA SYMPTOMATA LINEARVM CVRVARVM NON IN EODEM PLANO SITARVM INVESTIGANDI.

Dissertatio altera.

§. 1.

Quod si forte methodus ante tradita et doctrinae sphaericae innixa minus placeat, vtpote ex principiis alienis deducta; aliam hic sum propositurus methodum, ex principio magis directo petitam, qua omnia symptomata linearum curuarum non in eodem plano sitarum etiam multo succinctius et clarius demonstrari possunt quam methodo hactenus vsitata. Hanc autem methodum mihi suppeditauit consideratio illius plani, in quo bina curuae elementa contigua sunt sita, et in quo proinde tam tangens curuae quam eius radius curuaturae existere debent; vnde totam inuestigationem ex consideratione huius plani sum petiturus.

Tab. II. Fig. 4. §. 2. Constitutis igitur vt ante ternis axibus principalibus OA, OB, OC, quibus coordinatae, quoduis punctum curuae z determinantes, sint parallelae, scilicet

$$O\,x = x,\; x\,y = y,\; y\,z = z,$$

statim in calculum introducam illud planum, quod per bina curuae elementa contigua determinatur, siue in quo curua hoc loco incuruatur. Secet igitur istud planum ternos nostros axes in punctis u, v et w, ac ponamus distantias $O\,u = u$, $O\,v = v$, $O\,w = w$, quandoquidem his tribus quantitatibus positio plani penitus determinatur. Quatenus igitur curuae punctum z in hoc ipso plano reperitur, aequatio inter ternas coordinatus x, y, z erit vnius dimensionis, huius formae, $A\,x + B\,y + C\,z = D$, ad quam penitus determinandam transferamus primo punctum z in ipsum punctum u fietque $x = u$, $y = 0$ et $z = 0$, vnde erit $A\,u = D$. Simili modo, translato puncto z in ipsum punctum v, fiet $x = 0$, $y = v$, $z = 0$, vnde erit $B\,v = D$. Denique translato puncto z in ipsum punctum w, erit $z = w$, $x = 0$, $y = 0$, vnde erit $C\,w = D$. Cum igitur hinc fit

$$A = \frac{D}{u},\; B = \frac{D}{v},\; C = \frac{D}{w},$$

erit aequatio localis pro plano $u\,v\,w$ haec:

$$\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 1.$$

§. 3. Quod si nunc vt ante elementum curuae statuatur $= d\,s$, ac per hoc elementum reliqua ita definiantur, vt sit

$$d\,x = p\,d\,s,\; d\,y = q\,d\,s,\; d\,z = r\,d\,s,$$

erit vtique $p\,p + q\,q + r\,r = 1$, ac differentiando

$p\,d\,p$

$$p\,dp + q\,dq + r\,dr = 0.$$

Transferatur nunc punctum curuae z in proximum, cui respondeant coordinatae $x + dx$, $y + dy$, $z + dz$, quoniam istud punctum proximum denuo in plano $u\,v\,w$ est situm, erit etiam

$$\frac{x + dx}{u} + \frac{y + dy}{v} + \frac{z + dz}{w} = 1,$$

a qua aequatione si prior subtrahatur, habebimus:

$$\frac{dx}{u} + \frac{dy}{v} + \frac{dz}{w} = 0, \text{ ideoque } \frac{p}{u} + \frac{q}{v} + \frac{r}{w} = 0.$$

Denique quia etiam sequens elementum in idem planum incidere debet, erit etiam denuo differentiando

$$\frac{dp}{u} + \frac{dq}{v} + \frac{dr}{w} = 0.$$

Vlterius vero differentiando progredi non licet, quia sequentia elementa vtique in aliud planum incidere possunt.

§. 4. Consideratio igitur istius plani $u\,v\,w$ cum indole curuae coniuncta nobis suppeditauit has tres aequationes:

$$\text{I. } \frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 1.$$

$$\text{II. } \frac{p}{u} + \frac{q}{v} + \frac{r}{w} = 0.$$

$$\text{III. } \frac{dp}{u} + \frac{dq}{v} + \frac{dr}{w} = 0.$$

ex quibus aequationibus vicissim ternas quantitates u, v et w per coordinatas curuae earumque differentialia determinare poterimus. Hunc in finem adhibeamus hanc combinationem: II. dr — III. r, quae nobis praebebit

$$\frac{p\,dr - r\,dp}{u} + \frac{q\,dr - r\,dq}{v} = 0, \text{ siue}$$

$$\frac{p\,dr - r\,dp}{u} = \frac{r\,dq - q\,dr}{v},$$

vnde

vnde deducitur haec proportio:

$$\frac{1}{u} : \frac{1}{v} = (r\,d\,q - q\,d\,r) : (p\,d\,r - r\,d\,p).$$

Statuamus igitur

$$\frac{1}{u} = \frac{r\,d\,q - q\,d\,r}{t}, \text{ erit } \frac{1}{v} = \frac{p\,d\,r - r\,d\,p}{t},$$

qui valores in ſecunda aequatione ſubſtituti dant

$$r\,\frac{(p\,d\,q - q\,d\,p)}{t} + \frac{r}{w} = 0, \text{ vnde colligimus}$$

$$\frac{1}{w} = \frac{q\,d\,p - p\,d\,q}{t}.$$

§. 5. Ex his autem valoribus pro $\frac{1}{u}$, $\frac{1}{v}$, $\frac{1}{w}$ inuentis, ſi ii in prima aequatione ſubſtituantur, perueniemus ad hanc aequationem:

$$x(r\,d\,q - q\,d\,r) + y(p\,d\,r - r\,d\,p) + z(q\,d\,p - p\,d\,q) = t,$$

quae nobis igitur valorem quantitatis t exhibet, ita vt deinceps hac littera t tanquam cognita vti queamus, vbi quidem neceſſe erit valores formularum $\frac{1}{u}$, $\frac{1}{v}$, $\frac{1}{w}$ in calculum introducere.

Tab. II. Fig. 5. §. 6. Nunc igitur determinata poſitione plani $u\,v\,w$ inueſtigemus eius inclinationem ad terna noſtra plana principalia A O B, B O C, C O A. Ac primo quidem quia iſtud planum ſecat planum A O B per lineam $u\,v$, quae eſt $= \surd(u\,u + v\,v)$, ex O ducatur in hanc $u\,v$ perpendiculum O r, eritque $u\,r$: O u = Ov : O r, vnde fit O $r = \frac{u\,v}{\surd(u\,u + v\,v)}$. Iungatur nunc recta $r\,w$, et quia triangulum O $w\,r$ eſt verticale, recta $w\,r$ quoque normalis erit ad rectam $u\,v$, vnde angulus O $r\,w$ dabit inclinationem plani $u\,v\,w$ ad planum A O B; quare cum angulus wOr ſit rectus et O $w = w$, erit hypothenuſa

$r\,w$

$$r w = \sqrt{\left(\frac{u u v v + u u w w + v v w w}{u u + v v}\right)},$$

ex qua colligimus

$$\text{cof. } O r w = \frac{O r}{r w} = \frac{u v}{\sqrt{(u u v v + u u w w + v v w w)}}, \text{ fiue}$$

$$\text{cof. } O r w = \frac{1}{\sqrt{\left(1 + \frac{w w}{u u} + \frac{w w}{v v}\right)}} = \frac{1}{w \sqrt{\left(\frac{1}{u u} + \frac{1}{v v} + \frac{1}{w w}\right)}}.$$

Sicque innotefcit inclinatio plani $u v w$ ad planum A O B, quippe cuius cofinus eft $\frac{1}{w \sqrt{\left(\frac{1}{u u} + \frac{1}{v v} + \frac{1}{w w}\right)}}$. Simili modo per analogiam concluditur inclinatio plani $u v w$ ad planum B O C, cuius cofinus eft $\frac{1}{u \sqrt{\left(\frac{1}{u u} + \frac{1}{v v} + \frac{1}{w w}\right)}}$; ac denique inclinatio plani $u v w$ ad planum C O A, cuius cofinus eft $\frac{1}{v \sqrt{\left(\frac{1}{u u} + \frac{1}{v v} + \frac{1}{w w}\right)}}$.

§. 7. Cum igitur fit

$$\frac{1}{u} = \frac{r d q - q d r}{t},$$

$$\frac{1}{v} = \frac{p d r - r d p}{t} \text{ et}$$

$$\frac{1}{w} = \frac{q d p - p d q}{t}, \text{ erit}$$

$$\frac{1}{u u} + \frac{1}{v v} + \frac{1}{w w} = \frac{d p^2 (q q + r r) + d q^2 (p p + r r) + d r^2 (p p + q q)}{t t}$$
$$- \frac{2 p q d p d q - 2 p r d p d r - 2 q r d q d r}{t t},$$

quia vero eft

$$q q + r r = 1 - p p; \; p p + r r = 1 - q q \text{ et}$$
$$p p + q q = 1 - r r,$$

his valoribus substitutis formula ista in duas partes distribuetur, alteram positiuam, alteram negatiuam; ac positiua quidem erit $\frac{dp^2 + dq^2 + dr^2}{tt}$, termini vero negatiui erunt

$$\frac{-ppdp^2 - qqdq^2 - rrdr^2 - 2pqdpdq - 2prdpdr - 2qrdqdr}{tt},$$

quae forma cum manifesto sit quadratum huius formulae: $\frac{pdp + qdq + rdr}{t}$, ea ob

$$pdp + qdq + rdr = 0,$$

sponte euanescit, ita vt habeamus:

$$\sqrt{\left(\frac{1}{uu} + \frac{1}{vv} + \frac{1}{ww}\right)} = \sqrt{\left(\frac{dp^2 + dq^2 + dr^2}{t}\right)},$$

quamobrem inclinationes supra inuentae ita succincte exprimentur:

I. cos. incl. ad planum A O B $= \frac{qdp - pdq}{\sqrt{(dp^2 + dq^2 + dr^2)}}$.

II. cos. incl. ad planum B O C $= \frac{rdq - qdr}{\sqrt{(dp^2 + dq^2 + dr^2)}}$.

III. cos. incl. ad planam C O A $= \frac{pdr - rdp}{\sqrt{(dp^2 + dq^2 + dr^2)}}$.

Sicque iam praecipua symptomata sumus nacti, quae praecedente methodo demum post plures ambages sunt eruta.

§. 8. Quo nunc etiam reliqua symptomata facilius eliciamus, ponamus breuitatis gratia angulum $Ouv = \alpha$ atque habebimus

$$\text{sin.}\,\alpha = \frac{v}{\sqrt{(uu + vv)}},\ \text{cos.}\,\alpha = \frac{u}{\sqrt{(uu + vv)}}\ \text{et tang.}\,\alpha = \frac{v}{u}.$$

Deinde ponatur inclinatio seu angulus $Orw = \theta$, eritque

$$\text{sin.}\,\theta = \frac{Ow}{rw} = \frac{w\sqrt{(uu + vv)}}{\sqrt{(uuvv + uuww + vvww)}},$$

$$\text{cos.}\,\theta = \frac{Or}{rw} = \frac{uv}{\sqrt{(uuvv + uuww + vvww)}}\ \text{et}$$

$$\text{tang.}\,\theta = \frac{w\sqrt{(uu + vv)}}{uv},$$

vbi

vbi notandum eſt hos valores manere conſtantes, dum a puncto curuae z per duo elementa ſequentia progredimur.

§. 9. His praemiſſis inueſtigemus locum puncti z in ipſo plano $u\,v\,w$, et cum ſit $O\,x = x$, $x\,y = y$, $y\,z = z$, ex punctis x et y ad rectam $u\,v$ ducamus perpendicula $x\,p$ et $y\,q$, et quoniam $x\,u = u - x$ et angulus $x\,u\,p = \alpha$, erit $x\,p = (u - x)\,\text{ſin.}\,\alpha$ et $u\,p = (u - x)\,\text{cof.}\,\alpha$. Tum vero ex y in $x\,p$ demiſſo perpendiculo $y\,o$, erit $y\,o = y\,\text{ſin.}\,\alpha$ et $x\,o = y\,\text{cof.}\,\alpha$. Quare cum ſit $p\,q = y\,o$ et $y\,q = p\,o$, fiet interuallum Tab II Fig. 4.

$$u\,q = (u - x)\,\text{cof.}\,\alpha + y\,\text{ſin.}\,\alpha \text{ et}$$
$$y\,q = (u - x)\,\text{ſin.}\,\alpha - y\,\text{cof.}\,\alpha.$$

Nunc igitur quia $y\,z = z$ eſt verticalis, ducta recta $q\,z$ triangulum $y\,z\,q$ non ſolum plano A O B normaliter inſiſtit, ſed etiam rectae $y\,q$ et $z\,q$ ad rectam $u\,v$ erunt perpendiculares, et angulus $y\,q\,z$ ipſi inclinationi θ aequabitur, vnde erit recta

$$q\,z = \frac{(u - x)\,\text{ſin.}\,\alpha - y\,\text{cof.}\,\alpha}{\text{cof.}\,\theta},$$

vel etiam erit

$$q\,z = \frac{y\,z}{\text{ſin.}\,\theta} = \frac{z}{\text{ſin.}\,\theta},$$

vnde has duas formulas aequales eſſe neceſſe eſt, ex quo ſequitur fore

$$\frac{z\,\sqrt{(uu\,vv + uu\,ww + vv\,ww)}}{w\,\sqrt{(u\,u + v\,v)}} = \frac{(u - x)\,\text{ſin.}\,\alpha - y\,\text{cof.}\,\alpha)\,\sqrt{(uu\,vv + u\,u\,ww + vv\,ww)}}{u\,v},$$

ſiue

$$\frac{z}{w\,\sqrt{(u\,u + v\,v)}} = \frac{(u - x)\,v - u\,y}{u\,v\,\sqrt{(u\,u + v\,v)}},$$

quae aequatio reducitur ad hanc: $\frac{z}{w} = \frac{(u - x)}{u} - \frac{y}{v}$, quae eſt

ipſa

ipſa aequatio primo conſtituta, ſcilicet

$$\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 1.$$

§. 10. Quoniam igitur recta zq in ipſo plano uvw ad rectam uv eſt normalis, in hoc plano pro puncto z ſpectemus rectam uq tanquam abſciſſam et qz tanquam applicatam, et vocemus $uq = X$ et $qz = Y$, quocirca habebimus

$$X = (u - x)\operatorname{cof.}\alpha + y \operatorname{fin.}\alpha \text{ et}$$

$$Y = \frac{(u-x)\operatorname{fin.}\alpha - y\operatorname{cof.}\alpha}{\operatorname{cof.}\theta} = \frac{z}{\operatorname{cof.}\theta}.$$

Quae quo clarius perſpici queant, hoc planum uvw cum puncto z ſeorſim in tabulam coniiciamus.

Tab. II. Fig. 6. §. 11. Nunc igitur differentiando per elementum curuae zz' progrodiamur, atque ob

$$dx = p\,ds,\ dy = q\,ds \text{ et } dz = r\,ds,$$

quoniam quantitates u, v, w, cum angulis α et θ manent conſtantes, erit

$$dX = -p\,ds\operatorname{cof.}\alpha + q\,ds\operatorname{fin.}\alpha \text{ et}$$

$$dY = -\frac{p\,ds\operatorname{fin.}\alpha - q\,ds\operatorname{cof.}\alpha}{\operatorname{cof.}\theta} = \frac{r\,ds}{\operatorname{fin.}\theta},$$

quos duos valores ipſius dY pariter aequales eſſe neceſſe eſt, ideoque erit

$$-\frac{p\operatorname{fin.}\alpha - q\operatorname{cof.}\alpha}{\operatorname{cof.}\theta} = \frac{r}{\operatorname{fin.}\theta},$$

quae aequatio, ſubſtitutis valoribus, tranſit in hanc:

$$\frac{p}{u} + \frac{q}{v} + \frac{r}{w} = 0,$$

quae eſt ipſa aequatio ſecunda initio conſtituta. Praeterea, cum poſuerimus elementum curuae $= ds$, quod nunc ex ex coordinatis X et Y fit $ds = \surd(dX^2 + dY^2)$, necesse est vt euadat $dX^2 + dY^2 = ds^2$. Est vero

$$dX = \frac{ds(qv - pu)}{\surd(uu + uv)} \text{ et}$$

$$dY = -\frac{ds(pv + qu)\surd(uuvv + uuww + vvww)}{uv\surd(uu + vv}\,,$$

vel etiam

$$dY = \frac{r\,ds\surd(uuvv + uuww + vvww)}{w(\surd uu + vv)}.$$

Hinc autem erit

$$\frac{dX^2 + dY^2}{ds^2} = \frac{uuvv(qqvv - 2pquv + ppuu) + (ppvv + 2pquv) + qquu(qquu + uuww + vvww)}{uuvv(uu + vv)}\,,$$

quae expressio reducitur ad hanc:

$$\frac{dX^2 + dY^2}{ds^2} = \frac{pp(uu + ww)}{uu} + \frac{qq(vv + ww)}{vv} + \frac{2pqww}{uv}$$

$$= (pp + qq) + ww\left(\frac{p}{u} + \frac{q}{v}\right)^2.$$

Cum autem sit $\frac{p}{u} + \frac{q}{v} = -\frac{r}{w}$, resultabit haec forma:

$$\frac{dX^2 + dY^2}{ds^2} = pp + qq + rr = 1.$$

Sicque patet, reuera esse $dX^2 + dY^2 = ds^2$.

§. 12. Quod si iam statuamus $dY = P\,dX$, notum est radium osculi curuae, quatenus cadet in rectam $zN\Omega$, esse $z\Omega = -\frac{dX(1 + PP)^{\frac{3}{2}}}{dP}$, ita vt Ω sit centrum circuli nostram curuam in z osculantis. Modo autem vidimus reuera fieri $dX\surd(1 + PP) = ds$, vnde cum sit $dX = \frac{ds(qv - pu)}{\surd uu + vv}$, erit

$$\surd(1 + PP) = \frac{ds}{dX} = \frac{\surd(uu + vv)}{qv - pu}, \text{ siue}$$

$$\frac{1}{\surd(1 + PP)} = \frac{qv - pu}{\surd(uu + vv)}\,,$$

vnde

vnde differentiando colligimus:

$$-\frac{P\,dP}{(1+PP)^{\frac{3}{2}}} = \frac{v\,dq - u\,dp}{\sqrt{(uu+vv)}},$$

vnde deducitur

$$(1+PP)^{\frac{3}{2}} = \frac{P\,dP\sqrt{(uu+vv)}}{u\,dp - v\,dq},$$

atque hinc expressio pro radio osculi erit

$$z\Omega = \frac{P\,dX\sqrt{(uu+vv)}}{v\,dq - u\,dp} - \frac{dY\sqrt{(uu+vv)}}{v\,dq - u\,dp},$$

ac loco dY substituto valore obtinebitur tandem radius osculi

$$z\Omega = \frac{ds\,(pv+qu)\sqrt{(uuvv+uuww+vvww)}}{uv\,(u\,dp - v\,dq)}.$$

§. 13. Quo nunc hanc expressionem radii osculi ab elementis u, v et w liberemus, diuidamus numeratorem et denominatorem seorsim per uvw, et obtinebimus numeratorem

$$= ds\,(pv+qu)\sqrt{\left(\tfrac{1}{uu}+\tfrac{1}{vv}+\tfrac{1}{ww}\right)},$$

denominator vero erit $\frac{u\,dp - v\,dq}{w}$. Supra autem iam ostendimus esse

$$\sqrt{\left(\tfrac{1}{uu}+\tfrac{1}{vv}+\tfrac{1}{ww}\right)} = \frac{\sqrt{(dp^2+dq^2+dr^2)}}{t},$$

vnde fit numerator

$$= ds\left(\tfrac{pv}{t}+\tfrac{qu}{t}\right)\sqrt{(dp^2+dq^2+dr^2)}.$$

Cum nunc fit

$$\frac{v}{t} = \frac{1}{-p\,dr - r\,dp} \text{ et } \frac{u}{t} = \frac{1}{r\,dq - q\,dr},$$

erit noster numerator

$$ds\left(\frac{p}{p\,dr - r\,dp} + \frac{q}{r\,dq - q\,dr}\right)\sqrt{(dp^2 + dq^2 + dr^2)}$$
$$= \frac{r\,ds\,(p\,dq - q\,dp)}{(p\,dr - r\,dp)\,(r\,dq - q\,dr)}\sqrt{(dp^2 + dq^2 + dr^2)};$$

denominator vero, ob

$$\frac{u}{w} = \frac{q\,dp - p\,dq}{r\,dq - q\,dr} \text{ et } \frac{v}{w} = \frac{q\,dp - p\,dq}{p\,dr - r\,dp},$$

induet hanc formam:

$$(q\,dp - p\,dq)\left(\frac{dp}{r\,dq - q\,dr} - \frac{dq}{p\,dr - r\,dp}\right) =$$
$$(q\,dp - p\,dq)\,((p\,dp + q\,dq)\,dr - (dp^2 + dq^2)\,r).$$

Cum autem ſit

$$p\,dp + q\,dq = -r\,dr,$$

erit iſte denominator:

$$-\frac{r\,(q\,dp - p\,dq)\,(dp^2 + dq^2 + dr^2)}{(r\,dq - q\,dr)\,(p\,dr - r\,dp)}.$$

§. 14. Cum igitur euoluerimus tam numeratorem quam denominatorem, inde colligetur radius oſculi

$$z\,\Omega = \frac{ds}{\sqrt{(dp^2 + dq^2 + dr^2)}},$$

qui valor perfecte congruit cum eo qui ſuperiore methodo erat inuentus. Supereſt igitur vt eius quoque poſitionem reſpectu ternorum axium principalium definiamus. Hic autem vidimus eum cadere in normalem $z\,N$ productam, ſiquidem eius expreſſio fuerit poſitiua. Eſt vero ſubnormalis $q\,N = \frac{Y\,dY}{dX} = P\,Y$, atque ſi inſuper ducamus tangentem $z\,T$, erit ſubtangens $q\,T = \frac{Y\,dX}{dY} = \frac{Y}{P}$. At vero poſitionem huius tangentis $z\,T$ in praecedente diſſertatione tam ſimpliciter determinauimus, vt angulorum, ſub quibus ea ad axes principales $O\,A$, $O\,B$ et $O\,C$ inclinatur, coſinus ſint p, q et r.

§. 15.

§. 15. Confideremus primo tangentem $z\,T$, pro cuius pofitione refpectu rectae $u\,v$ habemus tangentem anguli $q\,T\,z = \frac{q\,z}{q\,T} = P = \frac{d\,Y}{d\,X}$; tum vero quia elementum curuae eft $d\,s$, fi ponamus breuitatis ergo angulum $q\,T\,z = \pi$, erit

$$\text{fin.}\ \tau = \frac{d\,Y}{d\,s} = -\frac{(p\,v + q\,u)\,\sqrt{(u\,u\,v\,v + u\,u\,w\,w + v\,v\,w\,w}}{u\,v\,\sqrt{(u\,u + v\,v)}} \text{ et}$$

$$\text{cof.}\ \tau = \frac{d\,X}{d\,s} = \frac{q\,v - p\,u}{\sqrt{(u\,u + v\,v)}},$$

vnde fit

$$\text{tang.}\ \tau = -\frac{(p\,v + q\,u)\,\sqrt{(u\,u\,v\,v + u\,u\,w\,w + v\,v\,w\,w)}}{u\,v\,(q\,v - p\,u)}.$$

Porro vero erit fubtangens $q\,T = \frac{Y}{\text{tang.}\,\tau}$, eft vero

$$Y = \frac{(v\,(u - x) - y\,u)\,\sqrt{(u\,u\,v\,v + u\,u\,w\,w + v\,v\,w\,w)}}{u\,v\,\sqrt{(u\,u + v\,v)}},$$

vnde fit fubtangens

$$q\,T = \frac{(v\,(u - x) - u\,y)\,(p\,u - q\,v)}{(p\,v + q\,u)\,\sqrt{(u\,u + v\,v)}},$$

quae fubtracta ab afciffa X relinquet fpatium

$$u\,T = \frac{p\,y + q\,(u - x)\,\sqrt{(u\,u + v\,v)}}{p\,v + q\,u}.$$

Denique fiat $d\,X : d\,s = q\,T$ ad $z\,T$,

hincque prodibit ipfa

$$\text{tang.}\ z\,T = -\frac{(v\,(u - x) - u\,y)}{p\,v + q\,u} = \frac{u\,y - v\,(u - x)}{p\,v + q\,u}.$$

§. 16. Eodem modo definiamus pofitionem normalis $z\,N$, ac primo quidem erit angulus $q\,N\,z = 90^\circ - \tau$; tum vero fubnormalis

$$q\,N = Y\,P = Y\,\text{tang.}\ \tau = \frac{(v\,(u - x) - u\,y)\,(p\,v + q\,u)\,(u\,u\,v\,v + u\,u\,w\,w + v\,v\,w\,w)}{u\,u\,v\,v\,(p\,u - q\,v)\,\sqrt{(u\,u + v\,v)}},$$

ipfa autem normalis erit

$z\,N$

$$z\,N = \frac{Y}{cos.\,\tau} = \frac{(v(u-x) - uy)\sqrt{(uuvv + uuww + vvww)}}{uv(qv - pu)}.$$

Tandem si subnormalem abscissae X addamus, prodibit interuallum

$$U\,N = \frac{(u-x)(pvv(uu+ww) + quvww)\sqrt{(uu+vv)}}{uuvv(pu-qv)} - \frac{y(quu(vv+ww) + puvww)\sqrt{(uu+vv)}}{uuvv(pu-qv)}.$$

§. 17. Transferamus nunc iterum hanc figuram in planum inclinatum $u\,v\,w$, et cum sit recta $z\,y$ ad planum A O B perpendicularis, ducta recta $y\,T$ erit triangulum $z\,y\,T$ ad y rectangulum, vnde anguli $y\,z\,T$ cosinus erit $\frac{z\,y}{z\,T}$. Hic autem angulus aequalis est illi, sub quo tang. $z\,T$ ad axem O C inclinatur, quia $z\,y$ ipsi O C est parallela. Cum igitur sit $z\,y = z$ et $z\,T = \frac{uy - v(u-x)}{pv + qu}$, at vero ex prima aequatione: Tab. II. Fig. 7.

$$\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = , \text{ fiat}$$

$$\frac{z}{w} = \frac{u-x}{u} - \frac{y}{v} = \frac{v(u-x) - uy}{uv}, \text{ erit}$$

$$z\,T = \frac{uvz}{w(pv+qu)};$$

tum vero ex secunda aequatione:

$$\frac{p}{u} + \frac{q}{v} + \frac{r}{w} = 0, \text{ erit}$$

$$\frac{r}{w} = -\frac{(pv+qu)}{uv},$$

quo valore substituto fit $z\,T = -\frac{z}{r}$, quocirca habebimus $cos.\,y\,z\,T = -r$. Supra autem vidimus cosinum anguli, quo tangens curuae ad axem O C inclinatur, esse $= +r$; verum notandum est ibi tangentem sursum fuisse productam, cum hic deorsum dirigatur, vnde signum cosinus mutatur.

 §. 18.

§. 18 Quoniam autem hic tantum de inclinatione agitur, quam tangens curuae tenet respectu trium axium, loco ipsius tangentis $z\,T$ aliam quamcunque ipsi parallelam substituere licebit, quippe quae ad axes nostros pariter erit inclinata. Hanc ob rem in plano inclinato Tab II. Fig. 8. $u\,v\,w$ ex u ducatur recta $u\,t$, tangenti curuae parallela, ita vt sit angulus $v\,u\,t = \tau$. Capiatur autem haec recta $u\,t$, calculi gratia, $= 1$, ita vt, si ex t in $u\,v$ demittatur perpendiculum $t\,d$, fit $u\,d = \text{cos.}\,\tau$ et $t\,d = \text{sin.}\,\tau$. Tum vero in ipso plano A O B ex d ad rectam $u\,v$ ducatur normalis $d\,e$, in eamque ex t demittatur perpendiculum $t\,e$, et quia angulus $t\,d\,e$ est inclinatio plani $u\,v\,w$ ad planum A O B, erit iste angulus $t\,d\,e = \theta$, vnde fit interuallum $d\,e = t\,d\,\text{cos.}\,\theta = \text{sin.}\,\tau\,\text{cos.}\,\theta$ et perpendiculum in normalem $t\,e = \text{sin.}\,\tau\,\text{sin.}\,\theta$; vbi notetur rectam $t\,e$ axi O C esse parallelam. Hinc si ducatur recta $u\,e$ triangulum $t\,e\,u$ ad e erit rectangulum, vnde anguli $u\,t\,e$ cosinus erit

$$\text{cos.}\,u\,t\,e = \frac{t\,e}{u\,t} = \text{sin.}\,\theta\,\text{sin.}\,\tau.$$

Supra autem vidimus esse

$$\text{sin.}\,\theta = \frac{w\,\surd(u\,u + v\,v)}{\surd(u\,u\,v\,v + u\,u\,w\,w + v\,v\,w\,w)};$$

tum vero erat

$$\text{sin.}\,\tau = -\frac{(p\,v + q\,u)\,\surd(u\,u\,v\,v + u\,u\,w\,w + v\,v\,w\,w)}{u\,v\,\surd(u\,u + v\,v)},$$

quae expressio ob

$$p\,v + q\,u = -r\,.\,\frac{u\,v}{w},$$

transformatur in hanc:

$$\text{sin.}\,\tau = \frac{r\,\surd(u\,u\,v\,v + u\,u\,w\,w + v\,v\,w\,w)}{w\,\surd(u\,u + v\,v)},$$

vnde fit

sin.

$$\sin.\theta \sin.\tau = r = \cos.\, u\,t\,e,$$

hoc eſt coſinus anguli, ſub quo tangens furſum ducta ad axem O C inclinatur $= r$, (Cf. §. 19.); recta autem $u\,e$ ſinum eiusdem anguli exhibebit, ita vt ſit

$$u\,e = \sqrt{(1 - r\,r)} = \sqrt{(p\,p + q\,q)},$$

quae etiam definiri poteſt ex triangulo $e\,d\,u$, in quo eſt

$$u\,d = \cos.\tau \text{ et } d\,e = \sin.\tau \cos.\theta,$$

vnde fit

$$u\,e = \sqrt{(1 - \sin.\theta^2 \sin.\tau^2)}.$$

Hinc autem deducitur tangens anguli $d\,u\,e$, ſcil.

$$\text{tang.}\, d\,u\,e = \frac{d\,e}{u\,d} = \cos.\theta \,\text{tang.}\,\tau,$$

ad quam euoluendam ponamus breuitatis gratia:

$$\sqrt{(u\,u\,v\,v + u\,u\,w\,w + v\,v\,w\,w)} = \Theta, \text{ eritque}$$

$$\cos.\theta = \frac{u\,v}{\Theta} \text{ et tang.}\,\tau = \frac{\Theta\,r}{w\,(q\,v - p\,u)},$$

vnde fit

$$\text{tang.}\, d\,u\,e = \frac{r\,.\,u\,v}{w\,(q\,v - p\,u)}.$$

Quare cum anguli O $u\,w$ tangens ſit $\frac{v}{u}$, hinc reperiemus tangentem anguli O $u\,e$, ſcil.

$$\text{tang. O}\,u\,e = \frac{v\,w\,(q\,v - p\,u) - r\,u\,u\,v}{u\,w\,(q\,v - p\,u) + r\,u\,v\,v},$$

cuius fractionis numerator ob $\frac{p}{u} + \frac{r}{w} = -\frac{q}{v}$, ſiue

$$u\,r + w\,p = -\frac{q.\,u\,w}{v},$$

reducitur ad hanc formam: $q\,w\,(u\,u + v\,v)$; denominator vero, ob

$$q\,w + r\,v = -\frac{v\,w\,p}{u},$$

reducitur ad: $-p\,w\,(u\,u + v\,v)$, ex quo colligitur

$$\text{tang. O}\,u\,e = -\frac{q}{p}.$$

§. 20.

§. 20. Iam ex e in Ou demittatur perpendiculum ef; et cum ſit $ue = \sqrt{(pp + qq)}$, tum vero

$$\text{ſin.}\, euf = \frac{-q}{\sqrt{(pp + qq)}} \text{ et } \text{coſ.}\, euf = \frac{p}{\sqrt{(pp + qq)}},$$

hinc erit

$$ef = -q \text{ et } uf = p.$$

Quod ſi nunc ducta concipiatur recta tf, ea erit ad Ou normalis, hinque triangulum tfu ad f rectangulum, ex quo obtinebitur coſinus anguli Out, ſub quo tangens tu ad axem OA inclinatur $= p$, prorſus vt ſupra. Eodem modo ſi ex w ducatur ug parallela et aequalis ipſi ef, erit quoque triangulum tug ad g rectangulum, vnde coſinus anguli tug, ſub quo noſtra tangens ad axem OB inclinatur $= -q$. Sicque has inclinationes, quae ſuperiore methodo ſponte ſe offerebant, hac methodo per plures ambages tandem ſunt erutae.

§. 21. His expeditis videamus etiam quomodo normalis ad curuam zN (fig. 6.) ad ternos axes inclinetur. Hunc in finem concipiatur quoque ex puncto u in plano uvw educta recta normali illi zN parallela, quae peitidem vnitati aequalis ſtatuatur: neque vero opus erit peculiarem figuram conſtituere. Nam ſi nunc recta ut pro directione huius normalis accipiatur, omnia manere poſſunt vt ante, ſi modo angulus dut capiatur $= 90° + \tau$. Sicque in praecedentibus formulis tantum opus erit ſcribere coſ. τ loco ſin. τ et ſin. τ loco coſ. τ. Sicque erit

$$td = \text{coſ.}\, \tau \text{ et } ud = -\text{ſin.}\, \tau.$$

Hincque porro erit

$$de = \text{coſ.}\, \tau\, \text{coſ.}\, \theta \text{ et } te = \text{coſ.}\, \tau\, \text{ſin.}\, \theta.$$

At

At vero per reductiones ante adhibitas habebimus

$$\text{fin.}\ \tau = \frac{\Theta r}{w \surd (u u + v v)} \text{ et } \text{cof.}\ \tau = \frac{q v - p u}{\surd (u u + v v)},$$

porro

$$\text{fin.}\ \theta = \frac{w \surd (u u + v v)}{\Theta} \text{ et } \text{cof.}\ \theta = \frac{u v}{\Theta},$$

quorum valorum ope erit

$$u d = - \frac{\Theta r}{w \surd (u u + v v)},$$

$$d t = \frac{u v}{\Theta},$$

$$d e = \frac{u v (q v - p u)}{\Theta \surd (u u + v v)} \text{ et}$$

$$t e = \frac{w (q v - p u)}{\Theta},$$

hincque

$$\text{tang.}\ d u e = - \frac{u v w (q v - p u)}{\Theta \Theta r} = \frac{d e}{u d} = - \frac{\text{cof.}\ \theta}{\text{tang.}\ \tau}.$$

§. 22. Nunc autem angulus $u t e$ exprimit inclinationem radii ofculi ad axem O C, cuius ergo cofinus erit $= \frac{w (q v - p u)}{\Theta}$; vbi cum fit

$$\Theta = \surd (u u v v + u u w w + v v w w), \text{ erit}$$

$$\Theta = u v w \surd \left(\frac{1}{u u} + \frac{1}{v v} + \frac{1}{w w}\right) = \frac{u v w}{t} \surd (d p^2 + d q^2 + d r^2),$$

vnde ille cofinus erit

$$= t \frac{(q v - p u)}{u v \surd (d p^2 + d q^2 + d r^2)}.$$

Diuidatur tam numerator quam denominator per $u v$, et haec expreffio euadet

$$\text{cof.}\ u t e = \frac{\left(\frac{q t}{u} - \frac{p t}{v}\right)}{\surd (d p^2 + d q^2 + d r^2)}.$$

§. 23. Supra autem vidimus effe

$$\frac{t}{u} = r d q - q d r \text{ et } \frac{t}{v} = p d r - r d p,$$

quibus valoribus introductis numerator noftrae formulae

erit

erit $d r$, vnde manifesto colligitur anguli sub quo radius osculi ad axem O C inclinatur cosinus

$$= \frac{-d r}{\sqrt{(d q^2 + d p^2 + d r^2)}},$$

vbi discrimen signi non est attendendum, propterea quod signum per se mutatur, siue radium osculi sursum vel deorsum tendere accipiamus.

§. 24. Cum igitur sit recta

$$t e = -\frac{d r}{\sqrt{(d p^2 + d q^2 + d r^2)}};$$

erit recta

$$u e = \frac{\sqrt{(d p^2 + d q^2)}}{\sqrt{(d p^2 + d q^2 + d r^2)}}.$$

at vero tang. $e u d = \frac{u v w (q v - p u)}{\Theta \Theta r}$. Ante autem erat

$$\Theta = \frac{u v w}{t} \sqrt{(d p^2 + d q^2 + d r^2)},$$

vnde ista tangens euadet

$$\frac{-(q v - p u) t t}{u v w r (d p^2 + d q^2 + d r^2)} = \frac{-\left(\frac{q t}{u} - \frac{p t}{v}\right) t}{w r (d p^2 + d q^2 + d r^2)},$$

cuius fractionis numerator transformatur in $+ t d r$, vnde ista tangens fiet

$$\text{tang. } d u e = \frac{d r (q d p - p d q)}{r (d p^2 + d q^2 + d r^2)}.$$

Hunc igitur angulum subtrahamus ab angulo O $u v$, cuius tangens est $\frac{v}{u} = \frac{r d q - q d r}{p d r - r d p}$, et factis omnibus reductionibus, quibus hactenus sumus vsi, tandem elicitur

$$\text{tang. O } u e = -\frac{d q}{d p}, \text{ ergo}$$

$$\text{sin.} = \text{O } u e \frac{-d q}{\sqrt{(d p^2 + d q^2)}} \text{ et cos. O } u e \frac{d p}{\sqrt{(d p^2 + d q^2)}}.$$

Cum igitur sit

$$u e = \frac{\sqrt{(d p^2 + d q^2)}}{\sqrt{(d p^2 + d q^2 + d r^2)}}, \text{ erit}$$

$u f = u e \cos. O u e = \frac{d p}{\sqrt{(d p^2 + d q^2 + d r^2)}}$ et

$e f = \frac{-d q}{\sqrt{(d p^2 + d q^2 + d r^2)}}$.

§. 25. Cum autem ſit $u t = 1$, recta $u f$ exprimit coſinum anguli $f u t$, quem radius oſculi cum axe O A conſtituit, recta vero $f e = u g$ exprimit coſinum anguli $g u t$, quem radius oſculi facit cum axe O B, quocirca habebimus:

$$\cos. \text{ incl. rad. osc. ad axem } OA = \frac{d p}{\sqrt{(d p^2 + d q^2 + d r^2)}},$$
$$\cos. \text{ incl. rad. osc. ad axem } OB = \frac{d q}{\sqrt{(d p^2 + d q^2 + d r^2)}},$$
$$\cos. \text{ incl. rad. osc. ad axem } OC = \frac{d r}{\sqrt{(d p^2 + d q^2 + d r^2)}}.$$

Ceterum iſta methodus ideo notatu maxime digna videtur, quod per reductiones hic expoſitas formulae maxime complexae ad ſimpliciſſimas, quaſi praeter omnem expectationem, reuocantur.

§. 26. Quo omnes iſtae reductiones facilius inſtitui queant, ſequentes proprietates probe notaſſe iuuabit. Cum enim ſit

$$\frac{t}{u} = r d q - q d r;$$
$$\frac{t}{v} = p d r - r d p;$$
$$\frac{t}{w} = q d p - p d q;$$

exiſtente vti aſſumſimus

$$t = x(r d q - q d r) + y(p d r - r d p) + z(q d p - p d q),$$

ita vt t denotet quantitatem differentialem, hinc erit

$$\text{I. } \frac{p t}{v} - \frac{q t}{u} = p(p d r - r d p) - q(r d q - q d r) = d r;$$

II.

$$\text{II. } \frac{qt}{w} - \frac{rt}{v} = q(qdp - pdq) - r(pdr - rdp) = dp;$$

$$\text{III. } \frac{rt}{u} - \frac{pt}{w} = r(rdq - qdr) - p(qdp - pdq) = dq.$$

Deinde vero ſequentes formulae etiam commodam reductionem patiuntur.

$$\text{I. } \frac{tdp}{v} - \frac{tdq}{u} = dp\,(pdr - rdp) - dq\,(rdq - qdr)$$
$$= -r\,(dp^2 + dq^2 + dr^2),$$

$$\text{II. } \frac{tdq}{w} - \frac{tdr}{v} = dq\,(qdp - pdq) - dr\,(pdr - rdp)$$
$$= -p\,(dp^2 + dq^2 + dr^2),$$

$$\text{III. } \frac{tdr}{u} - \frac{tdp}{w} = dr\,(rdq - qdr) - dp\,(qdp - pdq)$$
$$= -q\,(dp^2 + dq^2 + dr^2).$$

Praeterea vero hae reductiones notatu dignae ſunt:

$$\frac{tt}{uu} + \frac{tt}{vv} = dr^2 + rr\,(dp^2 + dq^2 + dr^2),$$
$$\frac{tt}{vv} + \frac{tt}{ww} = dp^2 + pp\,(dp^2 + dq^2 + dr^2),$$
$$\frac{tt}{ww} + \frac{tt}{uu} = dq^2 + qq\,(dp^2 + dq^2 + dr^2).$$

Tab. II. Fig. 9.

§. 27. Denique quoniam omnes has determinationes perduximus ad parallelepipeda circa axes principales exſtructa, cuiusmodi in hac figura exhibetur, vbi, prouti ternis lateribus O P, O Q, O R certi valores tribuuntur, diagonales O Ω poſitionem rectarum notabilium ad curvam pertinentium repraeſentant: quae hactenus hic ſunt allata ſequentibus parallelepipedis complecti licet. I.) Si parallelepipedi latera ſtatuantur

$$OP = x;\; OQ = y;\; OR = z;$$

diago-

diagonalis, quae erit $\sqrt{(xx + yy + zz)}$, dat positionem rectae ex puncto O ad curuae punctum z ductae.

II. Si parallelepipedi latera capiantur

$$OP = p;\ OQ = q;\ OR = r;$$

tum diagonalis $O\Omega$ dabit positionem tangentis curuae in puncto z, siue ista tangens parallela erit diagonali $O\Omega$; ipsa autem haec diagonalis erit

$$\sqrt{(pp + qq + rr)} = r.$$

III.) Si parallelepipedi latera statuantur:

$$OP = \frac{dp}{ds},\ OQ = \frac{dq}{ds},\ OR = \frac{dr}{ds},$$

tum diagonalis parallelogrammi dabit directionem radii osculi in puncto z, seu ipsi erit parallela: ipsa vero diagonalis erit $\frac{\sqrt{(dp^2 + dq^2 + dr^2)}}{ds}$, ita vt ipse radius osculi sit $\frac{1}{O\Omega}$. IV.) Si latera parallelepipedi statuantur:

$$OP = \frac{r\,dq - q\,dr}{ds},\ OQ = \frac{p\,dr - r\,dp}{ds},\ OR = \frac{q\,dp - p\,dq}{ds},$$

tum diagonalis $O\Omega$ erit normalis in planum in quo bina elementa curuae proxima incuruantur: ipsa autem diagonalis iterum erit $\frac{\sqrt{(dp^2 + dq^2 + dr^2)}}{ds}$. Postremum hoc theorema inde manifesto sequitur, quod normalis ad planum incuruationis perinde inclinatur ad ternos axes, atque ipsum planum inclinatur ad plana principalia opposita.