SPECVLATIONES SVPER FORMVLA INTEGRALI $\int \frac{x^n\,dx}{\sqrt{(aa-2bx+cxx)}}$.

VBI SIMVL EGREGIAE OBSERVATIONES CIRCA FRACTIONES CONTINVAS OCCVRRVNT.

Auctore
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§. 1.

Incipiamus a casu simplicissimo, quo $n=0$, et quaeramus integrale formulae $\frac{dx}{\sqrt{aa-2bx+cxx}}$, quae posito $x=\frac{b+z}{c}$ transit in hanc: $\frac{dz}{\sqrt{(aacc-bbc+czz)}}$, vbi duo casus distingui conuenit, prout c fuerit vel quantitas positiua vel negatiua. Sit igitur primo $c=+ff$ et formula nostra fiet $\frac{dz}{f\sqrt{(aaff-bb+zz)}}$, cuius integrale est $\frac{1}{f}\,l\,\frac{z+\sqrt{(aaff-bb+zz)}}{C}$, ideoque erit nostrum integrale

$$\frac{1}{\sqrt{c}}\,l\,\frac{cx-b+\sqrt{(aac-2bcx+ccxx)}}{C},$$

quod ergo, ita sumtum vt euanescat posito $x=0$, euadet

$$\frac{1}{\sqrt{c}}\,l\,\frac{cx-b+\sqrt{c(aa-2bx+cxx)}}{-b+a\sqrt{c}}.$$

At vero si c fuerit quantitas negatiua, puta $c=-gg$, formula differentialis per z expressa erit $\frac{dz}{g\sqrt{(aagg+bb-zz)}}$ cu-

cuius integrale est $\frac{1}{g}$ A sin. $\frac{z}{\sqrt{(aagg+bb)}}+C$; quare integrale ita sumtum vt euanescat posito $x=0$ fiet

$$=\frac{1}{g}\text{ A sin. }\frac{cx-b}{\sqrt{(aagg+bb)}}+\frac{1}{g}\text{ A sin. }\frac{b}{\sqrt{(aagg+bb)}}.$$

§. 2. Denotet nunc Π valorem formulae integralis $\int\frac{dx}{\sqrt{(aa-2bx+cxx)}}$ ita sumtum vt euanescat posito $x=0$, siue c fuerit quantitas positiua siue negatiua, ac si sit $c=+ff$ erit vti vidimus

$$\Pi=\frac{1}{f}\,l\,\frac{ffx-b+f\sqrt{aa-2bx+ffxx}}{af-b};$$

altero vero casu, quo $c=-gg$, erit

$$\Pi=-\frac{1}{g}\text{ A sin. }\frac{ggx+b}{\sqrt{(aagg+bb)}}+\frac{1}{g}\text{ A sin. }\frac{b}{\sqrt{(aagg+bb)}},$$

siue ambobus arcubus contractis habebimus

$$\Pi=\frac{1}{g}\text{ A sin. }\frac{bg\sqrt{(aa-2bx-ggxx)}-abg-ag^3x}{aagg+bb}.$$

Quoniam igitur mox ostendemus integrationem formulae generalis $\int\frac{x^n dx}{\sqrt{(aa-2bx+cxx)}}$ semper reduci posse ad casum $n=0$, si modo fuerit n numerus integer positiuus, omnia haec integralia per istum valorem Π exprimi poterunt.

§. 3. Iam post integrationem quantitati variabili x eiusmodi valorem constantem tribuamus, quo formula irrationalis $\sqrt{(aa-2bx+cxx)}$ ad nihilum redigatur, id quod fit si sumatur $x=\frac{b\pm\sqrt{(bb-aac)}}{c}$, ideoque duobus casibus. Ponamus pro vtroque casu functionem Π abire in Δ, ita vt casu $c=ff$ fit

$$\Delta=\frac{1}{f}\,l\,\frac{\sqrt{(bb-aaff)}}{af-b}=\frac{1}{f}\,l\sqrt{\frac{b+af}{b-af}};$$

pro altero autem casu, quo $c=-gg$

$$\Delta=\frac{1}{g}\text{ A sin. }\frac{\pm ag\sqrt{(bb+aagg)}}{aagg+bb}=\frac{1}{g}\text{ A sin. }\frac{ag}{\sqrt{(bb+aagg)}}.$$

Hos

Hos autem valores Δ in sequentibus casibus, quibus ipsa formula radicalis $\sqrt{(aa-2bx+cxx)}$ euanescit, potissimum sumus contemplaturi.

§. 4. Nunc ad sequentem casum progressuri, consideremus formulam $s=\sqrt{(aa-2bx+cxx)}-a$, vt scilicet euanescat facto $x=0$, et quoniam est

$$ds=\frac{-bdx+cxdx}{\sqrt{aa-2bx+cxx}}$$

erit viciſſim integrando

$$c\int\frac{xdx}{\sqrt{(aa-2bx+cxx)}}=b\int\frac{dx}{\sqrt{(aa-2bx+cxx)}}+s$$

vnde colligimus

$$\int\frac{xdx}{\sqrt{(aa-2bx+cxx)}}=\frac{b}{c}\Pi+\frac{\sqrt{(aa-2bx+xx)}-a}{c},$$

quare si post integrationem statuamus $x=\frac{b\pm\sqrt{(bb-aac)}}{c}$ quippe quibus casibus fit $\sqrt{(aa-2bx+cxx)}=0$ et $\Pi=\Delta$; fiet

$$\int\frac{xdx}{\sqrt{aa-2bx+cxx}}=\frac{b}{c}\Delta-\frac{a}{c}.$$

§. 5. Sumamus porro $s=x\sqrt{(aa-2bx+cxx)}$ fiet $ds=\frac{aadx-3bxdx+2cxxdx}{\sqrt{(aa-2bx+cxx)}}$, vnde viciſſim integrando colligitur

$$2c\int\frac{xxdx}{\sqrt{(aa-2bx+cxx)}}=3b\int\frac{xdx}{\sqrt{(aa-2bx+cxx)}}-aa\int\frac{dx}{\sqrt{(aa-2bx+cxx)}}+s$$

vnde statim pro casu $\sqrt{(aa-2bx+cxx)}=0$ deducimus

$$\int\frac{xxdx}{\sqrt{(aa-2bx+cxx)}}=\frac{(3bb-aac)}{2cc}\Delta-\frac{3ab}{2cc}.$$

§. 6. Iam ad altiores potestates ascensuri statuamus

mus $s = xx\sqrt{(aa-2bx+cxx)}$, et quia hinc fit

$$ds = \frac{2aax\,dx - 5bxx\,dx + 3cx^3\,dx}{\sqrt{(aa-2bx+cxx)}},\text{ erit}$$

$$3c\int\frac{x^3\,dx}{\sqrt{(aa-2bx+cxx)}} = 5b\int\frac{xx\,dx}{\sqrt{(aa-2bx+cxx)}}$$
$$-2aa\int\frac{x\,dx}{\sqrt{(aa-2bx+cxx)}} + s,$$

hincque porro pro casu quo post integrationem statuitur $x = \frac{b\pm\sqrt{(bb-aac)}}{c}$ habebitur

$$\int\frac{x^3\,dx}{\sqrt{(aa-2bx+cxx)}} = \left(\frac{5b^3-3aabc}{2c^3}\right)\Delta - \frac{15abb}{6c^3} + \frac{2a^4}{3cc}$$
$$= \left(\frac{5b^3}{2c^3} - \frac{3aab}{2cc}\right)\Delta - \frac{5abb}{2c^3} + \frac{2a^3}{3cc}.$$

§. 7. Simili modo fit $s = x^2\sqrt{(aa-2bx+cxx)}$, et quia hinc fit

$$ds = \frac{3aaxx\,dx - 7bx^3\,dx + 4cx^4\,dx}{\sqrt{(aa-2bx+cxx)}}$$

erit vicissim integrando

$$4c\int\frac{x^4\,dx}{\sqrt{(aa-2bx+cxx)}} = 7b\int\frac{x^3\,dx}{\sqrt{(aa-2bx+cxx)}}$$
$$-3aa\int\frac{xx\,dx}{\sqrt{(aa-2bx+cxx)}} + s;$$

tum igitur pro casu quo fit $\sqrt{(aa-2bx+cxx)} = 0$, habebimus

$$\int\frac{x^4\,dx}{\sqrt{(aa-2bx+cxx)}} = \left(\frac{35b^4}{8c^4} - \frac{15aabb}{4c^3} + \frac{3a^4}{8cc}\right)\Delta - \frac{55ab^3}{8c^4} + \frac{55a^3b}{24c^3}.$$

§. 8. Quo autem ordo in his formulis melius explorari possit, singulas exhibeamus per factores, quemadmodum ordine oriuntur, sine vlla abbreuiatione, atque hoc modo formulae integrales inuentae ita repraesententur:

$$\int\frac{dx}{\sqrt{(aa-2bx+cxx)}} = \Delta$$

$$\int\frac{x\,dx}{\sqrt{(aa-2bx+cxx)}} = \frac{b}{c}\Delta - \frac{a}{c}$$

 ∫

$$\int \frac{xx\,dx}{\sqrt{(aa-2bx+cxx)}} = \left(\frac{1.3bb}{1.2cc} - \frac{aa}{1.2c}\right)\Delta - \frac{1.3.ab}{1.2.cc}$$

$$\int \frac{x^3\,dx}{\sqrt{(aa-2bx+cxx)}} = \left(\frac{1.3.5b^3}{1.2.3c^3} - \frac{1.3.3aab}{1.2.3cc}\right)\Delta - \frac{1.3.5abb}{1.2.3c^3} + \frac{1.2.2a^3}{1.2.3cc}$$

$$\int \frac{x^4\,dx}{\sqrt{(aa-2bx+cxx)}} = \left(\frac{1.3.5.7b^4}{1.2.3.4c^4} - \frac{1.3.5.6aabb}{1.2.3.4c^4} + \frac{1.3.3a^4}{1.2.3.4cc}\right)\Delta$$
$$- \frac{1.3.5.7ab^3}{1.2.3.4c^4} + \frac{1.5.11a^3b}{1.2.3.4c^3}.$$

§. 9. Instituamus nunc in genere istam euolutionem, sumendo $s = x^n\sqrt{(aa-2bx+cxx)}$ et quia hinc fit

$$ds = \frac{naax^{n-1}dx - (2n+1)bx^n dx + (n+1)cx^{n+1}dx}{\sqrt{(aa-2bx+cxx)}}$$

inde vicissim integrando colligitur

$$(n+1)c\int\frac{x^{n+1}dx}{\sqrt{(aa-2bx+cxx)}} = (2n+1)b\int\frac{x^n dx}{\sqrt{(aa-2bx+cxx)}}$$
$$-naa\int\frac{x^{n-1}dx}{\sqrt{(aa-2bx+cxx)}} + x^n\sqrt{(aa-2bx+cxx)}.$$

Quod si vero iam ante elicuerimus

$$\int\frac{x^{n-1}dx}{\sqrt{(aa-2bx+cxx)}} = \mathrm{M}\Delta - \mathfrak{M} \text{ et}$$

$$\int\frac{x^n\,dx}{\sqrt{(aa-2bx+cxx)}} = \mathrm{N}\Delta - \mathfrak{N}$$

ita vt hae duae formulae sint cognitae, sequens ex iis ita determinabitur, vt sit

$$\int\frac{x^{n+1}\,dx}{\sqrt{(aa-2bx+cxx)}} = \left(\frac{(2n+1)b\mathrm{N}}{(n+1)c} - \frac{naa\mathrm{M}}{(n+1)c}\right)\Delta$$
$$-\frac{(2n+1)b\mathfrak{N}}{(n+1)c} + \frac{naa\mathfrak{M}}{(n+1)c}.$$

Hoc

Hoc igitur modo has integrationes quovsque libuerit continuare licet, dum ex binis quibusque sequens ope huius regulae formatur, ita vt omnia haec integralia vel a logarithmis vel ab arcubus circularibus pendeant, prouti coefficiens c fuerit vel positiuus vel negatiuus. Manifestum autem est istos valores assignari non posse, nisi exponens n fuerit numerus integer positiuus.

§. 10. Ex forma integrali modo inuenta, si post integrationem statuatur $x = \frac{b \pm \sqrt{(bb - aac)}}{c}$, vnde fit $s = 0$, erit

$$naa\int\frac{x^{n-1}dx}{\sqrt{(aa-2bx+cxx)}} = (2n+1)b\int\frac{x^n dx}{\sqrt{(aa-2bx+cxx)}}$$
$$-(n+1)c\int\frac{x^{n+1}dx}{\sqrt{(aa-2bx+cxx)}},$$

vnde si breuitatis gratia ponamus

$$\int\frac{x^{n-1}dx}{\sqrt{(aa-2bx+cxx)}} = P, \int\frac{x^n dx}{\sqrt{(aa-2bx+cxx)}} = Q,$$
$$\int\frac{x^{n+1}dx}{\sqrt{(aa-2bx+cxx)}} = R, \int\frac{x^{n+2}dx}{\sqrt{(aa-2bx+cxx)}} = S, \text{etc}$$

hae quantitates P, Q, R, S ita a se inuicem pendent vt sit

$$naaP = (2n+1)bQ - (n+1)cR;$$
$$(n+1)aaQ = (2n+3)bR - (n+2)cS;$$
$$(n+2)aaR = (2n+5)bS - (n+3)cT;$$
$$(n+3)aaS = (2n+7)bT - (n+4)cU;$$
$$(n+4)aaT = (2n+9)bU - (n+5)cW;$$
etc. etc.

Ex his relationibus deducuntur sequentes determinationes:

$$\frac{P}{Q} = \frac{(2n+1)b}{naa} - \frac{(n+1)c}{naaQ:R};$$
$$\frac{Q}{R} = \frac{(2n+3)b}{(n+1)aa} - \frac{(n+2)c}{(n+1)aaR:S};$$
$$\frac{R}{S} = \frac{(2n+5)b}{(n+2)aa} - \frac{(n+3)c}{(n+2)aaS:T};$$
$$\frac{S}{T} = \frac{(2n+7)b}{(n+3)aa} - \frac{(n+4)c}{(n+3)aaT:U};$$

etc. etc.

hinc igitur patet, singulas has fractiones $\frac{P}{Q}$, $\frac{Q}{R}$, $\frac{R}{S}$, etc. per sequentes satis commode determinari.

§. 11. Quod si iam in qualibet harum expressionum valores modo exhibiti successiue substituantur, pro fractione $\frac{P}{Q}$ impetrabimus fractionem continuam in infinitum progredientem, quae erit

$$naa\frac{P}{Q} = (2n+1)b - \cfrac{(n+1)^2aac}{(2n+3)b - \cfrac{(n+2)^2aac}{(2n+5)b - \cfrac{(n+3)^2aac}{(2n+7)b - \cfrac{(n+4)^2aac}{(2n+9)b - \text{etc.}}}}}$$

sicque peruenimus ad fractionem continuam satis concinnam et ordine perspicuo progredientem, cuius igitur valor semper vel per logarithmos (si fuerit $c > 0$), vel per arcus circulares (si fuerit $c < 0$) exprimi potest.

§. 12. Sumamus nunc $n = 1$ ac fiet

$$P = \int \frac{dx}{\sqrt{(aa - 2bx + cxx)}} = \Delta \text{ et}$$
$$Q = \int \frac{xdx}{\sqrt{(aa - 2bx + cxx)}} = \frac{b}{c}\Delta - \frac{a}{c},$$

qui casus nobis suppeditat sequentem fractionem continuam:

aac

$$\frac{aac\Delta}{b\Delta - a} = 3b - \cfrac{4aac}{5b - \cfrac{9aac}{7b - \cfrac{16aac}{9b - \cfrac{25aac}{11b - \text{etc.}}}}}$$

quae ob elegantiam omni attentione digna eſt cenſenda. Hic autem notaſſe iuuabit, ſi c fuerit numerus negatiuus tum omnes numeratores in hac fractione euadere poſitiuos.

§. 12. Fractio autem haec continua capite quaſi trunca videtur; vnde ſi ſuperne ei adiungatur membrum $b - aac$, ea adhuc concinnior eiusque valor ſimplicior reddetur. Si enim iſta fractio breuitatis gratia deſignetur littera S, ita vt ſit $S = \frac{aac\Delta}{b\Delta - a}$, adiecto iſto membro eius valor erit $b - \frac{aac}{S} = \frac{a}{\Delta}$; ſicque habebimus

$$\frac{a}{\Delta} = b - \cfrac{aac}{3b - \cfrac{4aac}{5b - \cfrac{9aac}{7b - \cfrac{16aac}{9b - \cfrac{25aac}{11b - \text{etc.}}}}}}$$

quae expreſſio eo magis eſt memorabilis, quod nulla adhuc via patet, qua talis fractionis continuae valor a priori inueniri poteſt.

§. 13. Euoluamus nunc ſeorſim binos caſus ſupra memoratos, et quos ſollicite a ſe inuicem diſtingui conue-

nit.

nit. Sit igitur primo $c = ff$, atque supra inuenimus fore

$$\Delta = \frac{1}{f} l \frac{\sqrt{(bb - aaff)}}{af - b},$$

vbi signum radicale ambigue accipi potest. Ante omnia igitur necesse est, vt sit $bb > aaff$, quia alioquin haec expressio euaderet imaginaria; duo ergo casus se offerunt, prouti b fuerit quantitas siue positiua siue negatiua. Priore casu, quo $b > 0$, atque adeo $b > af$, euidens est signo radicali tribui debere signum $-$, vt fiat

$$\Delta = \frac{1}{f} l \frac{\sqrt{(bb-aaff)}}{b-af} = \frac{1}{2f} l \frac{b+af}{b-af},$$

et iam habebimus istam summationem

$$\frac{2af}{l\frac{b+af}{b-af}} = b - \cfrac{aaff}{3b - \cfrac{4aaff}{5b - \cfrac{9aaff}{7b - \cfrac{16aaff}{9b - \text{etc.}}}}}$$

vnde cum sit $\frac{b+af}{b-af} > 1$, patet valorem huius expressionis fore positiuum.

§. 14. Sin autem fuerit b numerus negatiuus, siue si loco b scribatur $-b$, etiamnunc esse debet $b > af$; tum autem erit $\Delta = \frac{1}{2f} l \frac{b-af}{b+af}$, qui ergo logarithmus erit negatiuus, siue $\Delta = -\frac{1}{2f} l \frac{b+af}{b-af}$, vnde obtinebitur sequens aequatio:

$$\frac{-2af}{l\frac{b+af}{b-af}} = -b - \cfrac{aaff}{-3b - \cfrac{4aaff}{-5b - \cfrac{9aaff}{-7b - \cfrac{16aaff}{-9b - \text{etc.}}}}}$$

siue mutatis signis

$$\frac{2af}{l\frac{b+af}{b-af}} = b + \cfrac{aaff}{-3b + \cfrac{4aaff}{5b + \cfrac{9aaff}{-7b + \cfrac{16aaff}{9b + \text{etc.}}}}}$$

cuius ergo fractionis continuae summa aequalis est illi, quam in §. praecedente inuenimus. Ista autem aequalitas harum duarum expressionum calculum facienti mox fiet manifesta.

§. 15. Eodem modo euoluamus casum quo $c = -gg$, pro quo supra inuenimus $\Delta = \frac{1}{g}$ A sin. $\frac{ag}{\sqrt{(bb+aagg)}}$, qui valor per cosinum expressus dabit $\Delta = \frac{1}{g}$ A cos. $\frac{b}{\sqrt{bb+aagg}}$, vnde patet, per tangentem istum valorem adhuc fore simpliciorem: fit scilicet $\Delta = \frac{1}{g}$ A tang. $\frac{ag}{b}$, quam ob rem pro hoc casu prodit ista summatio

$$\frac{ag}{\text{A tang.}\frac{ag}{b}} = b + \cfrac{aagg}{3b + \cfrac{4aagg}{5b + \cfrac{9aagg}{7b + \cfrac{16aagg}{9b + \text{etc.}}}}}$$

vbi nulla amplius limitatione eſt opus.

De fractionibus continuis a logarithmis pendentibus.

§. 16. Perpendamus nunc etiam aliquos caſus ſpeciales, in vtraque forma contentos, et quoniam iam obſeruauimus, binas formas in §§. 13. et 14. inter ſe congruere, vtamur priori, qua erat

$$\frac{2af}{l\frac{b+af}{b-af}} = b - \cfrac{aaff}{3b - \cfrac{4aaff}{5b - \cfrac{9aaff}{7b - \text{etc.}}}}$$

ac primo conſideremus caſum, quo $b = af$, quippe quo euadit ſumma fractionis

$$\frac{2af}{l\frac{b+af}{b-af}} = 0 = b - \cfrac{bb}{3b - \cfrac{4bb}{5b - \cfrac{9bb}{7b - \text{etc.}}}}$$

quae

quae per reductionem facile mutatur in hanc:

$$0 = 1 - \cfrac{1}{3 - \cfrac{4}{5 - \cfrac{9}{7 - \cfrac{16}{9 - \text{etc.}}}}}$$

§. 17. In ista igitur forma nihilo aequali necesse est vt denominator primae fractionis sit $= 1$ ideoque

$$1 = 3 - \cfrac{4}{5 - \cfrac{9}{7 - \text{etc.}}} \quad \text{siue} \quad 0 = 2 - \cfrac{4}{5 - \cfrac{9}{7 - \text{etc.}}}$$

Hic igitur ob eandem rationem necesse est vt prior denominator fiat $= 2$, ita vt

$$2 = 5 - \cfrac{9}{7 - \cfrac{16}{9 - \text{etc.}}} \quad \text{siue} \quad 0 = 3 - \cfrac{9}{7 - \cfrac{16}{9 - \text{etc.}}}$$

Hic iterum primus denominator debet esse $= 3$ ideoque

$$3 = 7 - \cfrac{16}{9 - \cfrac{25}{11 - \text{etc.}}} \quad \text{siue} \quad 0 = 4 - \cfrac{16}{9 - \cfrac{25}{11 - \text{etc.}}}$$

Denuo igitur primus denominator esse debet $= 4$, ita vt $4 = 9 - \cfrac{25}{11 - \text{etc.}}$ atque hoc modo patet, istam relationem eodem ordine in infinitum locum habere, in quo ipso criterium veritatis huius aequationis est situm.

§. 18. Quoniam in hac forma numerus b maior esse debet quam af, statuamus nunc $b = 2af$, et nanciscemur sequentem summationem:

$$\frac{2af}{l\,x} = 2af - \cfrac{aaff}{6af - \cfrac{4aaff}{10af - \cfrac{9aaff}{14af - \text{etc.}}}}$$

quae reducitur ad hanc formam mere numericam

$$\frac{2}{l\,6} = 2 - \cfrac{1}{6 - \cfrac{4}{10 - \cfrac{9}{14 - \cfrac{16}{18 - \text{etc.}}}}}$$

§. 19. Simili modo omnes litterae ex calculo expelli possunt, si pro b accipiatur multiplum ipsius af. Sit enim in genere $b = naf$, ac prodit

$$\frac{2af}{l\frac{n+1}{n-1}} = naf - \cfrac{aaff}{3naf - \cfrac{4aaff}{5naf - \cfrac{9aaff}{7naf - \text{etc.}}}}$$

quae fractio reducitur ad formam sequentem:

$$\frac{2}{l\frac{n+1}{n-1}} = n - \cfrac{1}{3n - \cfrac{4}{5n - \cfrac{9}{7n - \text{etc.}}}}$$

vnde

vnde intelligitur, quemadmodum omnes logarithmos per fractiones continuas exprimi conueniat.

§. 20. Possent hic pro n numeri fracti accipi, tum autem priores termini in singulis membris prodirent fracti, quas quidem per reductionem ad integros reuocare liceret: verum huiusmodi casus facillime ex forma generali deriuari possunt, scribendo statim $b = n$ et $af = m$; tum enim habebimus

$$\frac{2m}{l\frac{n+m}{n-m}} = n - \cfrac{mm}{3n - \cfrac{4mm}{5n - \cfrac{9mm}{7n - \text{etc.}}}}$$

vnde si loco m scribatur $\sqrt{k}$, erit

$$\frac{2\sqrt{k}}{l\frac{n+\sqrt{k}}{n-\sqrt{k}}} = n - \cfrac{k}{3n - \cfrac{4k}{5n - \cfrac{9k}{7n - \text{etc.}}}}$$

§. 21. Hinc igitur omnium numerorum integrorum logarithmos hyperbolicos per fractiones continuas exprimere poterimus. Propositus igitur sit in genere numerus integer i, ac statuatur $\frac{n+m}{n-m} = i$, erit $\frac{n}{m} = \frac{i+1}{i-1}$. Capiatur ergo $n = i + 1$ et $m = i - 1$, atque habebimus

$$\frac{2(i-1)}{ii} = i+1-\cfrac{(i-1)^2}{3(i+1)-\cfrac{4(i-1)^2}{5(i+1)-\cfrac{9(i-1)^2}{7(i+1)-\cfrac{16(i-1)^2}{9(i+1)-\text{etc.}}}}}$$

vnde colligimus

$$li = \cfrac{2(i-1)}{i+1-\cfrac{(i-1)^2}{3(i+1)-\cfrac{4(i-1)^2}{5(i+1)-\cfrac{9(i-1)^2}{\text{etc.}}}}}$$

§. 22. Si huiusmodi fractiones desideremus pro logarithmis numerorum fractorum, statuamus $\frac{n+m}{n-m} = \frac{p}{q}$, vnde fit $n = p+q$ et $m = p-q$, quamobrem habebimus

$$l\frac{p}{q} = \cfrac{2(p-q)}{1(p+q)-\cfrac{1(p-q)^2}{3(p+q)-\cfrac{4(p-q)^2}{5(p+q)-\cfrac{9(p-q)^2}{7(p+q)-\text{etc.}}}}}$$

quae forma eo magis est notatu digna, quod satis commode adhiberi potest ad logarithmos proxime inuestigandos. Eo magis autem istae fractiones continuae conuergent, quo minor fuerit fractio $\frac{p-q}{p+q}$.

§. 23. Quo hoc exemplo illustremus, sumamus $p = 2$ et $q = 1$, vnde quidem non adeo vehemens conuergentia est expectanda, eritque

$l\,2 =$

$$l\,2 = \cfrac{2}{3 - \cfrac{1}{9 - \cfrac{4}{15 - \cfrac{9}{21 - \text{etc.}}}}}$$

vnde ſumendo tantum primum membrum $\frac{2}{3}$, in fractione decimali prodit 0, 666666, dum ex tabulis habetur $l\,2 =$ 0, 693147, vbi error iam ſatis eſt exiguus. Capiamus iam bina membra priora $\cfrac{2}{3 - \cfrac{1}{9}} = \frac{9}{13} = 0,6923$. Sumendo autem tria membra habebimus

$$\cfrac{2}{3 - \cfrac{1}{9 - \cfrac{4}{15}}} = \cfrac{2}{3 - \cfrac{15}{131}} = \frac{262}{378} = 0,693121$$

qui valor a veritate deficit quantitate 0, 000026. Multo promptior autem deprehendetur conuergentia, ſi ſumamus $p = 3$ et $q = 2$, vt habeamus

$$l\tfrac{3}{2} = \cfrac{2}{5 - \cfrac{1}{15 - \cfrac{4}{25 - \cfrac{9}{35 - \text{etc.}}}}}$$

cuius primum membrum dat $\frac{2}{5} = 0,400000$; reuera autem eſt $l\frac{3}{2} = 0,405465108$. Sumtis autem duobus membris $\cfrac{2}{5 - \cfrac{1}{15}}$ colligitur $l\frac{3}{2} = 0,40540$, vbi error tantum in

quin-

quintam figuram irrepit. Sumantur tria membra

$$\cfrac{2}{5-\cfrac{1}{15-\cfrac{4}{25}}} = \cfrac{2}{5-\cfrac{25}{371}} = 0,405464$$

vbi error demum in ſeptima figura ſe manifeſtat.

§. 24. Ob hunc inſignem vſum, qui ſe praeter expectationem obtulit, operae pretium erit talem inueſtigationem in genere expedire; atque in hunc finem vtatur formula inter litteras m et n ſupra §. 20. data, vbi fit

$$l\frac{n+m}{n-m} = \cfrac{2m}{n-\cfrac{mm}{3n-\cfrac{4mm}{5n-\cfrac{9mm}{7n-\cfrac{16mm}{9n-\text{etc.}}}}}}$$

vnde ſi capiamus tantum primum membrum, fiet propemodum $l\frac{n+m}{n-m} = \frac{2m}{n}$; ſumtis autem binis prioribus membris $\cfrac{2m}{n-\cfrac{mm}{3n}}$, erit iam propius $l\frac{n+m}{n-m} = \frac{6mn}{3nn-mm}$; ſumtis vero tribus membris erit

$$l\frac{n+m}{n-m} = \cfrac{2m}{n-\cfrac{mm}{3n-\cfrac{4mm}{5n}}} = \frac{30mnn-8m^3}{15n^3-9mmn}.$$

§. 25.

§. 25. Non adeo autem operose est has fractiones vlterius continuare: fractionibus enim iam inuentis praefigamus fractionem $\frac{0}{1}$, vt obtineamus hanc fractionum progressionem:

$$\overset{\text{I}}{\frac{0}{1}},\quad \overset{\text{II}}{\frac{2m}{n}},\quad \overset{\text{III}}{\frac{6mn}{3nn-mm}},\quad \overset{\text{IV}}{\frac{30mnn-8m^3}{15n^3-9mmn}},$$

cuius tam numeratores quam denominatores ex binis praecedentibus, ad similitudinem serierum recurrentium, formari possunt. Tertia scilicet ex prima et secunda formatur ope huius scalae relationis: $3n, -mm$; quarta vero formatur ex binis praecedentibus ope huius scalae relationis $5n, -4mm$. Pro quinta igitur vtendum erit hac scala: $7n, -9mm$, pro sexta hac: $9n, -16mm$; et ita porro. Hoc igitur modo facile reperitur fractio quinta

$$\text{V} = \frac{210mn^3-110m^3n}{105n^4-90mmnn+9m^4}$$

simili modo

$$\text{VI} = \frac{1980mn^4-1470m^3nn+128m^5}{945n^5-1050mmn^3+225m^4n} \text{ etc.}$$

§. 26. Hic autem imprimis notasse iuuabit, has fractiones continuo augeri, et per incrementa continuo minora ad veritatem accedere. Incrementa autem ista egregio ordine procedunt, vti videre hic licet

$$\text{II}-\text{I} = \frac{2m}{n};$$

$$\text{III}-\text{II} = \frac{2m^3}{n(3nn-mm)};$$

$$\text{IV}-\text{III} = \frac{2.4m^5}{(3nn-mm)(15n^3-9mmn)};$$

$$\text{V}-\text{IV} = \frac{2.4.9.m^7}{(15n^3-9mmn)(105n^4-90mmnn+9m^4)};$$

$$\text{VI}-\text{V} = \frac{2.4.9.16.m^9}{(105n^4-90mmnn+9m^4)(945n^5-1050mmn^3+225m^4n)}.$$

vnde

vnde patet, quo maior fuerit numerus n prae m, eo citius has differentias tam fieri exiguas, vt ſine errore negligi queant.

De fractionibus continuis ab arcubus circularibus pendentibus.

§. 27. Ex § 15. arcus circuli cuius tangens eſt $\frac{ag}{b}$ ita per fractionem exprimetur, vt fit

$$A\ \text{tang.}\ \frac{ag}{b} = \cfrac{ag}{b + \cfrac{aagg}{3b + \cfrac{4aagg}{5b + \cfrac{9aagg}{7b + \text{etc.}}}}}$$

Ponamus nunc ad ſimilitudinem ſuperiorum formarum $ag = m$ et $b = n$, atque habebimus

$$A\ \text{tang.}\ \frac{m}{n} = \cfrac{m}{n + \cfrac{mm}{3n + \cfrac{4mm}{5n + \cfrac{9mm}{7n + \text{etc.}}}}}$$

quae forma eo citius conuergit, quo maior fuerit numerus n prae m; vnde patet etiam hanc expreſſionem cum fructu ad calculum accommodari poſſe.

§. 28. Incipiamus a caſu quo $m = 1$ et $n = 1$, quo fit

A tang.

$$A\ \text{tang.}\ \frac{m}{n} = \frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{4}{5 + \cfrac{9}{7 + \cfrac{16}{9 + \text{etc.}}}}}}$$

quae quidem fractio non adeo conuergit; attamen videamus, quomodo paullatim ad veritatem accedat, quandoquidem nouimus esse $\frac{\pi}{4} = 0,785398163339$. Ac primum quidem membrum dabit $\frac{\pi}{4} = \frac{1}{1}$, (nimis magnum); duo membra praebent $\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1}{3}} = \frac{3}{4}$, (nimis paruum); tria membra dant $\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{4}{5}}} = \frac{19}{24} = 0,7916$, (nimis magnum). Sumantur quatuor membra, vt fiat

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{4}{5 + \cfrac{9}{7}}}} = \frac{40}{51} = 0,7843$$

vbi error demum in tertia figura deprehenditur. Ceterum haec fractio continua similis fere est illi, quam olim Brounckerus in medium protulit, quae ita se habebat:

$$\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\text{etc.}}}}}$$

Manifestum autem est nostram fractionem multo magis conuergere; neque minus concinna est censenda.

§. 29. Quo autem fractionem continuam magis conuergentem nanciscamur, statuamus A. tg. $\frac{m}{n} = 30°$, cuius tangens cum sit $\frac{1}{\sqrt{3}}$, ne numerus n fiat irrationalis sumamus $m = \sqrt{3}$ et $n = 3$, hinc igitur fiet

$$\frac{\pi}{6} = \sqrt{3}\cfrac{3}{3+\cfrac{3}{9+\cfrac{12}{15+\cfrac{27}{21+\cfrac{48}{27+\text{etc.}}}}}}$$

quae forma reducitur ad sequentem:

$$\frac{\pi}{6\sqrt{3}} = \cfrac{1}{3+\cfrac{1}{3+\cfrac{4}{15+\cfrac{9}{7+\cfrac{16}{27+\cfrac{25}{11+\text{etc.}}}}}}}$$

pro qua euoluenda quaeramus primo proxime valorem $\frac{\pi}{6\sqrt{3}}$, qui est 0,30222998. Nunc vero primum membrum praebet

praebet $\frac{\pi}{6\sqrt{3}} = 0,3333$; duo autem priora praebent

$$\frac{\pi}{6\sqrt{3}} = \cfrac{1}{3+\cfrac{1}{3}} = \frac{3}{10} = 0,3000;$$

tria membra dant.

$$\frac{\pi}{6\sqrt{3}} = \cfrac{1}{3+\cfrac{1}{3+\cfrac{4}{15}}} = \frac{49}{162} = 0,30247$$

vbi error quartam demum figuram afficit.

§. 30. Multo promptior autem conuergentia procurari poteſt, dum angulum rectum in duas partes ſecamus, quemadmodum olim oſtendi eſſe A tang. $\frac{1}{2}$ + A tang. $\frac{1}{3}$ = A tang. $1 = \frac{\pi}{4}$. Sic igitur duas fractiones continuas reperiemus, quarum ſumma dabit valorem ipſius $\frac{\pi}{4}$, quae erunt

A tang. $\frac{1}{2} = \cfrac{1}{2+\cfrac{1}{6+\cfrac{4}{10+\cfrac{9}{14+\text{etc.}}}}}$ et A tang. $\frac{1}{3} = \cfrac{1}{3+\cfrac{1}{9+\cfrac{4}{15+\cfrac{9}{21+\text{etc.}}}}}$

manifeſtum autem eſt has ambas fractiones, et potiſſimum poſteriorem, vehementer conuergere.

§. 31. Conuertamus vero etiam noſtram fractionem continuam generalem in fractiones communes; ac ex primo membro ſolo reperimus A tang. $\frac{m}{n} = \frac{m}{n}$; ex duobus membris prodit A tang. $\frac{m}{n} = \frac{3mn}{3nn+mm}$; tria membra prae-

bent

bent A tang. $\frac{m}{n} = \frac{15mnn + 4m^3}{15n^3 + 9mmn}$. Sumantur quatuor membra, vnde fit A tang. $\frac{m}{n} = \frac{105mn^3 + 55m^3n}{105n^4 + 90mmnn + 9m^4}$. Quod si nunc his fractionibus praefigatur vt supra $\frac{0}{1}$, orietur haec progressio:

$$\begin{array}{ccccc} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \\ \frac{0}{1}, & \frac{m}{n}, & \frac{3mn}{3nn + mm}, & \frac{15mnn + 4m^3}{15n^3 + 9mmn}, & \frac{105mn^3 + 55m^3n}{105n^4 + 90mmnn + 9m^4}, \end{array}$$

cuius singuli termini itidem ex praecedentibus binis secundum certam legem formari possunt, scilicet:

pro III scala relationis est $3n, + mm$
pro IV scala relationis est $5n, + 4mm$
pro V scala relationis est $7n, + 9mm$
etc. etc.

DEMON-