ACCVRATIOR EVOLVTIO FORMVLARVM PRO FILORVM FLEXIBILIVM AEQVILIBRIO ET MOTV INVENTARVM.

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§. 1.

Tab. IV. Fig. 6. Propositum sit filum quodcunque flexile $EZzF$, siue elasticitate praeditum siue destitutum, quod situm teneat in figura repraesentatum, quem, quatenus non in eodem plano continetur, referamus more solito ad ternos axes fixos OA, OB, OC inter se normales. Tum pro quouis fili puncto Z constitutis ternis coordinatis OX, XY, YZ, vocemus $OX = x$, $XY = y$, $Yz = z$; tum vero ipsa portio fili EZ vocetur $= s$, vt sit $ds^2 = dx^2 + dy^2 + dz^2$. Praeterea vero consideretur crassities fili vtcunque variabilis atque elementi $Zz = ds$ statuatur massula $= S\,ds$; ita vt S sit functio quaecunque data ipsius s. His positis fili elementum $Zz = ds$ sollicitetur a tribus viribus secundum directiones axibus parallelas, quae sint: $ZP = P$, $ZQ = Q$, $ZR = R$, elasticitas autem absoluta in loco Z designetur littera G, quae pro indole fili poterit esse vel constans vel vtcunque variabilis ab arcu s pendens.

§. 2.

§. 2. Quodsi iam istud filum fuerit in aequilibrio, tres sequentes aequationes sunt erutae:

$$\text{I. } \int dy \int P\, ds - \int dx \int Q\, ds = \frac{G\,(dy\,ddx - dx\,ddy)}{ds^3},$$

$$\text{II. } \int dz \int Q\, ds - \int dy \int R\, ds = \frac{G\,(dz\,ddy - dy\,ddz)}{ds^3},$$

$$\text{III. } \int dx \int R\, ds - \int dz \int P\, ds = \frac{G\,(dx\,ddz - dz\,ddx)}{ds^3}.$$

Hae autem tres aequationes ita a se inuicem pendent, vt binae iam tertiam in se inuoluant, ita vt sufficiat binas earum tantum resoluisse, quippe quibus totus status aequilibrii iam perfecte determinatur.

§. 3. Praeterea vero si tensio fili in puncto Z ponatur $= T$, pro eius determinatione inuenta est haec formula:

$$T = -\frac{dx}{ds}\int P\, ds - \frac{dy}{ds}\int Q\, ds - \frac{dz}{ds}\int R\, ds.$$

Tensio autem ista T exprimet eam vim, qua punctum Z secundum tangentem versus E trahi deberet ad aequilibrium conseruandam, si portio antecedens fuisset rescissa.

§. 4. Sin autem filum in motu versetur atque elapso tempore t, quod semper in minutis secundis exprimi assumimus, situm habeat in figura repraesentatum, tum ternae coordinatae x, y, z tanquam functiones binarum variabilium, scilicet arcus $EZ = s$ et temporis t spectari debent. Tum autem denotante g altitudinem ex qua grauia vno minuto secundo delabuntur, ex viribus sollicitantibus P, Q, R quaerantur sequentes valores deriuati:

$$P' = P - \frac{S}{2g}\left(\frac{ddx}{dt^2}\right);$$

$$Q' = Q - \frac{S}{2g}\left(\frac{ddy}{dt^2}\right);$$

$$R' = R - \frac{S}{2g}\left(\frac{ddz}{dt^2}\right);$$

vbi formulae $\left(\frac{ddx}{dt^2}\right)$, $\left(\frac{ddy}{dt^2}\right)$, $\left(\frac{ddz}{dt^2}\right)$, tantum ex variabilitate temporis t sunt determinandae, arcu s pro constante habito. Quo facto isti valores P', Q', R' denuo vt functiones tantum ipsius s spectentur et in superioribus formulis loco P, Q, R scribantur, vt prodeant sequentes aequationes:

$$\text{I. } \int dy \int P\,ds - \int dx \int Q\,ds - \tfrac{1}{2g}\int dy \int S\,ds \left(\tfrac{ddx}{dt^2}\right) + \tfrac{1}{2g}\int dx \int S\,ds \left(\tfrac{ddy}{dt^2}\right) = \frac{G(dy\,ddx - dx\,ddy)}{ds^3},$$

$$\text{II. } \int dz \int Q\,ds - \int dy \int R\,ds - \tfrac{1}{2g}\int dz \int S\,ds \left(\tfrac{ddy}{dt^2}\right) + \tfrac{1}{2g}\int dy \int S\,ds \left(\tfrac{ddz}{dt^2}\right) = \frac{G(dz\,ddy - dy\,ddz)}{ds^3},$$

$$\text{III. } \int dx \int R\,ds - \int dz \int P\,ds - \tfrac{1}{2g}\int dx \int S\,ds \left(\tfrac{ddz}{dt^2}\right) + \tfrac{1}{2g}\int dz \int S\,ds \left(\tfrac{ddx}{dt^2}\right) = \frac{G(dx\,ddz - dz\,ddx)}{ds^3}.$$

Iidem vero valores loco P, Q, R in formula pro tensione T data substituti praebebunt tensionem durante motu, quae ergo erit

$$T = -\tfrac{dx}{ds}\int P\,ds - \tfrac{dy}{ds}\int Q\,ds - \tfrac{dz}{ds}\int R\,ds + \tfrac{dx}{2g\,ds}\int S\,ds\left(\tfrac{ddx}{dt^2}\right) + \tfrac{dy}{2g\,ds}\int S\,ds\left(\tfrac{ddy}{dt^2}\right) + \tfrac{dz}{2g\,ds}\int S\,ds\left(\tfrac{ddz}{dt^2}\right).$$

§. 5. Quanquam autem in his formulis omnes plane motus, qui huiusmodi filis a viribus quibuscunque induci possunt continentur: tamen maxime dolendum est, ob defectum analyseos vix vllum fructum ex iis adhuc percipi potuisse. Quicquid enim mihi quidem inde elicere licuit ad motus reciprocos et quasi infinite paruos restringitur, cuiusmodi sunt motus chordarum, laminarum elasticarum et oscillationes funium libere suspensorum, vnde etiamnunc superfluum foret istas formulas diligentius euol-

vere,

vere, praeterquam quod tam arcto nexu cum formulis pro aequilibrio cohaerent, vt non difficile sit, quicquid de his reperietur, etiam ad illas transferre.

§. 6. Deinde vero etiam hoc loco elasticitatem fili seponam et tantum statum aequilibrii filorum perfecte flexibilium plenius sum inuestigaturus, quo facilius formulae inuentae ad omnes plane casus accommodari queant. Hoc modo tantum habebimus tres aequationes sequentes:

$$\text{I. } \int dy \int P\,ds = \int dx \int Q\,ds;$$
$$\text{II. } \int dz \int Q\,ds = \int dy \int R\,ds;$$
$$\text{III. } \int dx \int R\,ds = \int dz \int P\,ds;$$

quibuscum coniungamus formulam pro tensione datam:

$$T = -\frac{dx}{ds}\int P\,ds - \frac{dy}{ds}\int Q\,ds - \frac{dz}{ds}\int R\,ds.$$

Illae igitur aequationes differentiatae praebent istae:

$$\text{I. } dy \int P\,ds = dx \int Q\,ds,$$
$$\text{II. } dz \int Q\,ds = dy \int R\,ds,$$
$$\text{III. } dx \int R\,ds = dz \int P\,ds.$$

§. 7. Quod si harum trium aequationum prima ducatur in dz, secunda in dx, ac tertia in dy, eae inuicem additae producent aequationem $0 = 0$, quae, cum sit identica, declarat in binis aequationibus tertiam iam contineri, quaemadmodum iam supra monuimus. Hinc autem trium formularum integralium $\int P\,ds$, $\int Q\,ds$, $\int R\,ds$ binae quaelibet per tertiam definiri possunt, atque hinc habebimus

$$\int Q\,ds = \frac{dy}{dx}\int P\,ds \quad \text{et} \quad \int R\,ds = \frac{dz}{dx}\int P\,ds.$$

Simi-

Simili modo per formulam $\int Q\,ds$ erit

$$\int P\,ds = \frac{dx}{dy}\int Q\,ds \text{ et } \int R\,ds = \frac{dz}{dy}\int Q\,ds.$$

Porro vero per $\int R\,ds$ fiet

$$\int P\,ds = \frac{dx}{dz}\int R\,ds \text{ et } \int Q\,ds = \frac{dy}{dz}\int R\,ds.$$

§. 8. Hinc igitur tenſionem T per quamlibet harum formularum integralium, excluſis binis reliquis, exprimere poterimus. Primo enim erit per formulam $\int P\,ds$:

$$T = -\frac{dx}{ds}\int P\,ds - \frac{dy^2}{ds\,dx}\int P\,ds - \frac{dz^2}{ds\,dx}\int P\,ds = -\frac{ds}{dx}\int P\,ds,$$

propter $dx^2 + dy^2 + dz^2 = ds^2$. Eodem modo per formulam $\int Q\,ds$ reperiemus quoque $T = -\frac{ds}{dy}\int Q\,ds$ et per formulam $\int R\,ds$, $T = -\frac{ds}{dz}\int R\,ds$.

§. 9. Quod ſi ergo tenſionem T tanquam datam ſpectemus, noſtrae formulae integrales ita definientur vt ſit

$$\int P\,ds = -\frac{T\,dx}{ds},$$
$$\int Q\,ds = -\frac{T\,dy}{ds},$$
$$\int R\,ds = -\frac{T\,dz}{ds},$$

ex quarum differentiatione, ſi elementum ds conſtans capiamus, ipſas vires ſollicitantes $P\,ds$, $Q\,ds$ et $R\,ds$, ex tenſione T definire licebit hoc modo:

$$P\,ds = -\frac{dT\,dx - T\,ddx}{ds};$$
$$Q\,ds = -\frac{dT\,dy - T\,ddy}{ds};$$
$$R\,ds = -\frac{dT\,dz - T\,ddz}{ds};$$

atque hinc porro concludimus fore:

$P\,dx$

$$P\,dx = -\frac{dT\,dx^2}{ds^2} - \frac{T\,dx\,ddx}{ds^2};$$
$$Q\,dy = -\frac{dT\,dy^2}{ds^2} - \frac{T\,dy\,ddy}{ds^2};$$
$$R\,dz = -\frac{dT\,dz^2}{ds^2} - \frac{T\,dz\,ddz}{ds^2};$$

quae tres formulae coniunctae, cum ſit $dx^2 + dy^2 + dz^2 = ds^2$, hincque $dx\,ddx + dy\,ddy + dz\,ddz = 0$, ob ds conſtans, fient $P\,dx + Q\,dy + R\,dz = dT$.

§. 10. Euidens autem eſt hanc formulam $P\,dx + Q\,dy + R\,dz$, exprimere vim tangentialem, qua elementum ds ſecundum ipſam directionem Zz a ternis viribus $P\,ds$, $Q\,ds$, $R\,ds$ ſollicitatur. Quod ſi ergo hanc vim tangentialem deſignemus per $\Theta\,ds$, vt $\Theta\,ds = P\,dx + Q\,dy + R\,dz$, erit vtique $dT = -\Theta\,ds$, ideoque $T = C - \int \Theta\,ds$, quemadmodum ex natura rei conſtat, quandoquidem tenſio ſemper aequatur ſummae omnium virium tangentialium; ſicque viciſſim ex vi tangentiali $\Theta\,ds$ et tenſione T innoteſcent ternae vires ſollicitantes

$$P\,ds = +\Theta\,dx - \frac{T\,ddx}{ds};$$
$$Q\,ds = +\Theta\,dy - \frac{T\,ddy}{ds};$$
$$R\,ds = +\Theta\,dz - \frac{T\,ddz}{ds};$$

§. 11. Quod ſi iam praeter vim tangentialem $\Theta\,ds$ etiam vim normalem in calculum introducere velimus, eamque ponamus $= \Pi\,ds$, cuius directio non ſolum ad elementum Zz normalis eſt intelligenda, ſed etiam ad planum, in quo duo elementa contigua ſunt ſita, ita vt omnes vires ſollicitantes iunctim ſumtae ad has duas vires $\Theta\,ds$ et $\Pi\,ds$ reuocentur: quoniam vis tribus viribus ſollicitantibus $P\,ds$, $Q\,ds$ et $R\,ds$ aequiualens eſt

 $ds\sqrt{(P^2}$

$ds\sqrt{(P^2+Q^2+R^2)}$, at vero vires normalis et tangentialis Πds et Θds reducuntur ad hanc vnicam: $ds\sqrt{(\Pi^2+\Theta^2)}$; necesse est vt fiat $\sqrt{(\Pi^2+\Theta^2)}=\sqrt{(P^2+R^2+Q^2)}$ ideoque $\Pi^2=P^2+Q^2+R^2-\Theta^2$, siue

$$\Pi\Pi ds^2=(PP+QQ+RR)ds^2-\Theta\Theta ds^2.$$

§. 12. Supra autem vidimus esse

$$\Theta ds=Pdx+Qdy+Rdz,$$

vnde fit

$$\Theta\Theta ds^2=PPdx^2+QQdy^2+RRdz^2 \\ +2PQdxdy+2PRdxdz+QRdydz.$$

Tum vero loco ds^2 valorem suum $dx^2+dy^2+dz^2$ substituendo erit

$$(PP+QQ+RR)ds^2=\begin{cases}+(PP+QQ+RR)dx^2\\+(PP+QQ+RR)dy^2\\+(PP+QQ+RR)dz^2,\end{cases}$$

vnde si subtrahatur $\Theta^2 ds^2$, remanebit

$$\Pi\Pi ds^2=(QQ+RR)dx^2+(PP+RR)dy^2+(PP+QQ)dz^2 \\ -2PQdxdy-2PRdxdz-2QRdydz,$$

quae expressio manifesto reducitur ad summam trium sequentium quadratorum:

$$\Pi\Pi ds^2=(Pdy-Qdx)^2+(Qdz-Rdy)^2+(Rdx-Pdz)^2.$$

§. 13. Cum igitur supra inuenerimus

$$P=\frac{\Theta dx}{ds}-\frac{Tddx}{ds^2};$$
$$Q=\frac{\Theta dy}{ds}-\frac{Tddy}{ds^2};$$
$$R=\frac{\Theta dz}{ds}-\frac{Tddz}{ds^2};\text{ erit}$$

Pdy

$$P\,dy - Q\,dx = -\frac{T(dy\,ddx - dx\,ddy)}{ds^2},$$
$$Q\,dz - R\,dy = -\frac{T(dz\,ddy - dy\,ddz)}{ds^2},$$
$$R\,dx - P\,dz = -\frac{T(dx\,ddz - dz\,ddx)}{ds^2},$$

quarum formularum quadrata inuicem addita praebebunt:

$$\Pi\Pi\,ds^2 = \frac{TT}{ds^4}\left\{\begin{array}{l} +(dy\,ddx - dx\,ddy)^2 \\ +(dz\,ddy - dy\,ddz)^2 \\ +(dx\,ddz - dz\,ddx)^2 \end{array}\right\}.$$

§. 14. Quod si autem radius osculi curuae in puncto Z dicatur r, qui cadit in planum duorum elementorum contiguorum, alia occasione ostensum est esse

$$r = \frac{ds^3}{\sqrt{(dy\,ddx - dx\,ddy)^2 + (dz\,ddy - dy\,ddz)^2 + (dx\,ddz - dz\,ddx)^2}}$$

ergo radio osculi introducto erit $\Pi\Pi\,ds^2 = \frac{TT\,ds^2}{rr}$, ideoque $\Pi = \frac{T}{r}$, ita vt hoc modo simplicissime vis normalis ex sola tensione et radio osculi definiatur, dum vis tangentialis, vti ante inuenimus, est $\Theta = -\frac{dT}{ds}$.

§. 15. Quando ergo agitur de aequilibrio fili perfecte flexilis, cui in singulis elementis ds applicatae sint binae vires, altera tangentialis $= \Theta\,ds$, altera normalis $= \Pi\,ds$, quarum quidem haec cadat in planum in quo filum hoc loco incuruatur, praeterea vero tensio in hoc loco ponatur $= T$; tum status aequilibrii his duabus aequationibus determinatur; 1°). $T = \Pi r$ et 2°.) $dT = -\Theta ds$. Quare cum ex priore sit $dT = \Pi\,dr + r\,d\Pi$, elisa tensione T pro aequilibrio habebitur haec vnica aequatio: $r\,d\Pi + \Pi\,dr + \Theta\,ds = 0$; vnde si ambae vires Π et Θ per elementa curuae fuerint datae, haec aequatio naturam curuae determinabit. Sin autem ipsa curua fuerit data

cum

cum altera harum virium, inde altera reperietur. Veluti si data sit vis Π, erit

$$\Theta = -\frac{r\,d\Pi - \Pi\,dr}{ds} = -d.\frac{\Pi r}{ds};$$

sin autem altera Θ fuerit data, tum erit $\Pi = -\int\frac{\Theta\,ds}{r}$.

§. 16. Hinc iam facillime problema non ita pridem tractatum, quo quaerebatur status aequilibrii funis corpori cylindrico circumplicati et vtrinque a viribus tensi, multo generalius resolui poterit. Quando scilicet totus funis non in eodem plano versatur, atque etiam corpori cuiuscunque figurae fuerit circumuolutus, ita vt figura funis sit cognita; tum si frictio se habeat ad pressionem vt λ ad 1, ita vt sit $\Theta = \lambda\,\Pi$, prior aequatio statim praebet $r\,d\Pi + \Pi\,dr + \lambda\,\Pi\,ds = 0$, vnde fit

$$\frac{d\Pi}{\Pi} + \frac{dr}{r} + \frac{\lambda\,ds}{r} = 0,$$

vbi cum $\frac{ds}{r}$ denotet angulum elementarem, quo bina elementa proxima incuruantur, si ponamus $\int\frac{ds}{r} = \Phi$, habebimus integrando $\Pi\,r\,e^{\lambda\Phi} = C$, ideoque $\Pi = \frac{C\,e^{-\lambda\Phi}}{r}$, hinc $\Theta = \frac{\lambda\,C\,e^{-\lambda\Phi}}{r}$ et ipsa funis tensio $T = C e^{\lambda - \Phi}$. Quod si ergo in altero termino, vbi $\Phi = 0$, vis tendens fuerit $= M$, in altero autem termino, vbi $\Phi = \theta$, vis tendens $= N$: habebimus $M = C$, deinde $N = C\,e^{-\lambda\theta}$; vnde patet constantem C esse debere $= M$, ita vt $N = M\,e^{-\lambda\theta}$. Haec ergo solutio non solum conuenit cum ea quae nuper est inventa, sed etiam multo patet latius.

§. 17.

§. 17. Quando autem funis corpori cuiuscunque figurae circumplicatur, cuius quidem ſuperficies ſit laeuigata, funis tenſus ſuper ea ad aliam figuram ſe componere nequit, niſi quae ſit breuiſſima inter ſuos terminos. Ne igitur opus ſit hanc limitationem adiungere, res ita concipi poteſt, quaſi funis per canalem in ſuperficie excavatum circumduceretur, cui proinde figuram quamcunque tribuere licebit. Praeterea vero ſi totus funis in eodem plano eſſet applicatus, tum $\int \frac{ds}{r} = \Phi$ deſignaret amplitudinem curuae a fune formatae. At ſi funis non in eodem plano applicetur, idea amplitudinis quodammodo ceſſat: interim tamen Φ denotabit ſummam omnium angulorum elementarium $\frac{ds}{r}$, etiamſi non in idem planum cadant. Caeterum hinc patet vt ante, ſi $\lambda \Pi$ totum effectum frictionis denotet preſſioni Π reſpondentem, tum aequilibrium tam diu ſubſiſtere poſſe, quamdiu vis tendens maior M non maior fuerit quam $N e^{\lambda \Phi}$.

§. 18. Solutio autem tum tantum tam ſimplex euadit, quando vires ſollicitantes ſunt vel tangentiales vel normales. Quando autem hae vires ſecundum directiones ternorum axium OA, OB, OC agunt, quas litteris P, Q, R indicauimus, ita vt eae ſint functiones ternarum coordinatarum x, y, z; tum inueſtigatio curuae, quam hae vires filo flexili inducent, multo maiori laborat difficultate, quoniam aequationes primo inuentas a formulis integralibus liberari oportet. Interim tamen adhibendis ſequentibus artificiis negotium confici ſatis commode poterit, vbi quidem etiamnunc mentem ab elaſticitate abſtrahimus.

§. 19. Hic ſcilicet ſtatim in calculum introducemus ſequentes formulas analyticas:

$$\frac{dy\,ddx - dx\,ddy}{ds^3} = r,$$
$$\frac{dz\,ddy - dy\,ddz}{ds^3} = p,$$
$$\frac{dx\,ddz - dz\,ddx}{ds^3} = q,$$

quae litterae cum ſupra adhibitis non ſunt confundendae. Hinc igitur ſtatim radius oſculi curuae in puncto Z euadet $\frac{1}{\sqrt{pp+qq+rr}}$; deinde vero habebimus vt ſequitur:

$$\text{I. } d.\frac{dy}{dx} = -\frac{r\,ds^3}{dx^2}; \quad \text{II. } d.\frac{dx}{dy} = +\frac{r\,ds^3}{dy^2};$$
$$\text{III. } d.\frac{dz}{dy} = -\frac{p\,ds^3}{dy^2}; \quad \text{IV. } d.\frac{dy}{dz} = +\frac{p\,ds^3}{dz^2};$$
$$\text{V. } d.\frac{dx}{dz} = -\frac{q\,ds^3}{dz^2}; \quad \text{VI. } d.\frac{dz}{dx} = +\frac{q\,ds^3}{dx^2}.$$

Caeterum hae tres quantitates p, q, r a ſe inuicem ita pendent, vt ſit $p\,dx + q\,dy + r\,dz = 0$.

§. 20. His conſtitutis cum prima aequatio fuerit $dy\int P\,ds = dx\int Q\,ds$, erit $\frac{dy}{dx}\int P\,ds = \int Q\,ds$ et differentiando

$$-\frac{r\,ds^3}{dx^2}\int P\,ds + \frac{dy}{dx}P\,ds = Q\,ds,$$

vnde colligimus

$$\int P\,ds = \frac{-Q\,dx^2 + P\,dx\,dy}{r\,ds^2}.$$

Simili vero modo ſi aequationem $\int P\,ds = \frac{dx}{dy}\int Q\,ds$ differentiemus, prodibit

$$P\,ds = \frac{r\,ds^3}{dy^2}\int Q\,ds + \frac{Q\,dx\,ds}{dy};$$

vnde nanciſcimur

$$\int Q\,ds = \frac{P\,dy^2 - Q\,dx\,dy}{r\,ds^2},$$

ſimili modo ex reliquis aequationibus deducemus

$$\int Q\,ds = -\frac{R\,dy^2 + Q\,dy\,dz}{p\,ds^2}; \quad \int R\,ds = \frac{Q\,dz^2 - R\,dy\,dz}{p\,ds^2};$$
$$\int R\,ds = -\frac{P\,dz^2 + R\,dz\,dx}{q\,ds^2}; \quad \int P\,ds = \frac{R\,dx^2 - P\,dx\,dz}{q\,ds^2}.$$

§. 21. Quoniam igitur pro qualibet formula integrali geminos adepti sumus valores, iis inter se aequatis impetrabimus sequentes aequationes:

$$\frac{Pdy-Qdx}{r}=\frac{Rdx-Pdz}{q};$$
$$\frac{Qdx-Pdy}{r}=\frac{Rdy-Qdz}{p};$$
$$\frac{Rdy-Qdz}{p}=\frac{Pdz-Rdx}{q}.$$

quae omnes tres, ob $pdx+qdy+rdz=0$, reuocantur ad hanc vnicam: $Pp+Qq+Rr=0$. In hac ergo aequatione iam insignis status aequilibrii continetur proprietas; neque tamen ea status aequilibrii penitus exhauritur, sed insuper alia aequatione est opus ad solutionem completam perficiendam.

§. 22. Hunc in finem accipiatur pro lubitu unus valor cuiuspiam formulae integralis, ac prima quidem erat $\int P\,ds=\frac{dx(Pdy-Qdx)}{rds^2}$, quae denuo differentiata sumto elemento ds constante praebet

$$P\,ds=\frac{ddx(Pdy-Qdx)}{rds^2}+\frac{dx(Pddy-Qddx)}{rds^2}$$
$$+\frac{dx(dPdy-dQdx)}{rds^2}-\frac{dxdr(Pdy-Qdx)}{rrds^2},$$

quae per $rrds^2$ multiplicata ac omnibus terminis ad eandem partem translatis abit in hanc formam:

$$0=-Prrds^3+rddx(Pdy-Qdx)+rdx(Pddy-Qddx)$$
$$+rdx(dPdy-dQdx)-drdx(Pdy-Qdx).$$

In ista aequatione littera P ducitur in hanc formulam:

$$-rrds^3+r(dyddx+dxddy)-drdxdy,$$

quae loco rds^3 substituto valore fit $2rdxddy-drdxdy$; at vero littera Q ducitur in $drdx^2-2rdxddx$, vnde tota haec aequatio per dx diuisa erit

P(2r

$$P(2rddy-drdy)-Q(2rddx-drdx)+r(dP\,dy-dQ\,dx)=0$$

ex qua colligitur

$$\frac{dr}{r}=\frac{2Q\,ddx-2P\,ddy-dP\,dy+dQ\,dx}{Q\,dx-P\,dy},$$

ſiue commodius

$$\frac{dr}{r}=\frac{2Q\,ddx+dQ\,dx-2P\,ddy-dP\,dy}{Q\,dx-P\,dy},$$

quae eadem aequatio etiam ex ſecunda formula $\int Q\,ds$ elicitur. Eodem igitur modo ex reliquis obtinebitur

$$\frac{dp}{p}=\frac{2R\,ddy+dR\,dy-2Q\,ddz-dQ\,dz}{R\,dy-Q\,dz},$$

$$\frac{dq}{q}=\frac{2P\,ddz+dP\,dz-2R\,ddx-dR\,dx}{P\,dz-R\,dx}.$$

Quoniam autem vnica tantum aequatio, praeter inuentam $Pp+Qq+Rr=0$, ad aequilibrium determinandum requiritur, neceſſe eſt has tres aequationes modo inuentas ad vnicam reduci poſſe, ſi quidem in ſubſidium vocetur formula $p\,dx+q\,dy+r\,dz=0$.

§. 23. Quo haec clarius perſpiciantur in calculum introducamus tenſionem fili T, quandoquidem per eam ternas noſtras formulas integrales concinne expreſſas inuenimus: erat autem

$$\int P\,ds=-\frac{T\,dx}{ds};\ \int Q\,ds=-\frac{T\,dy}{ds};\ \int R\,ds=-\frac{T\,dz}{ds};$$

vnde differentiando elicimus

$$P=-\frac{dT\,dx-T\,ddx}{ds^2};\ Q=-\frac{dT\,dy-T\,ddy}{ds^2};\ R=-\frac{dT\,dz-T\,ddz}{ds^2},$$

vnde colligimus pro prima noſtra aequatione

$$Q\,dx-P\,dy=T\,r\,ds \text{ et } Q\,ddx-P\,ddy=-2r\,ds\,dt;$$

deinde vero hinc erit porro

$$dP=-\frac{ddT\,dx-2dT\,ddx-T\,d^3x}{ds^2} \text{ et } dQ=-\frac{ddT\,dy-2dT\,ddy-T\,d^3y}{ds^2};$$

vnde

vnde fit

$$dQ\,dx - dP\,dy = 2\,r\,ds\,dT + \frac{T(dy\,d^3x - dx\,d^3y)}{ds^2},$$

quibus valoribus substitutis prima aequatio fiet $\frac{dr}{r} - \frac{(dy\,d^3x - dx\,d^3y)}{r\,ds^3}$, et cum sit $dy\,ddx - dx\,ddy = r\,ds^3$, hinc fiet differentiando $dy\,d^3x - dx\,d^3y = dr\,ds^3$; vnde patet aequationem nostram fieri identicam, id quod mirum non est, cum introducta tensione T ipsae aequationes primitiuae

$$dy\int P\,ds = dx\int Q\,ds,$$
$$dz\int Q\,ds = dy\int R\,ds,$$
$$dx\int R\,ds = dz\int P\,ds,$$

iam sint identicae. Interim tamen quaelibet trium aequationum inuentarum continet vnam determinationem necessariam pro statu aequilibrii.

§. 24. Optime autem haec perspicientur, si quaestionem ad casum quendam particularem accommodemus. Consideremus igitur filum cylindro circulari ita circumuolutum, vt in eius superficie lineam breuissimam exhibeat, quam ergo figuram tanquam datam spectemus, et quaeramus vires P, Q, R, quae filo in singulis elementis applicatae ipsi hanc ipsam figuram inducere valeant. Sit radius istius cylindri $= 1$, et notum est pro quauis fili portione s coordinatas ita determinari, vt sit $x = \alpha \operatorname{cos}. s$, $y = \alpha \operatorname{sin}. s$ et $z = n\,s$, existente $\alpha = \sqrt{(1 - nn)}$ ideoque n fractione vnitate minore. Hinc ergo sumto elemento ds constante erit

$$dx = -\alpha\,ds \operatorname{sin}. s;\quad dy = \alpha\,ds \operatorname{cos}. s;\quad dz = n\,ds;$$

tum vero porro

$$ddx = -\alpha\,ds^2 \operatorname{cos}. s;\quad ddy = -\alpha\,ds^2 \operatorname{sin}. s;\quad ddz = 0;$$

 vnde

vnde pro p, q, r sequentes nanciscimur valores:

$$p = -\alpha n \sin. s;\ q = \alpha n \cos. s;\ r = -\alpha\alpha;$$

ex quibus valoribus vtique erit $p\,dx + q\,dy + r\,dz = 0$. Tum vero pro statu aequilibrii aequatio primo inuenta praebet hanc determinationem: $Pp + Qq + Rr = 0$, quae ergo fit $-\alpha n P \sin. s + \alpha n Q \cos. s - \alpha\alpha R = 0$. Altera vero conditio petatur ex aequatione nostra prima $\frac{dr}{r}$, vnde fiet

$$0 = \frac{2Q\,ds \cos. s + dQ \sin. s - 2P\,ds \sin. s + dP \cos. s}{Q \sin. s + P \cos. s},$$

at vero duae reliquae aequationes dabunt

$$\frac{ds \cos. s}{\sin. s} = -\frac{2\alpha R\,ds \sin. s + \alpha\,dR \cos. s - n\,dQ}{\alpha R \cos. s - nQ},$$

$$-\frac{ds \sin. s}{\cos. s} = \frac{2\alpha R\,ds \cos. s + \alpha\,dR \sin. s + n\,dP}{nP + \alpha R \sin. s}.$$

§. 25. Hae autem tres postremae aequationes vnicam continent determinationem, ad quod ostendendum eas primo ad formam simplicissimam reuocemus, eritque:

$$\text{I. } 0 = 2Q\,ds \cos. s + dQ \sin. s - 2P\,ds \sin. s + dP \cos. s;$$

$$\text{II. } \alpha R\,ds \cos. s^2 - nQ\,ds \cos. s = -2\alpha R\,ds \sin. s^2 + \alpha\,dR \sin. s \cos. s - n\,dQ \sin. s,$$

siue

$$\text{II. } 0 = -\alpha R\,ds\,(1 + \sin. s^2) + nQ\,ds \cos. s + \alpha\,dR \sin. s \cos. s - n\,dQ \sin. s$$

$$\text{III. } -nP\,ds \sin. s - \alpha R\,ds \sin. s^2 = 2\alpha R\,ds \cos. s^2 + \alpha\,dR \sin. s \cos. s + n\,dP \cos. s,$$

siue

$$\text{III. } 0 = \alpha R\,ds\,(1 + \cos. s^2) + nP\,ds \sin. s + \alpha\,dR \sin. s \cos. s + n\,dP \cos. s;$$

et

et quoniam prima conditio dedit

$$R = \frac{n\,Q\,\text{cof.}\,s - n\,P\,\text{fin.}\,s}{\alpha}, \text{ erit}$$

$$dR = \frac{n\,d\,Q\,\text{cof.}\,s - n\,Q\,d\,s\,\text{fin.}\,s - n\,d\,P\,\text{fin.}\,s - n\,P\,d\,s\,\text{cof.}\,s}{\alpha}.$$

Iam hi valores in fecunda et tertia aequatione fubftituti perducent ad ipfam aequationem primam; vnde patet has tres aequationes vnicae tantum aequiualere.

§. 26. Sufficit igitur primam euoluiffe, quae eft

$$0 = 2\,Q\,d\,s\,\text{cof.}\,s + d\,Q\,\text{fin.}\,s - 2\,P\,d\,s\,\text{fin.}\,s + d\,P\,\text{cof.}\,s,$$

vnde patet alteram quantitatum P et Q pro lubitu accipi poffe. Spectemus ergo quantitatem P tanquam cognitam et alteram Q quaeramus ex hac aequatione:

$$d\,Q\,\text{fin.}\,s + 2\,Q\,d\,s\,\text{cof.}\,s = 2\,P\,d\,s\,\text{fin.}\,s - d\,P\,\text{cof.}\,s,$$

quae per fin. s multiplicata integretur, vnde prodit

$$Q\,\text{fin.}\,s^2 = 2\int P\,d\,s\,\text{fin.}\,s^2 - \int d\,P\,\text{fin.}\,s\,\text{cof.}\,s,$$

eft autem per reductionem notiffimam:

$$\int d\,P\,\text{fin.}\,s\,\text{cof.}\,s = P\,\text{fin.}\,s\,\text{cof.}\,s - \int P\,d\,s\,(\text{cof.}\,s^2 - \text{fin.}\,s^2),$$

vnde erit

$$Q\,\text{fin.}\,s^2 = -P\,\text{fin.}\,s\,\text{cof.}\,s + \int P\,d\,s\,(\text{cof.}\,s^2 + \text{fin.}\,s^2)$$
$$= -P\,\text{fin.}\,s\,\text{cof.}\,s + \int P\,d\,s,$$

ita vt fit $Q = -\frac{P\,\text{cof.}\,s}{\text{fin.}\,s} + \frac{\int P\,d\,s}{\text{fin.}\,s^2}$, hincque porro

$$R = -\frac{n\,P}{\alpha\,\text{fin.}\,s} + n\,\text{cof.}\,s\,\frac{\int P\,d\,s}{\alpha\,\text{fin.}\,s^2},$$

ficque ex affumta quantitate P binae reliquae Q et R determinantur; vnde patet filum infinitis modis in hoc fitu in aequilibrio confiftere poffe.

§. 27. Quando autem vna trium virium P, Q et R vt cognita fpectatur, binae reliquae multo facilius ex ea

defi-

definire possunt per geminos valores quos ante pro formulis $\int P\,ds$, $\int Q\,ds$ et $\int R\,ds$ inuenimus; neque opus est ad vlteriorem differentiationem descendere. Ita si vis P fuerit data, tum gemini valores pro $\int P\,ds$ dati hoc casu euadent

$$\int P\,ds = Q\,\mathrm{sin.}\,s^2 + P\,\mathrm{sin.}\,s\,\mathrm{cos.}\,s = \frac{\alpha R\,\mathrm{sin.}\,s^2 + nP\,\mathrm{sin.}\,s}{n\,\mathrm{cos.}\,s},$$

vnde colligimus vt ante

$$Q = \frac{\int P\,ds - P\,\mathrm{sin.}\,s\,\mathrm{cos.}\,s}{\mathrm{sin.}\,s^2} = \frac{\int P\,ds}{\mathrm{sin.}\,s^2} - \frac{P\,\mathrm{cos.}\,s}{\mathrm{sin.}\,s} \text{ et}$$

$$R = \frac{n\,\mathrm{cos.}\,s\int P\,ds}{\alpha\,\mathrm{sin.}\,s^2} - \frac{nP}{\alpha\,\mathrm{sin.}\,s},$$

Simili modo si vim R pro data accipiamus, bini valores pro $\int R\,ds$ dati fient

$$\int R\,ds = \frac{-nQ}{\alpha\,\mathrm{sin.}\,s} + \frac{R\,\mathrm{cos.}\,s}{\mathrm{sin.}\,s} = \frac{-nP}{\alpha\,\mathrm{cos.}\,s} - \frac{R\,\mathrm{sin.}\,s}{\mathrm{cos.}\,s},$$

vnde reperitur:

$$P = \frac{\alpha R\,\mathrm{sin.}\,s}{n} - \frac{\alpha\,\mathrm{cos.}\,s}{n}\int R\,ds \text{ et } Q = \frac{\alpha R\,\mathrm{cos.}\,s}{n} - \frac{\alpha\,\mathrm{sin.}\,s\int R\,ds}{n},$$

vnde si fuerit, $R = 1$, quod euenit si axis cylindri sit verticalis et singula funis elementa a grauitate vrgeantur, tum binae reliquae vires requisitae ad aequilibrium erunt

$$P = \frac{\alpha\,\mathrm{sin.}\,s}{n} - \frac{\alpha(s+a)\,\mathrm{cos.}\,s}{n} \text{ et } Q = \frac{\alpha\,\mathrm{cos.}\,s}{n} - \frac{\alpha(s+a)\,\mathrm{sin.}\,s}{n}.$$

Ex quo patet, quaestionem etiamnunc esse indeterminatam, quoniam constantem a pro lubitu accipere licet.

§. 28. Quin etiam tensionem funis in singulis punctis pro arbitrio sumere licet; tum enim pro nostro casu, quem hic tractamus, formulae supra (§. 9.) exhibitae erunt

$$\int P\,ds = \alpha T\,\mathrm{sin.}\,s;\ \int Q\,ds = -\alpha T\,\mathrm{cos.}\,s;\ \int R\,ds = -nT,$$

vnde ipsae vires erunt

P =

$$P = \alpha \frac{dT}{ds} \sin. s + \alpha T \cos. s;$$
$$Q = -\alpha \frac{dT}{ds} \cos. s + \alpha T \sin. s;$$
$$R = -\frac{n\,dT}{ds},$$

quo ergo casu omnes tres vires perfecte determinantur; vnde si tensio vbique debeat esse constans, siue $T = D$, nostrae vires erunt $P = \alpha D \cos. s$; $Q = \alpha D \cos. s$; $R = o$, atque hinc manifestum est quemadmodum omnes huiusmodi quaestiones tractari conueniat.

De aequilibrio filorum elasticorum a viribus quibuscunque sollicitatorum.

§. 29. Formulae generales pro filis elasticis datae multo commodiores reddentur introductione litterarum p, q, r, quarum valores erant positi

$$q = \frac{dz\,ddy - dy\,ddz}{ds^3};\ q = \frac{dx\,ddz - dz\,ddx}{ds^3};\ r = \frac{dy\,ddx - dx\,ddy}{ds^3};$$

circa quas formulas iam supra notauimus fore $p\,dx + q\,dy + r\,dz = o$. Praeterea vero quoque euidens est fore $p\,ddx + q\,ddy + r\,ddz = o$; vnde ex differentiatione prioris aequalitatis oritur $dp\,dx + dq\,dy + dr\,dz = o$, id quod etiam inde patet, quod sumto elemento ds constante fit

$$dp = \frac{dz\,d^3y - dy\,d^3z}{ds^3};$$
$$dq = \frac{dx\,d^3z - dz\,d^3x}{ds^3};$$
$$dr = \frac{dy\,d^3x - dx\,d^3y}{ds^3};$$

hinc autem insuper sequitur fore

$$dp\,d^3x + dq\,d^3y + dr\,d^3z = o.$$

Has igitur insignes relationes inter quantitates p, q, r probe notasse iuuabit.

§. 30.

§. 30. Praeterea vero etiam formula irrationalis $\sqrt{(pp+qq+rr)}$ memorabilem reductionem suppeditat: fiet enim

$$ds^6(pp+qq+rr)=(dy^2+dz^2)ddx^2+(dx^2+dz^2)ddy^2+(dx^2+dy^2)+ddz^2$$
$$-2dxdyddxddy-2dxdzddxddz-2dydzddyddz,$$

cum igitur $dx^2+dy^2+dz^2=ds^2$, haec formula transformabitur in sequentem:

$$pp+qq+rr=\frac{ddx^2+ddy^2+ddz^2}{ds^4}-\frac{dx^2ddx^2-dy^2ddy^2-dz^2ddz^2}{ds^6}$$
$$-\frac{2dxdyddxddy-2dxdzddxddz-2dydzddyddz}{ds^6},$$

vbi membra negatiua manifesto continent quadratum formulae $dxddx+dyddy+dzddz$. Est vero

$$dxddx+dyddy+dzddz=dsdds,$$

nullo scilicet elemento pro constante assumto, hoc ergo valore substituto nascetur haec expressio satis concinna:

$$pp+qq+rr=\frac{ddx^2+ddy^2+ddz^2-dds^2}{ds^4};$$

quamobrem radius osculi curuae ad Z generalissime erit $\frac{ds^2}{\sqrt{(ddx^2+ddy^2+ddz^2-dds^2)}}$, vbi quodlibet elementum pro lubitu vt constans accipi poterit, ita vt valor expressionis hinc neutiquam pendeat, quandoquidem quouis casu differentialia secundi gradus sponte se mutuo tollunt. Veluti in casu ante tractato, vbi erat

$$dx=-\alpha ds\,\text{sin.}\,s;\; dy=\alpha ds\,\text{cos.}\,s;\; dz=nds;$$

existente $\alpha=\sqrt{(1-nn)}$, etiam elemento ds variabili sumto, fiet

$$ddx=-\alpha dds\,\text{sin.}\,s-\alpha ds^2\,\text{cos.}\,s;$$
$$ddy=\alpha dds\,\text{cos.}\,s-\alpha ds^2\,\text{sin.}\,s;$$
$$ddz=ndds,$$

quibus valoribus sustitutis reperietur:

ddx

$$d\,d\,x^2 + d\,d\,y^2 + d\,d\,z^2 - d\,d\,s^2 = \alpha\,\alpha\,d\,s^4;$$

ſicque radius oſculi erit $\frac{1}{\alpha}$.

§. 31. His praenotatis, ſi G exprimat elaſticitatem abſolutam fili in ſingulis punctis, quam hic tanquam conſtantem aſſumemus, ita vt filum vbique vniformiter craſſum ſit concipiendum; tum aequationes pro aequilibrio erunt

$$\text{I. } \int dy \int P\,ds - \int dx \int Q\,ds = G\,r.$$
$$\text{II. } \int dz \int Q\,ds - \int dy \int R\,ds = G\,p.$$
$$\text{III. } \int dx \int R\,ds - \int dz \int P\,ds = G\,q,$$

quae differentiatae praebent

$$dy \int P\,ds - dx \int Q\,ds = G\,dr,$$
$$dz \int Q\,ds - dy \int R\,ds = G\,dp,$$
$$dx \int R\,ds - dz \int P\,ds = G\,dq.$$

§. 32. Hinc primo tam $\int Q\,ds$ quam $\int R\,ds$ per $\int P\,ds$ exprimamus eritque

$$\int Q\,ds = \frac{dy}{dx}\int P\,ds - \frac{G\,dr}{dx} \text{ et } \int R\,ds = \frac{dz}{dx}\int P\,ds + \frac{G\,dq}{dx},$$

qui valores in formula pro tenſione T data ſubſtituti praebent hanc expreſſionem:

$$T = -\frac{dx}{ds}\int P\,ds - \frac{dy^2}{dx\,ds}\int P\,ds - \frac{dz^2}{dx\,ds}\int P\,ds - \frac{G\,dr\,dy}{dx\,ds} - \frac{G\,dq\,dz}{dx\,ds},$$

quae reducitur ad hanc formam:

$$T = -\frac{ds}{dx}\int P\,ds + \frac{G(dr\,dy - dq\,dz)}{dx\,ds},$$

vnde colligimus:

$$\int P\,ds = \frac{G(dr\,dy - dq\,dz)}{ds^2} - \frac{T\,dx}{ds}.$$

Similique modo reperietur:

$$\int Q\,ds = \frac{G(dp\,dz - dr\,dx)}{ds^2} - \frac{T\,dy}{ds},$$

$$\int R\,ds = \frac{G(dq\,dx - dp\,dy)}{ds^2} - \frac{T\,dz}{ds}.$$

Hinc autem differentiando porro fiet

$$P = \frac{G(dr\,ddy + dy\,ddr - dq\,ddz - dz\,ddq)}{ds^3} - \frac{T\,ddx - dT\,dx}{ds^2},$$

$$Q = \frac{G(dp\,ddz + dz\,ddp - dr\,ddx - dx\,ddr)}{ds^3} - \frac{T\,ddy - dT\,dy}{ds^3},$$

$$R = \frac{G(dq\,ddx + dx\,ddq - dp\,ddy - dy\,ddp)}{ds^3} - \frac{T\,ddz - dT\,dz}{ds^2}.$$

§. 33. Cum nunc ſit vis tangentialis, vt ſupra vidimus $= P\,dx + Q\,dy + R\,dz$, quam per $\Theta\,ds$ deſignauimus, erit

$$\Theta\,ds = -G(p\,dp + q\,dq + r\,dr) - dT,$$

quare cum ſit radius osculi curuae $= \frac{1}{\sqrt{(pp + qq + rr)}}$, ſi eum vocemus $= \varrho$, erit $pp + qq + rr = \frac{1}{\varrho\varrho}$, hincque

$$p\,dp + q\,dq + r\,dr = -\frac{d\varrho}{\varrho^3},$$

quo valore ſubſtituto fiet $\Theta\,ds = \frac{G\,d\varrho}{\varrho^3} - dT$.

Quod ſi porro vim normalem pro elemento ds ſtatuamus $= \Pi\,ds$, ſupra iam vidimus fore

$$\Pi\,ds = \sqrt{((P\,dy - Q\,dx)^2 + (Q\,dz - R\,dy)^2 + (R\,dx - P\,dz)^2)},$$

vbi ſi loco P, Q, R valores ſupra exhibitos ſubſtituere vellemus, in calculos inextricabiles delaberemur.

§. 34. Quoniam modo inuenimus

$$\int Q\,ds = \frac{dy}{dx}\int P\,ds - \frac{G\,dr}{dx},$$

erit differentiando

$$Q\,ds = -\frac{r\,ds^3}{dx^2}\int P\,ds + \frac{P\,ds\,dy}{dx} - \frac{G\,ddr}{dx} + \frac{G\,dr\,ddx}{dx^2},$$

vnde colligitur

$$\int \mathrm{P}\,ds = \frac{\mathrm{P}\,dx\,dy}{r\,ds^2} - \frac{\mathrm{Q}}{r}\,\frac{dx^2}{ds^2} + \frac{\mathrm{G}\,(dr\,ddx - dx\,ddr)}{r\,ds^3},$$ siue

$$\int \mathrm{P}\,ds = \frac{dx\,(\mathrm{P}\,dy - \mathrm{Q}\,dx)}{r\,ds^2} + \frac{\mathrm{G}\,(dr\,ddx - dx\,ddr)}{r\,ds^3},$$

hinc autem porro erit

$$\int \mathrm{Q}\,ds = \frac{dy\,(\mathrm{P}\,dy - \mathrm{Q}\,dx)}{r\,ds^2} - \frac{\mathrm{G}\,(dy\,ddr - dr\,ddy)}{r\,ds^3},$$

ex quibus per analogiam concludimus:

$$\int \mathrm{Q}\,ds = \frac{dy\,(\mathrm{Q}\,dz - \mathrm{R}\,dy)}{p\,ds^2} + \frac{\mathrm{G}\,(dp\,ddy - dy\,ddp)}{p\,ds^3},$$

$$\int \mathrm{R}\,ds = \frac{dz\,(\mathrm{Q}\,dz - \mathrm{R}\,dy)}{p\,ds^2} - \frac{\mathrm{G}\,(dz\,ddp - dp\,ddz)}{p\,ds^3},$$

$$\int \mathrm{R}\,ds = \frac{dz\,(\mathrm{R}\,dx - \mathrm{P}\,dz)}{q\,ds^2} + \frac{\mathrm{G}\,(dq\,ddz - dz\,ddq)}{q\,ds^3},$$

$$\int \mathrm{P}\,ds = \frac{px\,(\mathrm{R}\,dx - \mathrm{P}\,dz)}{q\,ds^2} - \frac{\mathrm{G}\,(dx\,ddq - dq\,ddx)}{q\,ds^2},$$

hocque modo pro qualibet formula integrali binos nacti sumus valores, qui inter se aequati praebent aequationem a signo integrali liberam. Hinc igitur satis patet, determinationem aequilibrii pro filis elasticis, quae a viribus quibuscunque sollicitantur, ad calculos non parum molestos perducere, ita vt in genere vix quicquam vlterius praestare liceat quam hic est factum. Quamobrem pro quouis casu oblato ac determinato dispiciendum erit, quomodo ope formularum hic traditarum commodissime ad solutionem perueniri queat. Ceterum iam obseruauimus motum huiusmodi filorum ne suscipi quidem posse, antequam insignia incrementa calculi fuerint detecta.