INVESTIGATIO CVRVARVM QVAE SIMILES SINT SVIS EVOLVTIS VEL PRIMIS, VEL SECVNDIS, VEL TERTIIS, VEL ADEO ORDINIS CVIVSCVNQVE.

Auctore
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§. 1.

Sit $a\,s$ curua quaesita ad axem $a\,r$ relata, qui ad curuam in a sit normalis, & statuatur curuae in s radius osculi $s\,s'$, erit s' punctum in euoluta prima, quae sit $a'\,s'$ et referatur ad axem $a'\,r'$ priori $a\,r$ normalem. Tum pro hac euoluta prima $a'\,s'$ sit $s'\,s''$ radius osculi in puncto s', erit s'' punctum in euoluta secunda $a''\,s''$, quae referatur ad axem $a''\,r''$ priori $a'\,r'$ normalem ideoque parallelum axi primo $a\,r$. Simili modo euolutae huius secundae $a''\,s''$ sit in puncto s'' radius osculi $s''\,s'''$, erit s''' punctum in euoluta tertia, quae referatur ad axem $a'''\,r'''$; hocque modo, quovsque libuerit, progredi licet. Hinc igitur primo ex natura euolutionis erit radius osculi $s\,s' = a\,a' + a'\,s'$; eodemque modo radius osculi $s'\,s'' = a'\,a'' + a''\,s''$; porro radius osculi $s''\,s''' = a''\,a''' + a'''\,s'''$; etc. Deinde quia singuli radii osculi sunt normales ad curuas, ad quas pertinent, sequentes autem tangunt: manifestum est, omnes angulos $a\,r\,s$, $a'\,r'\,s'$, $a''\,r''\,s''$, Tab. II. Fig. 1.

$a''' r''' s'''$, etc. esse inter se aequales: sunt vero isti anguli amplitudines arcuum $a s$, $a' s'$, $a'' s''$, $a''' s'''$ etc., vnde patet, omnes istos arcus sibi inuicem respondentes etiam esse aeque amplos.

§. 2. Cum igitur omnes arcus $a s$, $a' s'$, $a'' s''$, etc., sint aeque ampli, ponatur ista amplitudo $= \Phi$, cui ergo aequales erunt anguli $a' r' s'$, $a'' r'' s''$, $a''' r''' s'''$. Iam pro ipsa curua quaesita $a s$ vocetur arcus $a s = s$ et radius osculi $s s' = r$; tum vero pro prima euoluta $a' s'$ sit arcus $a' s' = s'$ et radius $s' s'' = r'$; similique modo pro euoluta secunda sit arcus $a'' s'' = s''$ et radius osculi $s'' s''' = r''$; eodemque modo denominationes fiant pro omnibus sequentibus euolutis. Praeterea vero ponantur interualla constantia $a a' = a$; $a' a'' = a'$; $a'' a''' = a''$; etc. quae simul radios osculi exhibent curuarum in punctis a', a'', a''' etc. Hinc igitur primo habebimus sequentes aequalitates:

$$r = a + s', \quad r' = a' + s'', \quad r'' = a'' + s''', \quad r''' = a''' + s'''', \text{ etc.}$$

vnde colliguntur sequentes valores:

$$s' = r - a, \quad s'' = r' - a', \quad s''' = r'' - a'', \quad s'''' = r''' - a''', \text{ etc.}$$

Tab. II. Fig. 2. §. 3. Cum nunc sit amplitudo arcus $a s$, seu angulus $a r s = \Phi$, ducto radio osculi proximo $\sigma \varrho \sigma'$, erit primo elementum $s \sigma = \partial s$ et angulus $a \varrho \sigma = \Phi + \partial \Phi$, vnde conficitur angulus $r s' \varrho = \partial \Phi$; hinc igitur fiet $\partial \Phi = \frac{\partial s}{r}$, ideoque $\partial s = r \partial \Phi$. Simili igitur modo pro curuis sequentibus erit $\partial s' = r' \partial \Phi$, $\partial s'' = r'' \partial \Phi$, $\partial s''' = r''' \partial \Phi$ etc. Ex superioribus autem fit $\partial s' = \partial r$, $\partial s'' = \partial r'$, $\partial s''' = \partial r''$, etc. quibus valoribus substitutis prodibunt sequentes aequationes:

$$\partial r = r' \partial \Phi; \quad \partial r' = r'' \partial \Phi; \quad \partial r'' = r''' \partial \Phi; \text{ etc.}$$

ex quibus sequuntur valores

$$r' = \frac{\partial r}{\partial \Phi}; \quad r'' = \frac{\partial r'}{\partial \Phi}; \quad r''' = \frac{\partial r''}{\partial \Phi}; \text{ etc.}$$

Quare

Quare ſi elementum $\partial \Phi$ pro conſtante accipiamus, omnes radios oſculi ſe inuicem inſequentes per differentialia primi radii oſculi r poterimus exprimere, quandoquidem erit

$$r' = \frac{\partial r}{\partial \Phi},\ r'' = \frac{\partial \partial r}{\partial \Phi^2},\ r''' = \frac{\partial^3 r}{\partial \Phi^3},\ r'''' = \frac{\partial^4 r}{\partial \Phi^4},\ \text{etc.}$$

§. 4. In genere igitur pro euoluta ordinis n erit radius oſculi $r^{(n)} = \frac{\partial^{(n)} r}{\partial \Phi^n}$; quamobrem ſi haec euoluta ſimilis eſſe debeat ipſi curuae quaeſitae, radius oſculi $r^{(n)}$ ſimili modo ſe habere debebit ad amplitudinem Φ, quo ſe habet r ad Φ, vnde cum amplitudo Φ vbique ſit eadem, neceſſe eſt vt ſit $r^{(n)} = C r$, vbi littera C inuoluit rationem ſimilitudinis, qua indicatur, quoties euoluta ordinis n maior minorue eſſe debeat quam ipſa curua quaeſita. Quoniam autem fieri poteſt vt euolutio in inuolutionem vertatur, his caſibus conſtanti C valorem negatiuum tribui conueniet; hanc ob rem, quo curua quaeſita ſimilis euadat ſuae euolutae ordinis n, ob $r^{(n)} = \pm C r$ erit aequatio pro curua quaeſita $C r = \frac{\partial^{(n)} r}{\partial \Phi^n}$, quae ergo aequatio plenam continet ſolutionem problematis propoſiti, totumque negotium redit ad reſolutionem huius aequationis differentialis ordinis n.

§. 5. Quoniam in hac aequatione quantitas r in vtroque termino vnicam habet dimenſionem, euidens eſt, ſi huic aequationi ſatisfaciant valores $r = P$, $r = Q$, $r = R$, etc. eidemque ſatisfacturum eſſe valorem $r = \alpha P + \beta Q + \gamma R$, ex qua conditione, poſtquam omnia integralia particularia fuerint inuenta, facili negotio colligetur integrale completum, quando ſcilicet numerus integralium particularium fuerit $= n$, quod ergo continebit relationem inter radium oculi r curuae quaeſitae

 eius-

eiusque amplitudinem Φ, ex qua quemadmodum aequationem inter coordinatas more solito elici oporteat deinceps sumus ostensuri.

§. 6. Quo igitur integralia particularia huius aequationis: $\frac{\partial^n r}{\partial \Phi^n} \pm C r = 0$ eruamus, facile patet, ei satisfacere huiusmodi valores: $r = A e^{\lambda \Phi}$, denotante e numerum cuius logarithmus hyperbolicus $= 1$; hinc enim erit vt sequitur:

$$\frac{\partial r}{\partial \Phi} = A \lambda e^{\lambda \Phi}; \quad \frac{\partial \partial r}{\partial \Phi^2} = A \lambda \lambda e^{\lambda \Phi}; \quad \frac{\partial^3 r}{\partial \Phi^3} = A \lambda^3 e^{\lambda \Phi}; \text{ etc.}$$

vnde in genere colligitur $\frac{\partial^n r}{\partial \Phi^n} = A \lambda^n e^{\lambda \Phi}$, quo valore substituto aequatio nostra euadit $A \lambda^n e^{\lambda \Phi} \pm C A e^{\lambda \Phi} = 0$, quae reducitur ad hanc formam: $\lambda^n \pm C = 0$, ex qua ergo aequatione omnes valores ipsius λ erui oportet; quae aequatio cum sit ordinis n, etiam totidem diuersos valores pro littera λ suppeditabit, quorum quilibet praebebit integrale particulare $r = A e^{\lambda \Phi}$. Hi ergo valores omnes in vnam summam collecti dabunt integrale completum.

§. 7. Cum igitur tota solutio ad hanc aequationem sit perducta: $\lambda^n \pm C = 0$, nihil aliud opus est, nisi vt huius aequationis radices siue reales siue imaginarii eruantur, id quod nulla amplius laborat difficultate, quod autem quo commodius fieri possit, loco C scribamus similem potestatem α^n, vt haec aequatio nobis sit resoluenda: $\lambda^n \pm \alpha^n = 0$. Nouimus autem formulae $\lambda^n - \alpha^n$ factorem trinomialem in genere esse

$$\lambda \lambda - 2 \alpha \lambda \cos. \tfrac{2 i \pi}{n} + \alpha \alpha,$$

alterius vero formae $\lambda^n + \alpha^n$ hunc fore factorem trinomialem:

$$\lambda \lambda - 2 \alpha \lambda \cos. \tfrac{(2 i + 1) \pi}{n} + \alpha \alpha.$$

Quod

Quodſi ergo breuitatis gratia ſcribamus ω, tam pro angulo $\frac{2i\pi}{n}$, quam pro $\frac{(2i+1)\pi}{n}$, vt habeamus hunc factorem: $\lambda\lambda - 2\alpha\lambda \cos.\omega + \alpha\alpha$, ex eo nihilo aequato colligitur

$$\lambda = \alpha(\cos.\omega \pm \sqrt{-1}\,\sin.\omega)$$

quae expreſſio totidem continet valores, quot numerus n habet vnitates.

§. 8. Hoc igitur valore pro λ in genere ſubſtituto aequatio pro curua quaeſita erit $r = A\, e^{\alpha\Phi\cos.\omega} \times e^{\pm\alpha\Phi\sqrt{-1}\sin.\omega}$, vbi factor poſtremus, in quo exponens eſt imaginarius, per notam reductionem, qua nouimus eſſe $e^{z\sqrt{-1}} = \cos.z + \sqrt{-1}\,\sin.z$, reducitur ad hanc formam:

$$\cos.\alpha\Phi\sin.\omega \pm \sqrt{-1}\,\sin.\alpha\Phi\sin.\omega,$$

ita vt in genere ſit

$$r = A\, e^{\alpha\Phi\cos.\omega}(\cos.\alpha\Phi\sin.\omega \pm \sqrt{-1}\,\sin.\alpha\Phi\sin.\omega).$$

§. 9. Quia haec formula duplicem inuoluit valorem, ob ſignum ambiguum, quo $\sqrt{-1}$ afficitur, mutato ſigno ſimili modo habebimus

$$r = B\, e^{\alpha\Phi\cos.\omega}(\cos.\alpha\Phi\sin.\omega \mp \sqrt{-1}\,\sin.\alpha\Phi\sin.\omega),$$

vnde ſi ponamus

$$A + B = \mathfrak{A} \text{ et } \pm A\sqrt{-1} \mp B\sqrt{-1} = \mathfrak{B},$$

erit ſublatis imaginariis

$$r = e^{\alpha\Phi\cos.\omega}(\mathfrak{A}\cos.\alpha\Phi\sin.\omega + \mathfrak{B}\sin.\alpha\Phi\sin.\omega).$$

Quoniam igitur pro ω ſemper habemus duas conſtantes arbitrarias $\mathfrak{A}$ et $\mathfrak{B}$, ex omnibus valoribus ipſius ω formabitur pro r expreſſio, quae continebit n conſtantes arbitrarias. At vero pro formula $\lambda^n - \alpha^n$ valores ipſius ω erunt ſequentes: $\frac{0\pi}{n}$, $\frac{2\pi}{n}$,

$\frac{4\pi}{n}$,

$\frac{4\pi}{n}$, $\frac{6\pi}{n}$, etc., pro altero autem caſu $\lambda^n + \alpha^n$ valores pro ω erunt $\frac{1\pi}{n}$, $\frac{3\pi}{n}$, $\frac{5\pi}{n}$, etc.

§. 10. Ponamus ad abbreuiandum $\alpha \,\text{coſ.}\, \omega = \zeta$ et $\alpha \,\text{ſin.}\, \omega = \eta$, vt ſit $\alpha\alpha = \zeta\zeta + \eta\eta$ et habebimus

$$r = e^{\zeta\Phi} (\mathfrak{A} \,\text{coſ.}\, \eta\Phi + \mathfrak{B} \,\text{ſin.}\, \eta\Phi).$$

Hinc iam poterimus etiam radios oſculi r', r'', r''', etc. pro ſingulis euolutis aſſignare. Cum enim ſit $r' = \frac{\partial r}{\partial \Phi}$, erit

$$r' = e^{\zeta\Phi} \begin{pmatrix} \mathfrak{A}\zeta \,\text{coſ.}\, \eta\Phi + \mathfrak{B}\zeta \,\text{ſin.}\, \eta\Phi \\ + \mathfrak{B}\eta \,\text{coſ.}\, \eta\Phi - \mathfrak{A}\eta \,\text{ſin.}\, \eta\Phi \end{pmatrix}$$

Quodſi igitur breuitatis gratia ponamus $\mathfrak{A}' = \mathfrak{A}\zeta + \mathfrak{B}\eta$ et $\mathfrak{B}' = \mathfrak{B}\zeta - \mathfrak{A}\eta$, habebimus

$$r' = e^{\zeta\Phi} (\mathfrak{A}' \,\text{coſ.}\, \eta\Phi + \mathfrak{B}' \,\text{ſin.}\, \eta\Phi).$$

Pro ſequentibus ponamus porro

$$\mathfrak{A}'' = \mathfrak{A}'\zeta + \mathfrak{B}'\eta = \mathfrak{A}(\zeta\zeta - \eta\eta) + 2\mathfrak{B}\zeta\eta \text{ et}$$

$$\mathfrak{B}'' = \mathfrak{B}'\zeta - \mathfrak{A}'\eta = \mathfrak{B}(\zeta\zeta - \eta\eta) - 2\mathfrak{A}\zeta\eta, \text{ erit}$$

$$r'' = \frac{\partial r'}{\partial \Phi} = e^{\zeta\Phi} (\mathfrak{A}'' \,\text{coſ.}\, \eta\Phi + \mathfrak{B}'' \,\text{ſin.}\, \eta\Phi).$$

Simili modo ponamus vlterius

$$\mathfrak{A}''' = \mathfrak{A}''\zeta + \mathfrak{B}''\eta = \mathfrak{A}(\zeta^3 - 3\zeta\eta\eta) + \mathfrak{B}(3\zeta\zeta\eta - \eta^3) \text{ et}$$

$$\mathfrak{B}''' = \mathfrak{B}''\zeta - \mathfrak{A}''\eta = \mathfrak{B}(\zeta^3 - 3\zeta\eta\eta) - \mathfrak{A}(3\zeta\zeta\eta - \eta^3) \text{ eritque}$$

$$r''' = e^{\zeta\Phi} (\mathfrak{A}''' \,\text{coſ.}\, \eta\Phi + \mathfrak{B}''' \,\text{ſin.}\, \eta\Phi),$$

ſimilique modo vlterius progredi licebit.

§. 11. Quo autem has formulas ad maiorem vniformitatem reducamus, reſtituamus loco ζ et η valores aſſumtos $\zeta = \alpha \,\text{coſ.}\, \omega$ et $\eta = \alpha \,\text{ſin.}\, \omega$, quo facto habebimus

$$\mathfrak{A}' = \alpha(\mathfrak{A} \,\text{coſ.}\, \omega + \mathfrak{B} \,\text{ſin.}\, \omega);$$

$$\mathfrak{B}' = \alpha(\mathfrak{B} \,\text{coſ.}\, \omega - \mathfrak{A} \,\text{ſin.}\, \omega);$$

$\mathfrak{B}'' =$

$$\mathfrak{A}'' = \alpha\,\alpha\,(\mathfrak{A}\,\text{cof.}\,2\,\omega + \mathfrak{B}\,\text{fin.}\,2\,\omega);$$
$$\mathfrak{B}'' = \alpha\,\alpha\,(\mathfrak{B}\,\text{cof.}\,2\,\omega - \mathfrak{A}\,\text{fin.}\,2\,\omega);$$
$$\mathfrak{A}''' = \alpha^3\,(\mathfrak{A}\,\text{cof.}\,3\,\omega + \mathfrak{B}\,\text{fin.}\,3\,\omega);$$
$$\mathfrak{B}''' = \alpha^3\,(\mathfrak{B}\,\text{cof.}\,3\,\omega - \mathfrak{A}\,\text{fin.}\,3\,\omega).$$

etc.

Hinc igitur pro euoluta ordinis n erit

$$\mathfrak{A}^{(n)} = \alpha^n\,(\mathfrak{A}\,\text{cof.}\,n\,\omega + \mathfrak{B}\,\text{fin.}\,n\,\omega);$$
$$\mathfrak{B}^{(n)} = \alpha^n\,(\mathfrak{B}\,\text{cof.}\,n\,\omega - \mathfrak{A}\,\text{fin.}\,n\,\omega).$$

Cum igitur fit vel $\omega = \frac{2\,i\,\pi}{n}$, vel $\omega = \frac{(2\,i+1)\,\pi}{n}$, erit priore cafu $n\,\omega = 2\,i\,\pi$, ideoque fin. $n\,\omega = 0$ et cof. $n\,\omega = 1$; pofteriore vero cafu erit $n\,\omega = (2\,i+1)\,\pi$, ideoque fin. $n\,\omega = 0$, at cof. $n\,\omega = -1$, quamobrem pro priore cafu erit $\mathfrak{A}^{(n)} = \alpha^n\,\mathfrak{A}$ et $\mathfrak{B}^{(n)} = \alpha^n\,\mathfrak{B}$, vnde fit

$$r^{(n)} = e^{\zeta\,\Phi}\,(\mathfrak{A}^{(n)}\,\text{cof.}\,\eta\,\Phi + \mathfrak{B}^{(n)}\,\text{fin.}\,\eta\,\Phi) \text{ ideoque}$$
$$r^{(n)} = \alpha^n\,e^{\zeta\,\Phi}\,(\mathfrak{A}\,\text{cof.}\,\eta\,\Phi + \mathfrak{B}\,\text{fin.}\,\eta\,\Phi),$$

qui valor fe habet ad r, vt $\alpha^n : 1$; pro pofteriore vero cafu erit $\mathfrak{A}^{(n)} = -\alpha^n\,\mathfrak{A}$ et $\mathfrak{B}^{(n)} = -\alpha^n\,\mathfrak{B}$, hincque nafcitur

$$r^{(n)} = -\alpha^n\,e^{\zeta\,\Phi}\,(\mathfrak{A}\,\text{cof.}\,\eta\,\Phi + \mathfrak{B}\,\text{fin.}\,\eta\,\Phi)$$

ergo $r^{(n)} = -\alpha^n\,r$, ficque pro vtroque cafu fimilitudo eft manifefta.

§. 12. Hoc igitur modo pro curua quaefita, quae in genere fuae euolutae ordinis n eft fimilis, aequationem nacti fumus inter eius radium ofculi r et amplitudinem Φ: imprimis igitur requiritur, vt hanc aequationem ad coordinatas orthogonales more folito reuocemus. Hunc in finem ad axem $a\,r$ ex curuae puncto s demittatur perpendiculum $s\,x$, ac vocentur abfciffa $a\,x = x$ et applicata $x\,s = y$, vt fit $\partial\,s^2 = \partial\,x^2 + \partial\,y^2$.

 Iam

Iam quia applicata xs inclinatur ad curuam as ſub angulo $asx = \Phi$, erit

$$\partial x = \partial s \text{ ſin. } \Phi \text{ et } \partial y = \partial s \text{ coſ. } \Phi;$$

quia igitur eſt

$$\partial s = r \partial \Phi = e^{\varsigma \Phi} \partial \Phi (\mathfrak{A} \text{ coſ. } \eta \Phi + \mathfrak{B} \text{ ſin. } \eta \Phi),$$

hinc ambas coordinatas x et y per amplitudinem Φ exprimere licebit ſequenti modo:

$$\partial x = e^{\varsigma \Phi} \partial \Phi \text{ ſin. } \Phi (\mathfrak{A} \text{ coſ. } \eta \Phi + \mathfrak{B} \text{ ſin. } \eta \Phi) \text{ et}$$
$$\partial y = e^{\varsigma \Phi} \partial \Phi \text{ coſ. } \Phi (\mathfrak{A} \text{ coſ. } \eta \Phi + \mathfrak{B} \text{ ſin. } \eta \Phi)$$

ad quas formulas integrandas notetur eſſe

$$\text{ſin. } \Phi \text{ coſ. } \eta \Phi = \tfrac{1}{2} \text{ ſin. } (\eta + 1) \Phi - \tfrac{1}{2} \text{ ſin. } (\eta - 1) \Phi;$$
$$\text{ſin. } \Phi \text{ ſin. } \eta \Phi = \tfrac{1}{2} \text{ coſ. } (\eta - 1) \Phi - \tfrac{1}{2} \text{ coſ. } (\eta + 1) \Phi;$$
$$\text{coſ. } \Phi \text{ coſ. } \eta \Phi = \tfrac{1}{2} \text{ coſ. } (\eta - 1) \Phi + \tfrac{1}{2} \text{ coſ. } (\eta + 1) \Phi;$$
$$\text{coſ. } \Phi \text{ ſin. } \eta \Phi = \tfrac{1}{2} \text{ ſin. } (\eta + 1) \Phi + \tfrac{1}{2} \text{ ſin. } (\eta - 1) \Phi.$$

His igitur valoribus ſubſtitutis, ambae noſtrae formulae in quatuor partes diſcerpantur, et integratione indicata fiet

$$x = \left\{ \begin{array}{l} \tfrac{1}{2}\mathfrak{A} \int e^{\varsigma \Phi} \partial \Phi \text{ ſin. } (\eta + 1) \Phi + \tfrac{1}{2}\mathfrak{B} \int e^{\varsigma \Phi} \partial \Phi \text{ coſ. } (\eta - 1) \Phi \\ -\tfrac{1}{2}\mathfrak{A} \int e^{\varsigma \Phi} \partial \Phi \text{ ſin. } (\eta - 1) \Phi - \tfrac{1}{2}\mathfrak{B} \int e^{\varsigma \Phi} \partial \Phi \text{ coſ. } (\eta + 1) \Phi \end{array} \right\};$$

$$y = \left\{ \begin{array}{l} \tfrac{1}{2}\mathfrak{A} \int e^{\varsigma \Phi} \partial \Phi \text{ coſ. } (\eta - 1) \Phi + \tfrac{1}{2}\mathfrak{B} \int e^{\varsigma \Phi} \partial \Phi \text{ ſin. } (\eta + 1) \Phi \\ +\tfrac{1}{2}\mathfrak{A} \int e^{\varsigma \Phi} \partial \Phi \text{ coſ. } (\eta + 1) \Phi + \tfrac{1}{2}\mathfrak{B} \int e^{\varsigma \Phi} \partial \Phi \text{ ſin. } (\eta - 1) \Phi \end{array} \right\}.$$

§. 13. Pro his integralibus inueniendis in ſubſidium vocentur iſtae integrationes generales:

$$\int e^{\varsigma \Phi} \partial \Phi \text{ ſin. } \lambda \Phi = - \frac{\lambda}{\varsigma\varsigma + \lambda\lambda} e^{\varsigma \Phi} \text{ coſ. } \lambda \Phi + \frac{\varsigma}{\varsigma\varsigma + \lambda\lambda} e^{\varsigma \Phi} \text{ ſin. } \lambda \Phi;$$
$$\int e^{\varsigma \Phi} \partial \Phi \text{ coſ. } \lambda \Phi = \frac{\lambda}{\varsigma\varsigma + \lambda\lambda} e^{\varsigma \Phi} \text{ ſin. } \lambda \Phi + \frac{\varsigma}{\varsigma\varsigma + \lambda\lambda} e^{\varsigma \Phi} \text{ coſ. } \lambda \Phi.$$

Hinc igitur erit

$x =$

$$x = \frac{e^{\zeta\Phi}}{2(\zeta\zeta+(\eta+1)^2)}[(\mathfrak{A}\zeta-\mathfrak{B}(\eta+1))\,\text{fin.}\,(\eta+1)\Phi-(\mathfrak{B}\zeta+\mathfrak{A}(\eta+1))\,\text{cof.}\,(\eta+1)\Phi]$$
$$-\frac{e^{\zeta\Phi}}{2(\zeta\zeta+(\eta-1)^2)}[(\mathfrak{A}\zeta-\mathfrak{B}(\eta-1))\,\text{fin.}\,(\eta-1)\Phi-(\mathfrak{B}\zeta+\mathfrak{A}(\eta-1))\,\text{cof.}\,(\eta-1)\Phi];$$

fimili modo reperietur

$$y = \frac{e^{\zeta\Phi}}{2(\zeta\zeta+(\eta+1)^2)}[(\mathfrak{A}\zeta-\mathfrak{B}(\eta+1))\,\text{cof.}(\eta+1)\Phi+(\mathfrak{B}\zeta+\mathfrak{A}(\eta+1))\,\text{fin.}\,(\eta+1)\Phi]$$
$$+\frac{e^{\zeta\Phi}}{2(\zeta\zeta+(\eta-1)^2)}[(\mathfrak{A}\zeta-\mathfrak{B}(\eta-1))\,\text{cof.}(\eta-1)\Phi+(\mathfrak{B}\zeta+\mathfrak{A}(\eta-1))\,\text{fin.}\,(\eta-1)\Phi].$$

Hic notetur, ob $\zeta = \alpha\,\text{cof.}\,\omega$ et $\eta = \alpha\,\text{fin.}\,\omega$ pro denominatoribus fore

$$\zeta\zeta+(\eta+1)^2 = \alpha\alpha+2\alpha\,\text{fin.}\,\omega+1 \text{ et}$$
$$\zeta\zeta+(\eta-1)^2 = \alpha\alpha-2\alpha\,\text{fin.}\,\omega+1.$$

§. 14. Cafus hic notatu dignus occurrit, quo fit $\omega = 0$, qui eft primus valor ipfius ω, quoties fuerit $r^{(n)} = +\alpha^n r$: hoc igitur cafu erit $\zeta = \alpha$ et $\eta = 0$, tum igitur erit $r = e^{\alpha\Phi}\mathfrak{A}$, hincque

$$x = \left\{ \begin{array}{l} \frac{e^{\alpha\Phi}}{2(\alpha\alpha+1)}[(\mathfrak{A}\alpha-\mathfrak{B})\,\text{fin.}\,\Phi-(\mathfrak{B}\alpha+\mathfrak{A})\,\text{cof.}\Phi] \\ +\frac{e^{\alpha\Phi}}{2(\alpha\alpha+1)}[(\mathfrak{A}\alpha+\mathfrak{B})\,\text{fin.}\,\Phi+(\mathfrak{B}\alpha-\mathfrak{A})\,\text{cof.}\,\Phi] \end{array} \right\};$$

$$y = \left\{ \begin{array}{l} \frac{e^{\alpha\Phi}}{2(\alpha\alpha+1)}[(\mathfrak{A}\alpha-\mathfrak{B})\,\text{cof.}\,\Phi+(\mathfrak{B}\alpha+\mathfrak{A})\,\text{fin.}\,\Phi] \\ +\frac{e^{\alpha\Phi}}{2(\alpha\alpha+1)}[(\mathfrak{A}\alpha+\mathfrak{B})\,\text{cof.}\,\Phi-(\mathfrak{B}\alpha-\mathfrak{A})\,\text{fin.}\,\Phi] \end{array} \right\};$$

quae expreffiones contrahuntur in fequentes formas fimplices:

$x =$

$$x = \frac{e^{\alpha\Phi}\mathfrak{A}}{(\alpha\alpha + 1)}(\alpha \sin.\Phi - \cos.\Phi) \text{ et}$$

$$y = \frac{e^{\alpha\Phi}\mathfrak{A}}{(\alpha\alpha + 1)}(\alpha \cos.\Phi + \sin.\Phi)$$

ſicque vnica tantum hoc caſu conſtans arbitraria $\mathfrak{A}$ ingreditur.

§. 15. Deinde etiam caſus ſingulari attentione dignus eſt, quo fit $\omega = \frac{\pi}{2} = 90°$, tum enim erit $\zeta = 0$ et $\eta = \alpha$, vnde habebimus $r = \mathfrak{A} \cos.\alpha\Phi + \mathfrak{B} \sin.\alpha\Phi$, hincque porro colligitur fore

$$x = \left\{\begin{array}{l} -\frac{1}{2(\alpha+1)}(\mathfrak{B} \sin.(\alpha+1)\Phi + \mathfrak{A} \cos.(\alpha+1)\Phi) \\ +\frac{1}{2(\alpha-1)}(\mathfrak{B} \sin.(\alpha-1)\Phi + \mathfrak{A} \cos.(\alpha-1)\Phi) \end{array}\right\}$$

$$y = \left\{\begin{array}{l} -\frac{1}{2(\alpha+1)}(\mathfrak{A} \sin.(\alpha+1)\Phi - \mathfrak{B} \cos.(\alpha+1)\Phi) \\ +\frac{1}{2(\alpha-1)}(\mathfrak{A} \sin.(\alpha-1)\Phi - \mathfrak{B} \cos.(\alpha-1)\Phi). \end{array}\right\}$$

§. 16. Hic caſus quo $\alpha = 1$ peculiarem euolutionem poſtulat, quia in partibus poſterioribus denominator euaneſcit; iſte autem caſus locum habet, quando euoluta ordinis n non ſolum ſimilis, verum adeo aequalis eſſe debet ipſi curuae quaeſitae, ita vt ſit $r^{(n)} = r$, ad quem caſum euoluendum ponatur $\alpha = 1 + \delta$, exſiſtente δ infinite paruo: tum igitur erit

$$\sin.(\alpha - 1)\Phi = \sin.\delta\Phi = \delta\Phi \text{ et}$$

$$\cos.(\alpha - 1)\Phi = \cos.\delta\Phi = 1 - \tfrac{1}{2}\delta\delta\Phi\Phi,$$

quibus valoribus introductis erit

$$x = -\tfrac{1}{4}(\mathfrak{B} \sin.2\Phi + \mathfrak{A} \cos.2\Phi) + \frac{\mathfrak{B}\Phi}{2} + \frac{\mathfrak{A}}{2\delta},$$

vbi terminum $\delta\delta\Phi\Phi$ omiſimus; tum vero etiam terminus conſtans $\frac{\mathfrak{A}}{2\delta}$ reiici poteſt, quoniam pro arbitrio conſtantem ad-

adiicere licet, quo facto erit

$$x = \frac{1}{2}\mathfrak{B}\Phi - \frac{1}{4}(\mathfrak{B}\,\text{sin.}\,2\Phi + \mathfrak{A}\,\text{cos.}\,2\Phi),$$

eodem modo

$$y = \frac{1}{2}\mathfrak{A}\Phi + \frac{1}{4}(\mathfrak{A}\,\text{sin.}\,2\Phi - \mathfrak{B}\,\text{cos.}\,2\Phi).$$

Hoc igitur casu etiam ipse angulus Φ in nostras formulas ingreditur.

§. 17. Non solum autem ex amplitudine Φ ambae coordinatae x et y per formulas finitas exprimi possunt, sed etiam ipse arcus curuae s. Cum enim sit $\partial s = r\,\partial\Phi$, ob

$$r = e^{\zeta\Phi}(\mathfrak{A}\,\text{cos.}\,\eta\Phi + \mathfrak{B}\,\text{sin.}\,\eta\Phi) \text{ erit}$$

$$s = \mathfrak{A}\int e^{\zeta\Phi}\partial\Phi\,\text{cos.}\,\eta\Phi + \mathfrak{B}\int e^{\zeta\Phi}\partial\Phi\,\text{sin.}\,\eta\Phi,$$

vnde sumtis integralibus per lemma praemissum erit

$$s = \frac{e^{\zeta\Phi}}{\zeta\zeta + \eta\eta}[(\mathfrak{A}\eta + \mathfrak{B}\zeta)\,\text{sin.}\,\eta\Phi + (\mathfrak{A}\zeta - \mathfrak{B}\eta)\,\text{cos.}\,\eta\Phi)]$$

siue ob $\zeta\zeta + \eta\eta = \alpha\alpha$ erit

$$s = \frac{e^{\zeta\Phi}}{\alpha\alpha}[(\mathfrak{A}\eta + \mathfrak{B}\zeta)\,\text{sin.}\,\eta\Phi + (\mathfrak{A}\zeta - \mathfrak{B}\eta)\,\text{cos.}\,\eta\Phi)].$$

§. 18. Quemadmodum istae formulae pro r et s et coordinatis x et y inuentae ad ipsam curuam quaesitam pertinent, ita si loco litterarum $\mathfrak{A}$ et $\mathfrak{B}$ scribantur $\mathfrak{A}'$ et $\mathfrak{B}'$, istae formulae naturam euolutae primae exhibebunt; similique modo si loco $\mathfrak{A}$ et $\mathfrak{B}$ scribantur litterae $\mathfrak{A}''$ et $\mathfrak{B}''$, eaedem formulae referentur ad euolutam secundam, et ita porro. Supra autem ostendimus esse

$\mathfrak{A}' = \alpha(\mathfrak{A}\,\text{cos.}\,\omega + \mathfrak{B}\,\text{sin.}\,\omega)$; $\mathfrak{B}' = \alpha(\mathfrak{B}\,\text{cos.}\,\omega - \mathfrak{A}\,\text{sin.}\,\omega)$;
$\mathfrak{A}'' = \alpha^2(\mathfrak{A}\,\text{cos.}\,2\omega + \mathfrak{B}\,\text{sin.}\,2\omega)$; $\mathfrak{B}'' = \alpha^2(\mathfrak{B}\,\text{cos.}\,2\omega - \mathfrak{A}\,\text{sin.}\,2\omega)$;
$\mathfrak{A}''' = \alpha^3(\mathfrak{A}\,\text{cos.}\,3\omega + \mathfrak{B}\,\text{sin.}\,3\omega)$; $\mathfrak{B}''' = \alpha^3(\mathfrak{B}\,\text{cos.}\,3\omega - \mathfrak{A}\,\text{sin.}\,3\omega)$;
etc. etc.

 vnde

vnde pro euoluta ordinis cuiuscunque λ erit

$$\mathfrak{A}^{(\lambda)} = \alpha^{\lambda} (\mathfrak{A} \operatorname{cof.} \lambda \omega + \mathfrak{B} \operatorname{fin.} \lambda \omega) \text{ et}$$
$$\mathfrak{B}^{(\lambda)} = \alpha^{\lambda} (\mathfrak{B} \operatorname{cof.} \lambda \omega - \mathfrak{A} \operatorname{fin.} \lambda \omega).$$

Quodfi ergo hi valores loco $\mathfrak{A}$ et $\mathfrak{B}$ fcribantur, formulae inventae valebunt pro euoluta ordinis λ.

§. 19. Quo has formulas adhuc fuccinctiores reddamus, ftatuamus $\mathfrak{A} = c \operatorname{fin.} \gamma$ et $\mathfrak{B} = c \operatorname{cof.} \gamma$, et formulae pro ipfa curua quaefita inuentae fequentes formas induent:

I. $r = c\, e^{\varsigma \Phi} \operatorname{fin.} (\gamma + \eta \Phi)$.

II. $s = \frac{c}{\alpha} e^{\varsigma \Phi} \operatorname{fin.} (\gamma - \omega + \eta \Phi)$.

III. $x = \frac{-c}{2(\alpha\alpha + 2\alpha \operatorname{fin.} \omega + 1)} e^{\varsigma \Phi} [\alpha \operatorname{cof.} (\gamma - \omega + (\eta + 1)\Phi) + \operatorname{fin.} (\gamma + (\eta + 1)\Phi)]$
$+ \frac{c}{2(\alpha\alpha - 2\alpha \operatorname{fin.} \omega + 1)} e^{\varsigma \Phi} [\alpha \operatorname{cof.} (\gamma - \omega + (\eta - 1)\Phi) - \operatorname{fin.} (\gamma + (\eta - 1)\Phi)]$.

IV. $y = \frac{c}{2(\alpha\alpha + 2\alpha \operatorname{fin.} \omega + 1)} e^{\varsigma \Phi} [\alpha \operatorname{fin.} (\gamma - \omega + (\eta + 1)\Phi) - \operatorname{cof.} (\gamma + (\eta + 1)\Phi)]$
$+ \frac{c}{2(\alpha\alpha - 2\alpha \operatorname{fin.} \omega + 1)} e^{\varsigma \Phi} [\alpha \operatorname{fin.} (\gamma - \omega + (\eta - 1)\Phi) + \operatorname{cof.} (\gamma + (\eta - 1)\Phi)]$.

§. 20. Pofitis autem loco $\mathfrak{A}$ et $\mathfrak{B}$ his valoribus affumtis $c \operatorname{fin.} \gamma$ et $\operatorname{cof.} \gamma$, fiet

$$\mathfrak{A}' = \alpha c \operatorname{fin.} (\gamma + \omega);$$
$$\mathfrak{B}' = \alpha c \operatorname{cof.} (\gamma + \omega).$$

Cum igitur pro euoluta prima fit radius osculi

$$r' = e^{\varsigma \Phi} (\mathfrak{A}' \operatorname{cof.} \eta \Phi + \mathfrak{B}' \operatorname{fin.} \eta \Phi),$$

habebimus

$$r' = \alpha c\, e^{\varsigma \Phi} \operatorname{fin.} (\gamma + \omega + \eta \Phi),$$

qui valor ex principali $r = c\, e^{\varsigma \Phi} \operatorname{fin.} (\gamma + \eta \Phi)$ oritur, fi ibi loco c fcribamus αc, loco γ vero $\gamma + \omega$, vnde fi in formulis

lis supra inuentis vbique loco c et γ scribamus αc et $\gamma + \omega$, deinde, quia etiam litterae ζ et η angulum ω involuunt, si pro valoribus sequentibus ipsius ω etiam loco ζ et η scribamus ζ' et η', et ita porro, eaedem formulae praebebunt naturam euolutae primae, cuius ergo elementa erunt

I. $r' = \alpha c e^{\zeta' \Phi} \sin. (\gamma + \omega + \eta' \Phi)$.

II. $s' = c' e^{\zeta' \Phi} \sin. (\gamma + \eta' \Phi)$.

III. $x' = \frac{-\alpha c}{2(\alpha\alpha + 2\alpha \sin.\omega + 1)} e^{\zeta' \Phi} [\alpha \cos.(\gamma + (\eta' + 1)\Phi) + \sin.(\gamma + \omega + (\eta' + 1)\Phi)]$
$\frac{+\alpha c}{2(\alpha\alpha - 2\alpha \sin.\omega + 1)} e^{\zeta' \Phi} [\alpha \cos.(\gamma + (\eta' - 1)\Phi) - \sin.(\gamma + \omega + (\eta' - 1)\Phi)]$.

IV. $y' = \frac{\alpha c}{2(\alpha\alpha + 2\alpha \sin.\omega + 1)} e^{\zeta' \Phi} [\alpha \sin.(\gamma + (\eta' + 1)\Phi) - \cos.(\gamma + \omega + (\eta' + 1)\Phi)]$
$\frac{+\alpha c}{2(\alpha\alpha - 2\alpha \sin.\omega + 1)} e^{\zeta \Phi} \alpha [\sin.(\gamma + (\eta' + 1)\Phi) + \cos.(\gamma + \omega + (\eta' - 1)\Phi)]$

§. 21. Consideremus nunc in genere euolutam ordinis λ, pro qua inuenimus radium osculi

$$r^{(\lambda)} = e^{\zeta \Phi} (\mathfrak{A}^{(\lambda)} \cos. \eta \Phi + \mathfrak{B}^{(\lambda)} \sin. \eta \Phi).$$

Nunc autem reperimus

$$\mathfrak{A}^{(\lambda)} = \alpha^{\lambda} (\mathfrak{A} \cos. \lambda \omega + \mathfrak{B} \sin. \lambda \omega) \text{ et}$$
$$\mathfrak{B}^{(\lambda)} = \alpha^{\lambda} (\mathfrak{B} \cos. \lambda \omega - \mathfrak{A} \sin. \lambda \omega),$$

siue etiam

$$\mathfrak{A}^{(\lambda)} = \alpha^{\lambda} c \sin. (\gamma + \lambda \omega) \text{ et}$$
$$\mathfrak{B}^{(\lambda)} = \alpha^{\lambda} c \cos. (\gamma + \lambda \omega),$$

ex quibus valoribus colligitur radius osculi

$$r^{(\lambda)} = \alpha^{\lambda} c e^{\zeta \Phi} \sin. (\gamma + \lambda \omega + \eta \Phi),$$

qui ex principali formatur, si in ea loco c et γ scribatur $\alpha^{\lambda} c$ et $\gamma + \lambda \omega$, quamobrem pro euoluta ordinis λ nanciscemur sequentia elementa:

I. $r^{(\lambda)} = \alpha^{\lambda} c e^{\zeta \Phi} \sin. (\gamma + \lambda \omega + \eta \Phi)$.

II. $s^{(\lambda)} = \alpha^{\lambda - 1} c e^{\zeta \Phi} \sin. (\gamma + (\lambda - 1) \omega + \eta \Phi)$.

III.

$$\text{III. } x^{(\lambda)} = \frac{-\alpha^\lambda c}{2(\alpha\alpha + 2\alpha \sin.\omega + 1)} e^{\xi\Phi}[\alpha \cos.(\gamma + (\lambda - 1)\omega + (\eta + 1)\Phi) + \sin.(\gamma + \lambda\omega + (\eta + \lambda\Phi)].$$

$$\frac{+\alpha^\lambda c}{2(\alpha\alpha - 2\alpha \sin.\omega + 1)} e^{\xi\Phi}[\alpha \cos.(\gamma + (\lambda - 1)\omega + (\eta - 1)\Phi) - \sin.(\gamma + \lambda\omega + (\eta - 1)\Phi)].$$

$$\text{IV. } y^{(\lambda)} = \frac{\alpha^\lambda c}{2(\alpha\alpha + 2\alpha \sin.\omega + 1)} e^{\xi\Phi}[\alpha \sin.(\gamma + (\lambda - 1)\omega + (\eta + 1)\Phi) - \cos.(\gamma + \lambda\omega + (\eta + 1)\Phi)].$$

$$\frac{+\alpha^\lambda c}{2(\alpha\alpha - 2\alpha \sin.\omega + 1)} e^{\xi\Phi}[\alpha \sin.(\gamma + (\lambda - 1)\omega + (\eta - 1)\Phi) + \cos.(\gamma + \lambda\omega + (\eta - 1)\Phi)].$$

§. 22. His igitur constitutis, si curua quaeratur, quae similis esse debeat suae euolutae ordinis n, quaestio bipartita est tractanda, prouti fuerit vel $r^{(n)} = + \alpha^n r$, vel $r^{(n)} = - \alpha^n r$; priore casu euoluta ordinis n directe dicatur similis ipsi curuae, posteriore vero casu inuerse similis. Tum vero pro priore casu loco ω sequentes habebimus angulos: $\frac{0\pi}{n}$, $\frac{2\pi}{n}$, $\frac{4\pi}{n}$, $\frac{6\pi}{n}$, etc. . . . $\frac{i\pi}{n}$, pro posteriore vero casu sequentes valores pro angulo ω sunt capiendi: $\frac{\pi}{n}$, $\frac{3\pi}{n}$, $\frac{5\pi}{n}$, $\frac{7\pi}{n}$, etc. . . . $\frac{(2i+1)\pi}{n}$, vnde pro vtroque casu tot valores pro ω sumi conueniet, quamdiu $2i$, vel $2i+1$ non superat denominatorem n, siquidem solutionem quaestionis completam desideremus.

§. 23. Quando autem pro ω plures adipiscimur valores, tum pro singulis quaternae formulae litterarum r, s, x, y euoluantur; et quia c et γ vicem gerunt quantitatum constantium per integrationem ingressarum, si pro primo valore ipsius ω vtamur litteris c et γ, pro secundo scribi conueniet c' et γ', pro tertio vero c'' et γ'', etc. quos valores omnes prorsus pro arbitrio assumere licet; omnes autem isti valores in vnam

vnam fummum collecti dabunt veros et completos valores quaternarum noftrarum quantitatum r, s, x et y. Sicque problema noftrum, in latiffimo fenfu acceptum, femper per formulas finitas ex amplitudine Φ refoluetur, ita vt aliae quantitates transfcendentes non occurrant, praeter quantitatem exponentialem $e^{\zeta\Phi}$ et finus cofinusque angulorum.

§. 24. Quo formulas pro coordinatis x et y inuentas ad maiorem vniformitatem perducamus, ex angulis $\gamma - \omega + (\eta \pm 1)\Phi$ litteram ω eximamus, et loco α cof. ω et α fin. ω reftituamus litteras ζ et η; hocque modo obtinebimus

$$x = \frac{-c}{2(\alpha\alpha + 2\alpha\,\text{fin.}\,\omega + 1)} e^{\zeta\Phi}[\zeta\,\text{cof.}(\gamma + (\eta+1)\Phi) + (\eta+1)\,\text{fin.}(\gamma + (\eta+1)\Phi)]$$
$$\frac{+c}{2(\alpha\alpha - 2\alpha\,\text{fin.}\,\omega + 1)} e^{\zeta\Phi}[\zeta\,\text{cof.}(\gamma + (\eta-1)\Phi) + (\eta-1)\,\text{fin.}(\gamma + (\eta-1)\Phi)].$$
$$y = \frac{c}{2(\alpha\alpha + 2\alpha\,\text{fin.}\,\omega + 1)} e^{\zeta\Phi}[\zeta\,\text{fin.}(\gamma + (\eta+1)\Phi) - (\eta+1)\,\text{cof.}(\gamma + (\eta+1)\Phi)]$$
$$\frac{+c}{2(\alpha\alpha - 2\alpha\,\text{fin.}\,\omega + 1)} e^{\zeta\Phi}[\zeta\,\text{fin.}(\gamma + (\eta-1)\Phi) - (\eta-1)\,\text{cof.}(\gamma + (\eta-1)\Phi)].$$

vbi duo tantum adhuc occurrunt diuerfi anguli $\gamma + (\eta + 1)\Phi$ et $\gamma + (\eta - 1)\Phi$, quae diuerfitas tolli poteft per iftas combinationes:

1°) y cof. $\Phi + x$ fin. $\Phi =$

$$\frac{c}{2(\alpha\alpha + 2\alpha\,\text{fin.}\,\omega + 1)} e^{\zeta\Phi}[\zeta\,\text{fin.}(\gamma + \eta\Phi) - (\eta+1)\,\text{cof.}(\gamma + \eta\Phi)]$$
$$\frac{+c}{2(\alpha\alpha - 2\alpha\,\text{fin.}\,\omega + 1)} e^{\zeta\Phi}[\zeta\,\text{fin.}(\gamma + \eta\Phi) - (\eta-1)\,\text{cof.}(\gamma + \eta\Phi)].$$

2°) y fin. $\Phi - x$ cof. $\Phi =$

$$\frac{c}{2(\alpha\alpha + 2\alpha\,\text{fin.}\,\omega + 1)} e^{\zeta\omega}[\zeta\,\text{cof.}(\gamma + \eta\Phi) + (\eta+1)\,\text{fin.}(\gamma + \eta\Phi)]$$
$$\frac{+c}{2(\alpha\alpha - 2\alpha\,\text{fin.}\,\omega + 1)} e^{\zeta\Phi}[\zeta\,\text{cof.}(\gamma + \eta\Phi) + (\eta-1)\,\text{fin.}(\gamma + \eta\Phi)].$$

§. 25. His igitur poftremis formulis, vtpote maxime concinnis, in applicatione ad cafus fpeciales vti conueniet, quandoquidem pro omnibus valoribus anguli ω amplitudo Φ ea-

 dem

dem manet. Inuentis autem pro quouis casu valoribus istarum formularum y cos. $\Phi + x$ sin. Φ et y sin. $\Phi - x$ cos. Φ, inde facile ipsas coordinatas x et y definire licet. Istae autem formulae in figura lineas satis memorabiles designant. Si enim ex punctis a et x in normalem $s\,r$ ducantur perpendicula $a\,p$ et $x\,z$, ex x vero in $a\,p$ perpendiculum $x\,q$, ex triangulo $x\,s\,z$, ob angulum $s\,x\,z = \Phi$, erit $s\,z = y$ sin. Φ et $x\,z = y$ cos. Φ; deinde vero ex triangulo $a\,x\,q$ fiet $a\,q = x$ sin. Φ et $x\,q = x$ cos. Φ, ex quibus colligitur recta $a\,p = y$ cos. $\Phi + x$ sin. Φ; at vero recta $s\,p = s\,z - x\,q = y$ sin. $\Phi - x$ cos. Φ. Quare si ad curuam in s ducamus tangentem $s\,t$, in eamque ex a perpendiculum demittamus $a\,t$, ac vocemus $a\,t = p$ et $s\,t = t$, erit

Tab. II. Fig. 3.

$$p = y \text{ sin. } \Phi - x \text{ cos. } \Phi \text{ et}$$
$$t = y \text{ cos. } \Phi + x \text{ sin. } \Phi.$$

Inuentis autem his duabus quantitatibus p et t, inde vicissim erit

$$x = t \text{ sin. } \Phi - p \text{ cos. } \Phi \text{ et}$$
$$y = p \text{ sin. } \Phi + t \text{ cos. } \Phi.$$

§. 26. Quodsi ergo praeter radium osculi r et arcum curuae s loco coordinatarum x et y istas binas quantitates s et p in calculum introducamus, pro curua quaesita $a\,s$ sequentes habebimus formulas satis concinnas:

$$\text{I. } r = c\, e^{\zeta\Phi} \text{ sin. } (\gamma + \eta\Phi).$$
$$\text{II. } s = \tfrac{c}{\alpha\alpha}\, e^{\zeta\Phi} [\zeta \text{ sin. } (\gamma + \eta\Phi) - \eta \text{ cos. } (\gamma - \eta\Phi)].$$
$$\text{III. } s = \frac{c}{2(\alpha\alpha + 2\alpha \text{ sin.}\,\omega + 1)} e^{\zeta\Phi}[\zeta \text{sin.}(\gamma+\eta\Phi) - (\eta+1) \text{ cos.}(\gamma+\eta\Phi)]$$
$$\frac{+\,c}{2(\alpha\alpha - 2\alpha \text{ sin.}\,\omega + 1)} e^{\zeta\Phi}[\zeta \text{ sin.}(\gamma+\eta\Phi) - (\eta-1) \text{cos.}(\gamma+\eta\Phi)].$$
$$\text{IV. } p = \frac{c}{2(\alpha\alpha + 2\alpha \text{ sin.}\,\omega + 1)} e^{\zeta\Phi}[\zeta \text{cos.}(\gamma+\eta\Phi) + (\eta+1) \text{ sin.}(\gamma+\eta\Phi)]$$
$$\frac{+\,c}{2(\alpha\alpha - 2\alpha \text{ sin.}\,\omega + 1)} e^{\zeta\Phi}[\zeta \text{cos.}(\gamma+\eta\Phi) + (\eta-1) \text{ sin.} (\gamma+\eta\Phi)].$$

Hinc igitur ipsae coordinatae x et y ita definientur, vt sit

$$x = t \text{ sin. } \Phi - p \text{ cos. } \Phi \text{ et}$$
$$y = p \text{ sin. } \Phi + t \text{ cos. } \Phi,$$

hoc-

hocque pacto omnia haec elementa per eundem angulum $\gamma + \eta \Phi$ determinantur.

§. 27. Quin etiam fimili modo tales formulae pro omnibus euolutis fatis fuccincte exhiberi poterunt. Quoniam enim pro euoluta ordinis λ, vt fupra vidimus, tantum opus eft vt loco c fcribatur $\alpha^\lambda c$, loco γ vero $\gamma + \lambda\omega$, formulae hoc modo fe habebunt:

I. $r^{(\lambda)} = \alpha^\lambda c e^{\zeta\Phi} \text{fin.}(\gamma + \lambda\omega + \eta\Phi)$.

II. $s^{(\lambda)} = \alpha^{\lambda-2} c e^{\zeta\Phi} [\zeta \text{fin.}(\gamma + \lambda\omega + \eta\Phi) - \eta \text{cof.}(\gamma + \lambda\omega + \eta\Phi)]$

III. $$t^{(\lambda)} = \frac{\alpha^\lambda c}{2(\alpha\alpha + 2\alpha \text{fin.}\omega + 1)} e^{\zeta\Phi} [\zeta \text{fin.}(\gamma + \lambda\omega + \eta\Phi) - (\eta + 1)\text{cof.}(\gamma + \lambda\omega + \eta\Phi)]$$
$$+ \frac{\alpha^\lambda c}{2(\alpha\alpha - 2\alpha \text{fin.}\omega + 1)} e^{\zeta\Phi} [\zeta \text{fin.}(\gamma + \lambda\omega + \eta\Phi) - (\eta - 1)\text{cof.}(\gamma + \lambda\omega + \eta\Phi)].$$

IV. $$p^{(\lambda)} = \frac{\alpha^\lambda c}{2(\alpha\alpha + 2\alpha \text{fin.}\omega + 1)} e^{\zeta\Phi} [\zeta \text{cof.}(\gamma + \lambda\omega + \eta\Phi) + (\eta + 1)\text{fin.}(\gamma + \lambda\omega + \eta\Phi)]$$
$$+ \frac{\alpha^\lambda c}{2(\alpha\alpha - 2\alpha \text{fin.}\omega + 1)} e^{\zeta\Phi} [\zeta \text{cof.}(\gamma + \lambda\omega + \eta\Phi) + (\eta - 1)\text{fin.}(\gamma + \lambda\omega + \eta\Phi)],$$

tum vero ipfae coordinatae ita definientur, vt fit

$$x^{(\lambda)} = t^{(\lambda)} \text{fin.}\Phi - p^{(\lambda)} \text{cof.}\Phi \text{ et}$$
$$y^{(\lambda)} = p^{(\lambda)} \text{fin.}\Phi + t^{(\lambda)} \text{cof.}\Phi.$$

§. 28. Cum littera α inuoluat rationem fimilitudinis, quam curua quaefita ad fuam euolutam ordinis n tenere debet, quandoquidem fingula elementa ipfius curuae quaefitae fe habere debent ad fingula elementa euolutae ordinis λ, vt 1 ad $\pm \alpha^n$, provti fcilicet haec euoluta vel directe vel inuerfe fimilis poftulatur: fi fumamus $\alpha = 1$, tum euoluta adeo curuae

 quaefiquaesitae aequalis prodibit, quem ergo casum seorsim euolui conueniet. Quia igitur tum fit $\zeta = \text{cos.}\,\omega$ et $\eta = \text{sin.}\,\omega$, ideoque $\zeta\zeta + \eta\eta = 1$, formulae pro euoluta ordinis λ modo exhibitae sequenti modo contrahentur:

$$r^{(\lambda)} = c\, e^{\zeta\Phi}\, \text{sin.}\,(\gamma + \lambda\,\omega + \eta\,\Phi)$$
$$s^{(\lambda)} = c\, e^{\zeta\Phi}\, [\zeta\, \text{sin.}\,(\gamma + \lambda\omega + \eta\Phi) - \eta\, \text{cos.}\,(\gamma + \lambda\omega + \eta\Phi)]$$
$$t^{(\lambda)} = \tfrac{c}{2\zeta}\, e^{\zeta\Phi}\,.\, \text{sin.}\,(\gamma + \lambda\,\omega + \eta\,\Phi),$$
$$p^{(\lambda)} = \tfrac{c}{2\zeta}\, e^{\zeta\Phi}\, \text{cos.}\,(\gamma + \lambda\,\omega + \eta\,\Phi),$$

vnde colligitur:

$$x^{(\lambda)} = \tfrac{c}{2\zeta}\, e^{\zeta\Phi}\, [\text{sin.}\Phi\, \text{sin.}(\gamma + \lambda\omega + \eta\Phi) - \text{cos.}\Phi\, \text{cos.}(\gamma + \lambda\omega + \eta\Phi)]$$
$$y^{(\lambda)} = \tfrac{c}{2\zeta}\, e^{\zeta\Phi}\, [\text{sin.}\Phi\, \text{cos.}(\gamma + \lambda\omega + \eta\Phi) + \text{cos.}\Phi\, \text{sin.}(\gamma + \lambda\omega + \eta\Phi)],$$

vbi notandum tam arcum s quam ambas coordinatas sequenti modo contrahi posse:

$$s^{(\lambda)} = c\, e^{\zeta\Phi}\, \text{sin.}\,(\gamma + (\lambda - 1)\,\omega + \eta\,\Phi)$$
$$x^{(\lambda)} = -\tfrac{c}{2\zeta}\, e^{\zeta\Phi}\, \text{cos.}\,(\gamma + \lambda\,\omega + (\eta + 1)\,\Phi)$$
$$y^{(\lambda)} = \tfrac{c}{2\zeta}\, e^{\zeta\Phi}\, \text{sin.}\,(\gamma + \lambda\,\omega + (\eta + 1)\,\Phi).$$

§. 29. Has formulas autem imprimis ad ipsam curuam quaesitam accommodari conueniet, quae cum se habere debeat ad suam euolutam ordinis n, vt $1 : \pm\alpha^n$, ante omnia quaerantur cuncti valores anguli ω, qui pro similitudine directa sunt: $\frac{0\pi}{n}$, $\frac{2\pi}{n}$, $\frac{4\pi}{n}$, $\frac{6\pi}{n}$, etc., pro similitudine autem inuersa: $\frac{\pi}{n}$, $\frac{3\pi}{n}$, $\frac{5\pi}{n}$, $\frac{7\pi}{n}$, etc. pro quibus scribamus breuitatis gratia ω, ω', ω'', ω''', etc., ex iisque formemus sequentes formulas:

$$\zeta = \alpha\, \text{cos.}\,\omega;\ \zeta' = \alpha\, \text{cos.}\,\omega';\ \zeta'' = \alpha\, \text{cos.}\,\omega'';\ \text{etc.}$$
$$\eta = \alpha\, \text{sin.}\,\omega;\ \eta' = \alpha\, \text{sin.}\,\omega';\ \eta'' = \alpha\, \text{sin.}\,\omega'';\ \text{etc.}$$

Simili modo loco constantium c et γ, quae ipsi angulo ω responſpondent, pro ſequentibus angulis ſcribamus c', γ'; c'', γ''; c''', γ'''; etc. quibus notatis pro ſingulis ω, ω', ω'', ω''', etc. colligantur ex formilis ſupra datis: 1) omnes valores ipſius r, qui ſint R, R′, R″, R‴. etc. 2) valores ipſius s, qui ſint S, S′, S″, S‴, etc. 3) valores ipſius x, qui ſint X, X′, X″, X‴, etc. 4) valores ipſius y, qui ſint Y, Y′, Y″, Y‴, etc. 5) valores ipſius t, qui ſint T, T′, T″, T‴, etc. 6) valores ipſius p, qui ſint P, P′, P″, P‴, etc. Hincque ſolutio problematis completa continebitur ſequentibus formulis:

$$1^\circ.\ r = R + R' + R'' + R''' + \text{etc.}$$

$$2^\circ.\ s = S + S' + S'' + S''' + \text{etc.} + A,$$

$$3^\circ.\ x = X + X' + X'' + X''' + \text{etc.} + B,$$

$$4^\circ.\ y = Y + Y' + Y'' + Y''' + \text{etc.} + C,$$

$$5^\circ.\ t = T + T' + T'' + T''' + \text{etc.} + C \operatorname{cof.} \Phi + B \operatorname{fin.} \Phi,$$

$$6^\circ.\ p = P + P' + P'' + P''' + \text{etc.} + C \operatorname{fin.} \Phi - B \operatorname{cof.} \Phi,$$

vbi litterae A, B, C, deſignant conſtantes per vltimas integrationes ingreſſas.

I. De curuis
quae ſuis euolutis primis ſint ſimiles.

§. 30. Cum hic ſit $n = 1$, formula principalis reſoluenda erit $\lambda \pm \alpha = 0$, vnde vel $\lambda = + \alpha$, vel $\lambda = - \alpha$, ita vt ſufficiat alterutrum tantum horum caſuum euoluere, quoniam alter inde naſcitur ſumto α negatiuo. Cum igitur fuerit $r = c\, e^{\lambda\Phi}$, hoc caſu habebimus $r = c\, e^{\alpha\Phi}$, qua ergo aequatione inter radium oſculi r et amplitudinem Φ natura curuae quaeſitae iam perfecte exprimitur; neque opus eſt angulum ω, qui hoc caſu foret $= 0$, introducere, quia hoc caſu factor formulae generalis $\lambda^n - \alpha^n$ tantum eſt ſimplex.

§. 31. Hoc igitur casu, ob $\partial s = r\partial\Phi$, erit $\partial s = c\partial\Phi e^{\alpha\Phi}$, cuius integrale praebet $s = \frac{c}{\alpha} e^{\alpha\Phi} + A$, vbi si constans A ita definiatur, vt pro amplitudine $\Phi = 0$ etiam ipse arcus s euanescat, quemadmodum in figura repraesentatur, vbi angulo $ars = \Phi$ respondet arcus $as = s$, erit $s = \frac{c}{\alpha}(e^{\alpha\Phi} - 1)$, qua lege secundum figuram etiam coordinatas $ax = x$ et $xs = y$ determinari conueniet. Cum igitur sit $\partial x = \partial s$ sin. Φ et $\partial y = \partial s$ cos. Φ, erit $\partial x = ce^{\alpha\Phi}\partial\Phi$ sin. Φ et $\partial y = ce^{\alpha\Phi}\partial\Phi$ cos. Φ, vnde integratione secundum Lemma §. 13. datum peracta fiet

$$x = \frac{-c}{\alpha\alpha+1} e^{\alpha\Phi}(\text{cos. }\Phi - \alpha \text{ sin. }\Phi) + \frac{c}{\alpha\alpha+1},$$
$$y = \frac{+c}{\alpha\alpha+1} e^{\alpha\Phi}(\text{sin. }\Phi + \alpha \text{ cos. }\Phi) - \frac{\alpha c}{\alpha\alpha+1},$$

vnde patet, si amplitudo Φ fuerit quam minima, tum fore $x = \frac{1}{2}c\Phi\Phi$ et $y = c\Phi$. Hinc vero denique erit

$$t = \frac{\alpha c}{\alpha\alpha+1} e^{\alpha\Phi} + \frac{c}{\alpha\alpha+1}(\text{sin. }\Phi - \alpha \text{ cos. }\Phi),$$
$$p = \frac{c}{\alpha\alpha+1} e^{\alpha\Phi} - \frac{c}{\alpha\alpha+1}(\text{cos. }\Phi + \alpha \text{ sin. }\Phi).$$

Tab. II. Fig. 4. §. 32. Cum sit $s + \frac{c}{\alpha} = \frac{c}{\alpha} e^{\alpha\Phi}$, erit $s + \frac{c}{\alpha} = \frac{r}{\alpha}$, vnde patet, si curua sa retro continuetur, vsque ad certum punctum o, vt arcus ao fiat $= \frac{c}{\alpha}$, tum fore arcum a puncto o sumtum, scilicet $oas = \frac{r}{\alpha}$, ita vt iste arcus oas ad radium osculi in s datam teneat rationem, scilicet vt $1 : \alpha$, ideoque radius osculi in ipso puncto o euanescat, ex quo iam facile concludere licet, hanc curuam esse spiralem logarithmicam centrum suum in puncto o habentem, ad quod demum peractis infinitis spiris pertingit. Quod quo clarius appareat, accuratius quaeramus hoc punctum o, pro quo ergo sumi debet $s = -\frac{c}{\alpha}$, tum autem pro amplitudine habetur ista aequatio: $\frac{c}{\alpha} e^{\alpha\Phi} = 0$, siue $e^{\alpha\Phi} = 0$, vnde fit $\Phi = \infty$. Quamobrem, si

curua

curua sa retro continuetur per amplitudinem infinitam, tum ea in ipſo puncto o terminabitur; ex quo intelligitur, curuam circa punctum o infinitas ſpiras continuo minores abſoluere. Ponatur igitur $\Phi = -\infty$, vt coordinatae x et y nobis hoc punctum o declarent; tum autem ob $e^{\alpha\Phi} = o$ fiet $x = \frac{c}{\alpha\alpha+1}$ et $y = -\frac{\alpha c}{\alpha\alpha+1}$. Iſtud ergo punctum o infra axem ar erit ſitum, ex quo ſi ad axem ducatur normalis op, tum erit $ap = \frac{c}{\alpha\alpha+1}$ et $po = \frac{\alpha c}{\alpha\alpha+1}$. Quod ſi iam ex puncto o in applicatam sx productam demittatur perpendiculum oq, erit

$$oq = ap - x = \frac{c}{\alpha\alpha+1} e^{\alpha\Phi} (\text{coſ.}\,\Phi - \alpha\,\text{ſin.}\,\Phi) \text{ et}$$

$$sq = y + op = \frac{c}{\alpha\alpha+1} e^{\alpha\Phi} (\text{ſin.}\,\Phi + \alpha\,\text{coſ.}\,\Phi).$$

Quodſi iam ducatur recta os ſecans axem ar in puncto u, erit $os = \frac{c}{\sqrt{(\alpha\alpha+1)}} e^{\alpha\Phi}$, ſiue $os = \frac{r}{\sqrt{(\alpha\alpha+1)}}$. Hinc ſi vocetur angulus $qos = aus = \psi$, erit

$$\text{tang.}\,\psi = \frac{qs}{oq} = \frac{\text{ſin.}\,\Phi + \alpha\,\text{coſ.}\,\Phi}{\text{coſ.}\,\Phi - \alpha\,\text{ſin.}\,\Phi}.$$

Quoniam igitur angulus $ars = \Phi$, erit angulus $rsu = \psi - \Phi$, conſequenter

$$\text{tang.}\,rsu = \frac{\text{tang.}\,\psi - \text{tang.}\,\Phi}{1 + \text{tang.}\,\psi\,\text{tang.}\,\Phi} = \alpha.$$

Quoniam igitur angulus asr eſt rectus, erit etiam angulus aso conſtans, eiusque cotangens $= \alpha$, ſiue tangens $= \frac{1}{\alpha}$. Quamobrem, cum omnes rectae ex puncto o ad curuam eductae ad ipſam curuam aequaliter inclinentur, manifeſtum eſt, hanc curvam eſſe logarithmicam ſpiralem, circa centrum o deſcriptam, ſub angulo obliquitatis cuius tangens $= \frac{1}{\alpha}$. Quodſi ergo curua quaeſita aequalis eſſe debeat ſuae euolutae, ita vt ſit $\alpha = 1$, curua ſatisfaciens erit logarithmica ſpiralis ſemi-rectangula, vti iam dudum eſt demonſtratum.

§. 33.

§. 33. Alter casus, quo pro α accipitur valor negativus, ab isto aliter non differt, nisi quod amplitudo Φ in negatiuam mutatur; vnde etiam curua satisfaciens erit eadem, scilicet spiralis logarithmica, hoc tantum discrimine, quod nunc arcus $a\,o$ ad axem $a\,r$ refertur. Quoniam autem ambo hi casus in sequentibus quaestionibus simul occurrere possunt, pro vtroque singula elementa hic conspectui exponamus.

Pro casu $\lambda = \alpha$.

$$r = c\,e^{\alpha\Phi}$$
$$s = \frac{c}{\alpha}(e^{\alpha\Phi} - 1)$$
$$x = \frac{c}{\alpha\alpha+1}\,e^{\alpha\Phi}(\alpha\,\text{sin.}\,\Phi - \text{cos.}\,\Phi) + \frac{c}{\alpha\alpha+1}$$
$$y = \frac{c}{\alpha\alpha+1}\,e^{\alpha\Phi}(\text{sin.}\,\Phi + \alpha\,\text{cos.}\,\Phi) - \frac{\alpha c}{\alpha\alpha+1}$$
$$t = \frac{\alpha c}{\alpha\alpha+1}\,e^{\alpha\Phi} + \frac{c}{\alpha\alpha+1}(\text{sin.}\,\Phi - \alpha\,\text{cos.}\,\Phi)$$
$$p = \frac{c}{\alpha\alpha+1}\,e^{\alpha\Phi} - \frac{c}{\alpha\alpha+1}(\text{cos.}\,\Phi + \alpha\,\text{sin.}\,\Phi)$$

Pro casu $\lambda = -\alpha$.

$$r = c\,e^{-\alpha\Phi}$$
$$s = \frac{c}{\alpha}(1 - e^{-\alpha\Phi})$$
$$x = -\frac{c}{\alpha\alpha+1}\,e^{-\alpha\Phi}(\text{cos.}\,\Phi + \alpha\,\text{sin.}\,\Phi) + \frac{c}{\alpha\alpha+1}$$
$$y = \frac{c}{\alpha\alpha+1}\,e^{-\alpha\Phi}(\text{sin.}\,\Phi - \alpha\,\text{cos.}\,\Phi) + \frac{\alpha c}{\alpha\alpha+1}$$
$$t = -\frac{\alpha c}{\alpha\alpha+1}\,e^{-\alpha\Phi} + \frac{c}{\alpha\alpha+1}(\text{sin.}\,\Phi + \alpha\,\text{cos.}\,\Phi)$$
$$p = \frac{c}{\alpha\alpha+1}\,e^{-\alpha\Phi} - \frac{c}{\alpha\alpha+1}(\text{cos.}\,\Phi - \alpha\,\text{sin.}\,\Phi).$$

§. 34. Quemadmodum hic curua quaesita similis est suae euolutae primae in ratione $1:\alpha$, ita quoque similis erit suae euolutae secundae, in ratione $1:\alpha\alpha$, parique modo etiam suae euolutae tertiae, in ratione $1:\alpha^3$, et ita porro, vnde manifestum

tum est logarithmicam spiralem semper quaestioni satisfacere, cuicunque euolutarum quaesita similis requiratur, quae autem solutio tantum est particularis, quandoquidem praeter eam etiam infinitae aliae lineae curuae assignari possunt, quae similes sint suis euolutis cuiusque ordinis, quamobrem pro solutione completa quouis casu omnes plane curuae quaesito satisfacientes inuestigari debebunt.

II. De curuis quae suis euolutis secundis *directe* sint similes, vbi $r'' = \alpha^2 r$.

§. 35. Cum ergo hoc casu sit $\lambda\lambda = \alpha\alpha$, pro λ duos statim habemus valores reales, qui sunt $\lambda = +\alpha$ et $\lambda = -\alpha$, tum vero pro radio osculi curuae quaesitae hanc habebimus aequationem: $r = \mathfrak{A} e^{\alpha\Phi} \pm \mathfrak{B} e^{-\alpha\Phi}$, hincque pro euoluta secunda fit

$$r'' = \alpha\alpha \mathfrak{A} e^{\alpha\Phi} + \alpha\alpha \mathfrak{B} e^{-\alpha\Phi},$$

vbi pro $\mathfrak{A}$ et $\mathfrak{B}$ quantitates quascunque constantes accipere licet; ex quo manifestum, si alterutra earum euanescat, pro curva satisfaciente, prorsus vt casu superiore, prodituram esse logarithmicam spiralem. Pro varia igitur relatione inter has constantes $\mathfrak{A}$ et $\mathfrak{B}$ innumerae videntur curuae diuersae quaestioni satisfacientes resultare; interim tamen eas omnes ad duas tantum species reuocare licet. Quoniam enim axis $a r$, a quo amplitudinem Φ computamus, prorsus arbitrio nostro relinquitur, dum curua eadem plane manet, hoc axe vtcunque mutato amplitudo Φ quopiam angulo arbitrario augebitur vel minuetur, qui angulus si sit $= \theta$, formula inuenta ad eandem curuam pertinebit, etiamsi loco Φ scribamus $\Phi + \theta$, quo facto erit

$$r = \mathfrak{A} e^{\alpha\theta} . e^{\alpha\Phi} \pm \mathfrak{B} e^{-\alpha\theta} . e^{-\alpha\Phi},$$

vbi manifesto angulum θ semper ita assumere licebit, vt fiat $\mathfrak{A}\, e^{\alpha\theta} = \mathfrak{B}\, e^{-\alpha\theta}$, sumendo scilicet $\theta = \frac{1}{2\alpha}\, l\, \frac{\mathfrak{B}}{\mathfrak{A}}$. Quod si ergo axem hoc modo constituamus, ac breuitatis gratia sumamus $\mathfrak{A}\, e^{\alpha\theta} = \mathfrak{B}\, e^{-\alpha\theta} = c$, nostra aequatio erit $r = c\,(e^{\alpha\Phi} \pm e^{-\alpha\Phi})$, in qua vnica quantitas constans c inest, vnde ob signum ambiguum $\pm$ duae tantum curuae diuersae exoriri sunt censendae, quas seorsim euolui conueniet.

I°. Euolutio casus $r = c\,(e^{\alpha\Phi} + e^{-\alpha\Phi})$.

§. 36. Hic ergo ambo casus ante tractati iunctim occurrunt, ita vt tantum opus sit pro singulis elementis binos valores supra exhibitos coniungere, vnde sequentes formulas nanciscemur:

$$r = c\, e^{\alpha\Phi} + c\, e^{-\alpha\Phi}$$
$$s = \frac{c}{\alpha}\, e^{\alpha\Phi} - \frac{c}{\alpha}\, e^{-\alpha\Phi}$$
$$x = \frac{c}{\alpha\alpha+1}\,[\alpha\, \text{fin.}\,\Phi\,(e^{\alpha\Phi} - e^{-\alpha\Phi}) - \text{cof.}\,\Phi\,(e^{\alpha\Phi} + e^{-\alpha\Phi})] + \frac{2c}{\alpha\alpha+1}$$
$$y = \frac{c}{\alpha\alpha+1}\,[\alpha\, \text{cof.}\,\Phi\,(e^{\alpha\Phi} - e^{-\alpha\Phi}) + \text{fin.}\,\Phi\,(e^{\alpha\Phi} + e^{-\alpha\Phi})]$$
$$z = \frac{\alpha c}{\alpha\alpha+1}\,(e^{\alpha\Phi} - e^{-\alpha\Phi}) + \frac{2c}{\alpha\alpha+1}\,\text{fin.}\,\Phi$$
$$p = \frac{c}{\alpha\alpha+1}\,(e^{\alpha\Phi} + e^{-\alpha\Phi}) - \frac{2c}{\alpha\alpha+1}\,\text{cof.}\,\Phi.$$

§. 37. Hic primum obseruo, posita amplitudine $\Phi = 0$ radium osculi curuae in ipso puncto a fore $= 2c$, vbi simul coordinatae x et y euanescunt. Sumta autem amplitudine Φ infinite parua, fiet $s = 2c\Phi$, cui applicata y debet esse aequalis; abscissa autem x ex formula notissima, qua in ipso vertice a subnormalis $\frac{y\,\partial y}{\partial x}$ semper aequatur radio, qui hic est $2c$, definietur: erit enim $\frac{4cc\Phi\,\partial\Phi}{\partial x} = 2c$, hincque $\partial x = 2c\Phi\,\partial\Phi$, ergo integrando $x = c\Phi\Phi$, quare cum sit $\Phi = \frac{y}{2c}$, erit pro

porti-

portiuncula nostrae curuae circa punctum a, $x = \frac{yy}{4c}$, siue $yy = 4cx$, quae ergo curua congruet cum parabola, cuius parameter $= 4c$, ita vt saltem pro ipso initio axis ra simul sit diameter nostrae curuae.

§. 38. Vtrum autem iste axis ar quoque sit diameter totius curuae quam quaerimus, videamus, examinaturi num sumto angulo Φ negatiuo abscissa x retineat eundem valorem, applicata vero in sui negatiuam abeat? Scribamus ergo $-\Phi$ loco Φ, ac reperiemus

$$x = \frac{c}{\alpha\alpha+1}[\alpha \sin.\Phi(e^{\alpha\Phi} - e^{-\alpha\Phi}) - \cos.\Phi(e^{\alpha\Phi} + e^{-\alpha\Phi})] + \frac{2c}{\alpha\alpha+1},$$

qui valor a praecedente prorsus non discrepat: at vero applicata euadet

$$y = -\frac{c}{\alpha\alpha+1}[\alpha \cos.\Phi(e^{\alpha\Phi} - e^{-\alpha\Phi}) + \sin.\Phi(e^{\alpha\Phi} + e^{-\alpha\Phi})],$$

quae expressio vtique prioris est negatiua; vnde patet, nostrum axem ar curuam quaesitam in duas partes similes et aequales diuidere, ita vt sufficiat alterutrum tantum ramum explorasse. Quia igitur sumto $\Phi = 90°$ tangens curuae axi euadit parallela, sumto autem $\Phi = 180°$ ea ad axem iterum fit normalis, quae vicissitudo perpetuo continget, dum amplitudo Φ angulo recto increscit: euidens est, ramum curuae as in infinitum continuatum per infinitas spiras reuolui, atque adeo absolutis aliquot spiris in ipsam logarithmicam spiralem degenerare. Quando enim amplitudo Φ iam totam circuli circumferentiam aliquoties sumtam superabit, formula $e^{-\alpha\Phi}$ tantum non in nihilum abit, sicque fiet $r = ce^{\alpha\Phi}$, quae ipsam logarithmicam spiralem inuoluit.

§. 39. Ad naturam huius curuae penitius perscrutandam capiamus in axe interuallum $ao = \frac{2c}{\alpha\alpha+1}$, quod ergo mi- Tab. II. Fig. 5.

nus erit quam radius osculi $aa' = 2c$, interuallo $oa' = \frac{2\alpha\alpha c}{\alpha\alpha+1}$; cum igitur sit $ax = x$ et $xs = y$, erit interuallum

$$ox = \frac{c}{\alpha\alpha+1}[\text{cos.}\,\Phi\,(e^{\alpha\Phi} + e^{-\alpha\Phi}) - \alpha\,\text{sin.}\,\Phi\,(e^{\alpha\Phi} - e^{-\alpha\Phi})],$$

quare cum sit angulus $ars = \Phi$, si ponamus angulum $aos = \psi$, erit

$$\text{tang.}\,\psi = \frac{xs}{ox} = \frac{\alpha\,\text{cos.}\,\Phi\,(e^{\alpha\Phi} - e^{-\alpha\Phi}) + \text{sin.}\,\Phi\,(e^{\alpha\Phi} + e^{-\alpha\Phi})}{\text{cos.}\,\Phi\,(e^{\alpha\Phi} + e^{-\alpha\Phi}) - \alpha\,\text{sin.}\,\Phi\,(e^{\alpha\Phi} - e^{-\alpha\Phi})}.$$

Quodsi ergo breuitatis gratia statuamus

$$e^{\alpha\Phi} + e^{-\alpha\Phi} = P \text{ et } e^{\alpha\Phi} - e^{-\alpha\Phi} = Q,$$

habebimus

$$\text{tang.}\,\psi = \frac{\alpha Q\,\text{cos.}\,\Phi + P\,\text{sin.}\,\Phi}{P\,\text{cos.}\,\Phi - \alpha Q\,\text{sin.}\,\Phi} = \frac{P\,\text{tang.}\,\Phi + \alpha Q}{P - \alpha Q\,\text{tang.}\,\Phi},$$

tum autem erit

$$ox = \frac{c}{\alpha\alpha+1}(P\,\text{cos.}\,\Phi - \alpha Q\,\text{sin.}\,\Phi) \text{ et}$$

$$xs = \frac{c}{\alpha\alpha+1}(P\,\text{sin.}\,\Phi + \alpha Q\,\text{cos.}\,\Phi),$$

vnde colligitur

$$os^2 = \frac{cc}{(\alpha\alpha+1)^2}(PP + \alpha\alpha QQ).$$

Est vero

$$PP + \alpha\alpha QQ = (1+\alpha\alpha)(e^{2\alpha\Phi} + e^{-2\alpha\Phi}) + 2(1-\alpha\alpha),$$

ideoque

$$os^2 = \frac{cc}{(\alpha\alpha+1)}(e^{2\alpha\Phi} + e^{-2\alpha\Phi}) + \frac{2cc(1-\alpha\alpha)}{(\alpha\alpha+1)^2}.$$

Praeterea vero per eosdem valores P et Q erit $r = cP$ et $s = \frac{cQ}{\alpha}$.

§. 40. Quodsi iam quaeramus angulum θ ita, vt sit $\text{tang.}\,\theta = \frac{\alpha Q}{P}$, erit

$$\text{tang.}\,\psi = \frac{\text{tang.}\,\Phi + \text{tang.}\,\theta}{1 - \text{tang.}\,\theta\,\text{tang.}\,\Phi} = \text{tang.}\,(\theta + \Phi),$$

vnde

vnde sequitur fore $\psi = \theta + \Phi$, hincque porro angulum $osr = \theta$. Ponamus porro $\sqrt{(PP + \alpha\alpha QQ)} = R$, vt fiat recta $os = \frac{c}{\alpha\alpha+1} R$. Si igitur ex o in rectam sr ducatur perpendiculum oq, ob sin. $\theta = \frac{\alpha Q}{R}$ et cos. $\theta = \frac{P}{R}$ erit $oq = \frac{\alpha c Q}{\alpha\alpha+1}$ et $sq = \frac{cP}{\alpha\alpha+1}$. Cum igitur sit $r = cP$, hinc ista insignis nostrae se prodit curuae proprietas, vt si ex puncto o in rectam sr, quae est normalis ad curuam, demittatur perpendiculum oq, semper sit interuallum $sq = \frac{r}{\alpha\alpha+1}$, quod ergo se habebit ad ipsum radium osculi r, vt $1 : \alpha\alpha + 1$, ex qua conditione per methodum tangentium inuersam ista curua inuestigari poterit, id quod iam passim est factum, ita vt haec curua Geometris non prorsus sit ignota.

§. 41. Consideremus nunc etiam huius curuae euolutam primam, pro qua, vt supra vidimus, erit radius osculi $r' = \frac{\partial r}{\partial \Phi} = \alpha c (e^{\alpha\Phi} - e^{-\alpha\Phi})$; vnde patet, hanc euolutam primam ipsam illam esse curuam, quam casu altero mox sumus euoluturi, id quod ipsa rei natura postulat. Cum enim curua quaesita similis esse debeat euolutae suae secundae, necesse est vt eius euoluta prima similis sit euolutae tertiae. Referat igitur figura euolutam primam $a's'$, exsistente $aa' = 2c$, quae ergo in a' habebit cuspidem, ita vt arcus illi similis sit $a'\sigma'$, tum vero huius curuae $a's'$ euoluta sit $a's''$, quae cum sit euoluta secunda ipsius curuae as, etiam illi similis erit, at situ duplici modo inuerso repraesentata, ita vt hanc euolutum potius *bis inuersam* appellari conueniret quam directam. Tab. II. Fig. 6.

2°. Euolutio casus $r = c(e^{\alpha\Phi} - e^{-\alpha\Phi})$.

§. 42. Pro hoc igitur casu formulas supra pro $r = ce^{-\alpha\Phi}$ inuentas, ab iis subtrahi debent, quae pertinebant ad

 casum

casum $c\,e^{\alpha\phi}$, quo facto nanciscemur sequentes formulas:

$$r = c\,(e^{\alpha\phi} - e^{-\alpha\phi}),$$

$$s = \frac{c}{\alpha}\,(e^{\alpha\phi} + e^{-\alpha\phi}) - \frac{2c}{\alpha},$$

$$x = \frac{c}{\alpha\alpha+1}\,[\alpha\,\text{sin.}\,\phi\,(e^{\alpha\phi} + e^{-\alpha\phi}) - \text{cos.}\,\phi\,(e^{\alpha\phi} - e^{-\alpha\phi})],$$

$$y = \frac{c}{\alpha\alpha+1}\,[\alpha\,\text{cos.}\,\phi\,(e^{\alpha\phi} + e^{-\alpha\phi}) + \text{sin.}\,\phi\,(e^{\alpha\phi} - e^{-\alpha\phi})] - \frac{2\alpha c}{\alpha\alpha+1}.$$

Hic ergo si iterum statuamus

$$P = e^{\alpha\phi} + e^{-\alpha\phi} \text{ et } Q = e^{\alpha\phi} - e^{-\alpha\phi},$$

erit succinctius

$$r = c\,Q,$$

$$s = \frac{c}{\alpha}\,(P - 2),$$

$$x = \frac{c}{\alpha\alpha+1}\,(\alpha\,P\,\text{sin.}\,\phi - Q\,\text{cos.}\,\phi),$$

$$y = \frac{c}{\alpha\alpha+1}\,(\alpha\,P\,\text{cos.}\,\phi + Q\,\text{sin.}\,\phi) - \frac{2\alpha c}{\alpha\alpha+1}.$$

§. 43. In ipso ergo curuae initio a radius osculi erit $r = 0$, vnde iam concludere licet, curuam in puncto a habere cuspidem. Sumta enim amplitudine ϕ infinite parua, fiet $s = \alpha c\,\phi\phi$, hincque $\partial s = 2\alpha c\,\phi\,\partial\phi$, vnde cum sit

$$\partial x = \partial s\,\text{sin.}\,\phi = \phi\,\partial s \text{ et}$$

$$\partial y = \partial s\,\text{cos.}\,\phi = \partial s,$$

integrando colligimus: $x = \frac{2}{3}\alpha c\,\phi^3$ et $y = \alpha c\,\phi\phi$, vnde fit $y^3 = \frac{9}{4}\alpha c\,x\,x$, quae est aequatio pro parabola cubicali secunda, vnde iam concludere licet, curuam hanc talem habere figuram (fig. 7.), ita vt cuspide perpendiculariter super axe $a\,r$ insistat et portio continuata $a\,\sigma$ ad amplitudines negatiuas sit referenda. Sumto autem ϕ negatiuo fiet

Tab. II. Fig. 7.

$$x = -\frac{c}{\alpha\alpha+1}\,(\alpha\,P\,\text{sin.}\,\phi - Q\,\text{cos.}\,\phi),$$

qui valor est praecedentis negatiuus, ita vt pro similibus punctis

ctis s et σ abscissae in contrariam partem vergant, applicata vero in σ erit

$$\sigma\xi = y = \frac{c}{\alpha\alpha+1}(\alpha P \operatorname{cos.}\Phi + Q \operatorname{sin.}\Phi) - \frac{2\alpha c}{\alpha\alpha+1},$$

(quia sumto Φ negatiuo quantitates P et cos. Φ eundem valorem retinent, quantitates vero Q et sin. Φ fiunt negatiuae) qui valor conuenit cum praecedente. Sicque recta ac, ad axem in a normalis, simul erit diameter nostrae curuae.

§. 44. Sumatur nunc in diametro ac retro producto punctum o, vt sit $ao = \frac{2\alpha c}{\alpha\alpha+1}$, ex s porro ad diametrum ducatur normalis sy, et ob $ay = xs = y$ erit

$$oy = \frac{c}{\alpha\alpha+1}(\alpha P \operatorname{cos.}\Phi + Q \operatorname{sin.}\Phi) \text{ et}$$
$$sy = x = \frac{c}{\alpha\alpha+1}(\alpha P \operatorname{sin.}\Phi - Q \operatorname{cos.}\Phi).$$

Hinc ergo si ducatur recta so, secans axem in puncto u, erit

$$os = \frac{c}{\alpha\alpha+1}\sqrt{(\alpha\alpha P P + Q Q)} = \frac{cS}{\alpha\alpha+1},$$

posito $S = \sqrt{(\alpha\alpha P P + Q Q)}$. Vocetur nunc etiam angulus $soy = osx = \psi$, eritque

$$\operatorname{tang.}\psi = \frac{sy}{oy} = \frac{\alpha P \operatorname{sin.}\Phi - Q \operatorname{cos.}\Phi}{\alpha P \operatorname{cos.}\Phi + Q \operatorname{sin.}\Phi} = \frac{\alpha P \operatorname{tang.}\Phi - Q}{\alpha P + Q \operatorname{tang.}\Phi}.$$

Introducamus nunc angulum θ, vt sit $\operatorname{tang.}\theta = \frac{Q}{\alpha P}$, eritque

$$\operatorname{tang.}\psi = \frac{\operatorname{tang.}\Phi - \operatorname{tang.}\theta}{1 + \operatorname{tang.}\Phi \operatorname{tang.}\theta} = \operatorname{tang.}(\Phi - \theta),$$

ideoque $\psi = \Phi - \theta$. Cum nunc sit angulus $sru = \Phi$, hincque angulus $rsx = 90° - \Phi$, erit angulus $osr = 90° - \Phi + \psi$, quamobrem fiet iste angulus $osr = 90° - \theta$, vnde concluditur angulus $aso = \theta$, qui ergo est angulus, quem recta os cum ipsa curua as constituit, cuius ergo tangens est $= \frac{Q}{\alpha P}$, hincque $\operatorname{sin.}\theta = \frac{Q}{S}$ et $\operatorname{cos.}\theta = \frac{\alpha P}{S}$.

§. 45.

§. 45. Quodsi iam ex puncto o in radium osculi ss' ducamus perpendiculum op, ob $os = \frac{cS}{\alpha\alpha+1}$ erit

$$op = os \text{ cos. } \theta = \frac{\alpha c P}{\alpha\alpha+1} \text{ et}$$

$$sp = os \text{ sin. } \theta = \frac{c Q}{\alpha\alpha+1},$$

quare cum sit radius osculi $ss' = r = cQ$, erit interuallum $sp = \frac{r}{\alpha\alpha+1}$, sicque erit $sp : ss' = 1 : \alpha\alpha+1$. Vnde patet, hanc curuam respectu puncti o eadem gaudere proprietate, quam supra pro curua priori inuenimus. Ita, si quaeratur curva as talis, vt si ex puncto fixo in radium osculi demittatur perpendiculum op, oporteat esse $sr : ss' = 1 : \alpha\alpha+1$, tam curua praecedens, quam ea quam nunc inuenimus, quaestioni satisfacient, ex quo iam insignis affinitas inter has duas curuas elucet, dum altera similis est euolutae alterius. Ceterum notasse iuuabit inter arcum $as = s$ et perpendiculum op istam relationem intercedere: $s + \frac{2c}{\alpha} = \frac{\alpha\alpha+1}{\alpha\alpha} \cdot op$.

§. 46. Ducamus nunc etiam rectam os' ad euolutam curuae as, et cum sit $ss' = r = cQ$, erit interuallum $ps' = \frac{\alpha\alpha c Q}{\alpha\alpha+1}$, vnde ob $op = \frac{\alpha c P}{\alpha\alpha+1}$, fiet

$$os' = \frac{\alpha c}{\alpha\alpha+1} \sqrt{(PP + \alpha\alpha QQ)} = \frac{\alpha c}{\alpha\alpha+1} \cdot R,$$

prouti scilicet supra posuimus $R = \sqrt{(PP + \alpha\alpha QQ)}$, ex quo patet fore $os : os' = S : \alpha R$. Quodsi iam porro vocemus angulum $s'op = \xi$, erit tang. $\xi = \frac{ps'}{op} = \frac{\alpha Q}{P}$; quare cum sit angulus $sop = \theta$, fiet angulus $sos' = \theta + \xi$, ideoque eius tangens

$$= \frac{\text{tang.}\,\theta + \text{tang.}\,\xi}{1 - \text{tang.}\,\theta\ \text{tang.}\,\xi} = \frac{1+\alpha\alpha}{\alpha} \cdot \frac{PQ}{PP - QQ} = \frac{\alpha\alpha+1}{4\alpha}(e^{2\alpha\Phi} - e^{-2\alpha\Phi}).$$

Quodsi ergo statuatur $\alpha = 1$, vt curua quaesita aequalis fiat suae euolutae secundae fiet

$$S = R = \sqrt{(PP + QQ)} = \sqrt{2(e^{2\Phi} + e^{-2\Phi})},$$

hoc

hoc ergo casu erit

$$os = os' = \tfrac{1}{2}cR = \tfrac{1}{2}c\sqrt{2(e^{2\Phi} + e^{-2\Phi})},$$

anguli autem sos' tangens $= \tfrac{1}{2}(e^{2\Phi} - e^{-2\Phi})$. Praeterea vero pro hoc casu $\alpha = 1$ habebimus

$$r = c(e^{\Phi} - e^{-\Phi}),$$
$$s = c(e^{\Phi} + e^{-\Phi} - 2),$$

ipsae vero coordinatae erunt

$$ax = sy = x = \tfrac{1}{2}c[\text{sin.}\Phi(e^{\Phi} + e^{-\Phi}) - \text{cos.}\Phi(e^{\Phi} - e^{-\Phi})] \text{ et}$$
$$xs = ay = y = \tfrac{1}{2}c[\text{cos.}\Phi(e^{\Phi} + e^{-\Phi}) + \text{sin.}\Phi(e^{\Phi} - e^{-\Phi})] - c.$$

Sicque hoc casu interuallum ao erit $= c$.

§. 47. Si simili modo pro curua casus praecedentis statuamus $\alpha = 1$, pro ea habebimus

$$r = c(e^{\Phi} + e^{-\Phi}),$$
$$s = c(e^{\Phi} - e^{-\Phi}),$$
$$ax = x = \tfrac{1}{2}c[\text{sin.}\Phi(e^{\Phi} - e^{-\Phi}) - \text{cos.}\Phi(e^{\Phi} + e^{-\Phi})] + c,$$
$$xs = y = \tfrac{1}{2}c[\text{cos.}\Phi(e^{\Phi} - e^{-\Phi}) + \text{sin.}\Phi(e^{\Phi} + e^{-\Phi})].$$

Tab. II. Fig. 5.

Sicque etiam hoc casu erit interuallum $ao = c$, at radius osculi in puncto $a = 2c$. Hae autem duae curuae hac insigni proprietate erunt praeditae, vt altera alterius sit euoluta.

Tab. III. Fig. 8.

§. 48. Quo autem relatio inter has duas curuas maxime memorabiles, quarum altera alterius est euoluta, clarius perspiciatur, ambas coniunctim in eadem figura repraesentemus, quae cum ad communem diametrum referantur, sit recta caa' iste diameter, et as curua posteriore loco inuenta, quae ergo in a habebit cuspidem, cuius curuae si radius osculi in s sit recta

$s\sigma$, erit σ punctum in eius euoluta $a\sigma$. Huius vero curuae quia radius osculi in a est $aa' = 2c$, referat curua $a's'$ euolutam curuae $a\sigma$, quae ergo similis et aequalis primae curuae as, quamque radius osculi $\sigma s'$ in puncto s' tanget. Denique ducto istius curuae radio osculi $\sigma' s'$, is eius euolutam $a'\sigma'$ in puncto σ' tanget, eritque pariter curua $a'\sigma'$ similis et aequalis curuae $a\sigma$. Manifestum igitur est omnes arcus hic exhibitos as, $a\sigma$, $a's'$, $a'\sigma'$, esse aeque amplos, ideoque eorum amplitudinem communem $= \Phi$. Quare si ex punctis s, σ, s' et σ' ad communem diametrum ducantur normales sy, $\sigma\xi$, $s'y'$, $\sigma'\xi'$, elementa harum duarum curuarum sequenti modo se habebunt:

I. Pro curuis as et $a's'$.

1.) Arcus $as = a's' = c(e^{\Phi} + e^{-\Phi}) - 2c$,

2.) Radius osculi in s et $s' = c(e^{\Phi} - e^{-\Phi})$,

cui aequales sunt $s\sigma$ et $s'\sigma'$,

3.) 4.) Coord. $\begin{cases} ay = a'y' = \frac{1}{2}c\,[\cos.\Phi\,(e^{\Phi} + e^{-\Phi}) + \sin.\Phi\,(e^{\Phi} - e^{-\Phi})] - c, \\ sy = s'y' = \frac{1}{2}c\,[\sin.\Phi\,(e^{\Phi} + e^{-\Phi}) - \cos.\Phi\,(e^{\Phi} - e^{-\Phi})]. \end{cases}$

II. Pro curuis $a\sigma$ et $a'\sigma'$.

1.) Arcus $a\sigma = a'\sigma' = c(e^{\Phi} - e^{-\Phi})$,

2.) Radius osculi in σ et $\sigma' = c(e^{\Phi} + e^{-\Phi})$,

3.) 4.) Coordin. $\begin{cases} a\xi = a'\xi' = \frac{1}{2}c\,[\sin.\Phi\,(e^{\Phi} - e^{-\Phi}) - \cos.\Phi\,(e^{\Phi} + e^{-\Phi})] + c, \\ \xi\sigma = \xi'\sigma' = \frac{1}{2}c\,[\cos.\Phi\,(e^{\Phi} - e^{-\Phi}) + \sin.\Phi\,(e^{\Phi} + e^{-\Phi})]. \end{cases}$

Cete-

Ceterum ambae iſtae curuae as et $a\sigma$ per infinitos gyros continuo creſcentes in infinitum excurrent.

III. De curuis

quae ſuis euolutis ſecundis *inuerſe* ſunt ſimiles, vbi $r'' = -\alpha^2 r$.

§. 49. Cum igitur hic ſit $\lambda\lambda + \alpha\alpha = 0$, ob $n = 2$ erit $\omega = \frac{\pi}{2}$, hinc $\zeta = 0$ et $\eta = \alpha$, et formulae pro hoc caſu erunt ſequentes:

$$r = c\,\text{ſin.}\,(\gamma + \alpha\Phi),$$

$$s = -\tfrac{c}{\alpha}\,\text{coſ.}\,(\gamma + \alpha\Phi) + \tfrac{c}{\alpha}\,\text{coſ.}\,\gamma,$$

$$x = -\tfrac{c}{2(\alpha+1)}\,\text{ſin.}\,(\gamma + (\alpha+1)\Phi) + \tfrac{c}{2(\alpha-1)}\,\text{ſin.}\,(\gamma + (\alpha-1)\Phi) - \tfrac{c}{\alpha\alpha-1}\,\text{ſin.}\,\gamma,$$

$$y = -\tfrac{c}{2(\alpha+1)}\,\text{coſ.}\,(\gamma + (\alpha+1)\Phi) - \tfrac{c}{2(\alpha-1)}\,\text{coſ.}\,(\gamma + (\alpha-1)\Phi) + \tfrac{\alpha c}{\alpha\alpha-1}\,\text{coſ.}\,\gamma.$$

§. 50. Ex harum formularum prima $r = c\,\text{ſin.}(\gamma + \alpha\Phi)$, in qua reliquae omnes continentur, ſtatim patet, perinde eſſe ſiue α poſitiue ſiue negatiue accipiatur, quoniam poſterior caſus eodem redit, ac ſi α maneret poſitiuum, amplitudo vero Φ negatiue caperetur, vnde ſufficiet quantitatem α perpetuo vt poſitiuam conſideraſſe. Deinde quia amplitudinem Φ pro lubitu augere ſiue diminuere licet, dum curua prorſus eadem manet, manifeſtum eſt curuam eandem eſſe proditurum, quicunque valor ipſi γ tribuatur; quamobrem ſumamus $\gamma = 0$, et formulae pro curua quaeſita ſequenti modo contrahentur:

$$r = c \sin. \alpha \Phi,$$
$$s = -\frac{c}{\alpha} \cos. \alpha \Phi + \frac{c}{\alpha},$$
$$x = -\frac{c}{2(\alpha+1)} \sin. (\alpha + 1) \Phi + \frac{c}{2(\alpha-1)} \sin. (\alpha - 1) \Phi,$$
$$y = -\frac{c}{2(\alpha+1)} \cos. (\alpha+1)\Phi - \frac{c}{2(\alpha-1)} \cos. (\alpha-1)\Phi + \frac{\alpha c}{\alpha\alpha - 1}.$$

§. 51. Hic ergo vnica constans arbitraria inest c, quae, siue maior siue minor accipiatur, nihil mutat in natura ipsius curuae. Omnis igitur varietas orietur ex quantitate α, qua ratio similitudinis continetur, cum pro euoluta secunda esse debeat $r'' = -\alpha\alpha r$; vnde patet, si fuerit $\alpha > 1$, tum euolutam secundam maiorem fore ipsa curua quaesita, contra autem minorem, si accipiatur $\alpha < 1$; sumto autem $\alpha = 1$, euoluta secunda adeo ipsi curuae prodire debet aequalis. Quamobrem hic tres casus euolui conueniet, quos ergo singulos seorsim tractemus.

1°. Euolutio casus $\alpha = 1$.

§. 52. Hoc casu singulare phaenomenon statim se offert in formulis pro x et y inuentis, quia ibi denominator $\alpha - 1$ euanescit. Quia autem hoc casu angulus $(\alpha - 1)\Phi$ fit infinite paruus, eius sinus erit $(\alpha - 1)\Phi$, at vero cosinus $= 1$, quo obseruato sequentes nanciscemur formulas:

$$r = c \sin. \Phi;$$
$$s = c(1 - \cos. \Phi);$$
$$x = -\frac{c}{4} \sin. 2\Phi + \frac{c\Phi}{2};$$
$$y = -\frac{c}{4} \cos. 2\Phi + \frac{c}{4};$$

quos posteriores valores facilius immediate reperire licet. Cum enim sit $\partial s = r \partial \Phi = c \partial \Phi \sin. \Phi$, erit

$$\partial x = \partial s \sin. \Phi = c \partial \Phi \sin. \Phi^2 = \tfrac{1}{2} c \partial \Phi (1 - \cos. 2\Phi) \text{ et}$$
$$\partial y = \partial s \cos. \Phi = c \partial \Phi \sin. \Phi \cos. \Phi = \tfrac{1}{2} c \partial \Phi \sin. 2\Phi,$$

vnde

vnde integrando colligitur

$$x = \tfrac{1}{2} c \Phi - \tfrac{1}{4} c \sin. 2\Phi \text{ et}$$

$$y = \tfrac{1}{4} c - \tfrac{1}{4} c \cos. 2\Phi.$$

§. 53. Hinc primo patet in ipſo puncto a, vbi $\Phi = 0$, etiam radium oſculi curuae r fore $= 0$, et curuam in hoc puncto cuſpidem eſſe habituram, a qua porro per arcum $a\sigma$ retro eſt continuanda, pro quo amplitudo Φ negatiue ſumi debet, vnde in puncto a erit radius oſculi $= - c \sin. \Phi$ et arcus $a\sigma = c(1 - \cos. \Phi)$, quippe qui per legem continuitatis iterum fit poſitiuus, propterea quod ſurſum vergit, id quod ex valore ipſius y patet, cuius ſignum non mutatur; at vero abſciſſa x in negatiuam abit, ideoque in partem contrariam $\alpha\xi = ax$; ex quo patet, curuam in a habere diametrum ac axi normalem. Deinde idem euenit quoties Φ fuerit Φ vel π, vel 2π, vel 3π, vel 4π etc., quippe quibus caſibus omnibus fit tam $r = 0$ quam $y = 0$; at vero ſumto $\Phi = \pi$ fit $x = \frac{c\pi}{2}$; tum vero ex $\Phi = 2\pi$ fit $x = c\pi$; ſimilique modo ſumto $\Phi = 3\pi$ erit $x = \frac{3}{2} c\pi$, et ita porro: vnde patet, in omnibus his punctis radium oſculi euaneſcere, haecque puncta ſuper axe ſecundum aequalia interualla eſſe diſpoſita $\frac{1}{2} c\pi$, prorſus vti in cycloide ſuper axe deſcripta euenit. In punctis autem intermediis, vbi eſt vel $\Phi = \frac{1}{2}\pi$, vel $\Phi = \frac{3}{2}\pi$, vel $\Phi = \frac{5}{2}\pi$ etc. vbique applicata y euadet maxima $= \frac{c}{2}$, quae ergo exhibebit diametrum circuli, qui ſuper axe voluendo cycloidem deſcribit: haec enim altitudo $\frac{1}{2} c$ ſe habet ad interuallum cuſpidum $\frac{1}{2} c\pi$, vt $1 : \pi$ hoc eſt vt diameter ad peripheriam. Tab. III. Fig. 9.

§. 54. Quo autem clarius appareat, hanc curuam reuera eſſe cycloidem, conſideremus radium oſculi in puncto s,

 qui

qui fit $s s' = c$ fin. Φ, cuius interfectio cum axe affumto fit r, et angulus $a r s = \Phi$. Iam quia inuenimus applicatam

$$x s = y = \tfrac{1}{4} c (1 - \text{cof. } 2\Phi) = \tfrac{1}{2} c \text{ fin. } \Phi^2,$$

erit primo recta

$$s r = \frac{y}{\text{fin. } \Phi} = \tfrac{1}{2} c \text{ fin. } \Phi = \tfrac{1}{2} r,$$

ficque patet, radium ofculi $s s'$ in puncto r bifecari, quae eft notiffima proprietas cycloïdis. Porro vero ob $\frac{xs}{rx} =$ tang. Φ erit $r x = \frac{1}{2} c$ fin. Φ cof. $\Phi = \frac{1}{4} c$ fin. 2Φ; quare cum fit $a x = x = \frac{1}{2} c \Phi - \frac{1}{4} c$ fin. 2Φ, erit interuallum $a r = \frac{1}{2} c \Phi$, ficque erit recta $a r$ ad normalem $r s$ vt Φ : fin. Φ.

Tab, III. Fig. 10. §. 55. Erigatur nunc ex r perpendiculum $r o = \frac{1}{4} c$, et ex o in $s r$ ducatur normalis $o p$, atque ob angulum $s r o = 90° - \Phi$, ideoque $r o p = \Phi$, erit $r p = \frac{1}{4} c$ fin. $\Phi = \frac{1}{2} r s$, ita vt punctum p in medium rectae $r s$ incidat, vnde etiam recta $o s$ aequalis erit ipfi $r o = \frac{1}{4} c$. Quod fi iam centro o radio $o r$ defcribatur circulus per puncta r et s tranfiens, longitudo arcus $r s$ erit $\frac{1}{2} c \Phi$, ob angulum $r o s = 2\Phi$, vnde patet, iftum arcum $r s$ aequalem effe diftantiae $a r$. Sicque manifeftum eft noftram curuam effe cycloidem prouolutione circuli, cuius radius $o r = \frac{1}{4} c$ ideoque diameter $= \frac{1}{2} c$, fuper axe $a r$ defcriptam, quae ergo fuae euolutae fecundae eft aequalis. Quod autem etiam euoluta prima fit fimilis cyclois, ex eius radio ofculi facile intelligitur, qui cum in genere fit $r' = \frac{\partial r}{\partial \Phi}$, erit $r' = c$ cof. $\Phi = c$ fin. $(90° - \Phi)$, ita vt in euoluta tantum amplitudo ab alio termino computetur. Haec autem omnia inuulgus maxime funt nota.

2°. Euolutio cafus quo $\alpha < 1$.

§. 56. Hoc ergo cafu, quo euoluta fecunda minor eft quam ipfa curua, in ratione $\alpha\alpha : 1$, formulae noftrae ita fe habe-

habebunt:

$$r = c\,\text{fin.}\,\alpha\Phi,$$

$$s = \frac{2c}{\alpha}\,\text{fin.}\,\tfrac{1}{2}\alpha\Phi^2,$$

$$x = -\frac{c}{2(1+\alpha)}\,\text{fin.}\,(1+\alpha)\Phi + \frac{c}{2(1-\alpha)}\,\text{fin.}\,(1-\alpha)\Phi,$$

$$y = -\frac{c}{2(1+\alpha)}\,\text{cof.}\,(1+\alpha)\Phi + \frac{c}{2(1-\alpha)}\,\text{cof.}\,(1-\alpha)\Phi - \frac{\alpha c}{1-\alpha\alpha},$$

vnde patet radium ofculi r toties euanefcere, ideoque curuam cufpidem effe habituram, quoties fuerit vel $\Phi = 0$, vel $\Phi = \frac{\pi}{\alpha}$, vel $\Phi = \frac{2\pi}{\alpha}$, vel $\Phi = \frac{3\pi}{\alpha}$, vel in genere $\Phi = \frac{i\pi}{\alpha}$; maximum autem valorem effe adepturum, fcilicet $r = \pm c$, vbi fuerit vel $\Phi = \frac{\pi}{2\alpha}$, vel $\Phi = \frac{3\pi}{2\alpha}$, vel $\Phi = \frac{5\pi}{2\alpha}$, etc.

§. 57. Ex his intelligitur, vti in cafu praecedente, curuam habituram effe diametrum ac, in puncto a ad axem normalem, ad quem ex s normaliter ducatur $sy = ax = x$, vt fit $ay = xs = y$. Iam in hac recta ca, retro producta, capiatur interuallum $ao = \frac{\alpha c}{1-\alpha\alpha}$, vt fiat Tab. III. Fig. 11.

$$oy = -\frac{c}{2(1+\alpha)}\,\text{cof.}\,(1+\alpha)\Phi + \frac{c}{2(1-\alpha)}\,\text{cof.}\,(1-\alpha)\Phi$$

exfiftente

$$sy = -\frac{c}{2(1+\alpha)}\,\text{fin.}\,(1+\alpha)\Phi + \frac{c}{2(1-\alpha)}\,\text{fin.}\,(1-\alpha)\Phi,$$

vnde ducta recta os erit

$$os^2 = \frac{cc}{4(1+\alpha)^2} - \frac{cc}{2(1-\alpha\alpha)}\,\text{cof.}\,2\alpha\Phi + \frac{cc}{4(1-\alpha)^2},\ \text{fiue}$$

$$os^2 = \frac{cc(1+\alpha\alpha)}{2(1-\alpha\alpha)^2} - \frac{cc}{2(1-\alpha\alpha)}\,\text{cof.}\,2\alpha\Phi,$$

ita vt fit

$$os = \frac{c}{1-\alpha\alpha}\sqrt{[\tfrac{1}{2}(1+\alpha\alpha) - (1-\alpha\alpha)\,\text{cof.}\,2\alpha\Phi)]}.$$

Hinc igitur pro omnibus cufpidibus, vbi eft $\alpha\Phi = i\pi$, ideoque cof. $2\alpha\Phi = +1$, erit $os = \frac{\alpha c}{1-\alpha\alpha} = oa$, ficque omnes cu-

fpiſpides reperientur in peripheria circuli centro o radio $o\,a$ deſcripti. Deinde vero omnia puncta, vbi radius oſculi fit maximus, quod euenit ſi fuerit $2\,\alpha\,\Phi = (2\,i + 1)\,\pi$, ideoque coſ. $2\,\alpha\Phi = -1$, a puncto o remota erunt interuallo $o\,s = \frac{c}{1-\alpha\alpha}$, quod praecedens interuallum $a\,o$ ſuperat quantitate $\frac{c}{1+\alpha}$, quare omnia iſta puncta reperientur in peripheria circuli centro o deſcripti, cuius radius eſt $\frac{c}{1-\alpha\alpha} = \frac{\alpha\,c}{1-\alpha\alpha} + \frac{c}{1+\alpha}$.

§. 58. Vocemus nunc angulum $a\,o\,s = \psi = o\,s\,x$ eritque

$$\text{tang. } \psi = \frac{y\,s}{y\,o} = \frac{(1-\alpha)\,\text{ſin.}\,(1+\alpha)\,\Phi - (1+\alpha)\,\text{ſin.}\,(1-\alpha)\,\Phi}{(1-\alpha)\,\text{coſ.}\,(1+\alpha)\,\Phi - (1+\alpha)\,\text{coſ.}\,(1-\alpha)\,\Phi},$$

quae expreſſio, euoluendo angulos $(1+\alpha)\Phi$ et $(1-\alpha)\Phi$, transformatur in hanc:

$$\text{tang. } \psi = \frac{\alpha\,\text{ſin.}\,\Phi\,\text{coſ.}\,\alpha\,\Phi - \text{coſ.}\,\Phi\,\text{ſin.}\,\alpha\,\Phi}{\alpha\,\text{coſ.}\,\Phi\,\text{coſ.}\,\alpha\,\Phi + \text{ſin.}\,\Phi\,\text{ſin.}\,\alpha\,\Phi} = \frac{\alpha\,\text{tang.}\,\Phi - \text{tang.}\,\alpha\,\Phi}{\alpha + \text{tang.}\,\Phi\,\text{tang.}\,\alpha\,\Phi},$$

vnde loca ſingularum cuſpidum haud difficulter detegentur.

§. 59. Ad hanc formulam magis euoluendam introducamus angulum θ, vt ſit tang. $\theta = \frac{1}{\alpha}$ tang. $\alpha\,\Phi$, eritque

$$\text{tang. } \psi = \frac{\text{tang.}\,\Phi - \text{tang.}\,\theta}{1 + \text{tang.}\,\Phi\,\text{tang.}\,\theta} = \text{tang.}\,(\Phi - \theta),$$

ita vt ſit angulus $o\,s\,x = \Phi - \theta$. Quoniam igitur angulus $x\,s\,r$ eſt $90° - \Phi$, hinc fiet angulus $o\,s\,r = 90° - \theta$, conſequenter angulus $a\,s\,o = \theta$, qui ergo angulus euaneſcit, ſi fuerit vel $\Phi = 0$, vel $\Phi = \frac{i\,\pi}{\alpha}$, contra vero recta $o\,s$ ad curuam erit normalis, ſiue $\theta = 90°$, quoties fuerit $\alpha\Phi = \frac{\pi}{2}$, vel $\frac{3\,\pi}{2}$, vel $\frac{5\,\pi}{2}$.

§. 60. Demittamus nunc ex o in radium oſculi perpendiculum $o\,p$, et poſito breuitatis gratia $o\,s = z$, ob angulum $o\,s\,p = 90° - \theta$ fiet $o\,p = z\,\text{coſ.}\,\theta$ et $s\,p = z\,\text{ſin.}\,\theta$. Iam centro

tro o radio $o\,a$ describatur circulus, radium osculi secans in u, vt sit $o\,u = \frac{\alpha c}{1-\alpha\alpha}$, eritque

$$u\,p^2 = o\,u^2 - o\,p^2 = \frac{\alpha\alpha c c}{(1-\alpha\alpha)^2} - z\,z\,\text{cos}.\,\theta^2.$$

Quodsi iam in valore ipsius $z\,z$ loco cos. $2\,\alpha\,\Phi$ scribamus valorem cos. $\alpha\,\Phi^2$ — sin. $\alpha\,\Phi^2$, fiet

$$z\,z = \frac{c c}{(1-\alpha\alpha)^2}(\text{sin}.\,\alpha\,\Phi^2 + \alpha\alpha\,\text{cos}.\,\alpha\,\Phi^2);$$

quia autem tang. $\theta = \frac{1}{\alpha}$ tang. $\alpha\,\Phi$, erit

$$\text{cos}.\,\theta^2 = \frac{\alpha\alpha\,\text{cos}.\,\alpha\,\Phi^2}{\text{sin}.\,\alpha\,\Phi^2 + \alpha\alpha\,\text{cos}.\,\alpha\,\Phi^2}$$

quibus valoribus substitutis fiet

$$z\,z\,\text{cos}.\,\theta^2 = \frac{\alpha\alpha c c\,\text{cos}.\,\alpha\,\Phi^2}{(1-\alpha\alpha)^2},$$

similique modo

$$z\,z\,\text{sin}.\,\theta^2 = \frac{c c\,\text{sin}.\,\alpha\,\Phi^2}{(1-\alpha\alpha)^2},\ \text{vnde fit}$$

$$u\,p^2 = \frac{\alpha\alpha c c\,\text{sin}.\,\Phi^2}{(1-\alpha\alpha)^2}\ \text{ideoque}\ u\,p = \frac{\alpha c\,\text{sin}.\,\alpha\,\Phi}{1-\alpha\alpha},$$

vnde patet fore angulum $u\,o\,p = \alpha\,\Phi$. Quare cum sit

$$s\,p = z\,\text{sin}.\,\theta = \frac{c\,\text{sin}.\,\alpha\,\Phi}{1-\alpha\alpha},$$

erit tota distantia, seu recta

$$s\,u = s\,p - u\,p = \frac{c\,\text{sin}.\,\alpha\,\Phi}{1+\alpha}.$$

§. 61. Cum igitur sit radius osculi $s\,s' = c$ sin. $\alpha\,\Phi$, euidens est rectas $s\,u$ et $s\,p$ ad eum vbique constantem tenere rationem: erit enim $s\,u = \frac{r}{1+\alpha}$ et $s\,p = \frac{r}{1-\alpha\alpha}$, ita vt sit $s\,u : s\,s' = 1 : 1+\alpha$ et $s\,p : s\,s' = 1 : 1-\alpha\alpha$ et $s\,u : s\,p = 1-\alpha : 1$. Porro vero erit $s'u = \frac{\alpha r}{1+\alpha}$ et $s'p = \frac{\alpha\alpha r}{1-\alpha\alpha}$. Haec autem sola conditio, quod radius osculi $s\,s'$ a circulo in u ita secatur, vt interuallum su ad ipſum radium oſculi ss' datam teneat rationem, ſufficit, ad euincendum, curuam noſtram eſſe epicycloidem, ſuper circulo immobili, cuius radius oa, a circulo mobili, cuius diameter $= \frac{c}{1+\alpha}$ deſcriptam, cuius curuae Phaenomena paſſim abunde ſunt expoſita.

§. 62. Ceterum quoniam inuenimus angulum $uop = \alpha\Phi$, erit angulus $oup = 90° - \alpha\Phi$; praeterea vero ob angulum $osp = 90° - \theta$, erit angulus $sop = \theta$, hincque colligitur angulus $sou = \theta - \alpha\Phi$, cui ſi addatur angulus $aos = \psi = \Phi - \theta$, prodibit angulus $acu = (1-\alpha)\Phi$, qui ductus in radium $oa = \frac{\alpha c}{1-\alpha\alpha}$ praebet ipſum arcum $au = \frac{\alpha c \Phi}{1+\alpha}$. Erat vero recta $su = \frac{c \,\text{ſin.}\, \alpha\Phi}{1+\alpha}$, ita vt ſe habeat arcus au ad rectam su vt angulus $\alpha\Phi$ ad ſuum ſinum. Quodſi iam radius ou producatur vsque in q, vt ſit $uq = \frac{c}{1+\alpha}$, ob angulum $suq = 90° - \alpha\Phi$ erit $su = uq \,\text{ſin.}\, \alpha\Phi = uq \,\text{coſ.}\, suq$; vnde patet, rectam qs fore ad us normalem, ideoque curuam tangere. Quare ſi circa diametrum $uq = \frac{c}{1+\alpha}$ deſcribatur circulus, is primo circulum au tanget in u, tum vero per ipſum punctum s tranſibit, vnde ob angulum $uqs = \alpha\Phi$, erit arcus $us = \frac{c}{1+\alpha} \,\text{ſin.}\, \alpha\Phi$ ideoque aequalis $\frac{\alpha c \Phi}{1+\alpha}$. Ex quo manifeſtum eſt, noſtram curuam eſſe epicycloidem, prouolutione circuli mobilis usq, cuius diameter $= \frac{c}{1+\alpha}$, ſuper circulo immobili au, cuius radius $= \frac{\alpha c}{1-\alpha\alpha}$ generatam. Hinc porro, quia peripheria circuli mobilis eſt $\frac{\pi}{1+\alpha}$, capiamus in circulo immobili arcum ab illi aequalem, eritque b punctum, quo circulus mobilis poſt integram reuolutionem peruenit et hic nouam cuſpidem formabit. Pro hoc igitur puncto b erit angulus $aob = \frac{\pi(1-\alpha)}{\alpha}$.

Tab. III. Fig. 12.

3.

3°. Euolutio casus quo $\alpha > 1$.

§. 63. Omnes euolutiones hic eaedem manent vt in articulo praecedente, hoc tantum discrimine, vt loco $1-\alpha$ scribi debeat $-(\alpha-1)$. Hinc igitur statim patet, punctum o hoc casu supra axem nostrum ar cadere, ita vt sit $ao = \frac{\alpha c}{\alpha\alpha-1}$. Ex hoc igitur centro o radio ao describatur circulus au radium osculi ss' secans in u, qui nobis referet circulum immobilem, super cuius peripheria concaua alter circulus, cuius radius erit vt ante $\frac{c}{1+\alpha}$, mobilis prouoluitur, circulus autem iste mobilis pro puncto s ita erit situs, vt immobilem in puncto u tangat simulque per punctum s transeat. Hoc igitur casu, si radius osculi ss' retro continuetur, in eumque ex o perpendiculum demittatur op, erit vt ante $sp = \frac{r}{\alpha\alpha-1}$ et $su = \frac{r}{1+\alpha}$, porro $s'u = \frac{\alpha r}{1+\alpha}$ et $s'p = \frac{\alpha\alpha r}{\alpha\alpha-1}$. Quamdiu ergo diameter circuli mobilis $\frac{c}{1+\alpha}$ minor est quam diameter circuli immobilis $\frac{2\alpha c}{1+\alpha}$, ille intra circulum mobilem suas prouolutiones peraget et eas curuas describet, quae sub nomine hypocycloidum sunt notae. Sin autem circulus mobilis maior sit quam immobilis, tota curua extra circulum immobilem cadet, dum antea tota intra eum erat sita. Casus autem quo ambo circuli fiunt aequales, hoc est $\frac{1}{1+\alpha} = \frac{2\alpha}{\alpha\alpha-1}$, quia tum foret $\alpha = -1$, locum habere nequit, quoniam supra iam valores negatiuos ipsius α exclusimus. Atque ob eandem rationem etiam casus, quibus circulus mobilis maior fieret quam immobilis, excluduntur, quia fieret $\alpha - 1 > 2\alpha$. Sic igitur patet, alias curuas non satisfacere, praeter epicycloides et hypocycloides. Tab. III. Fig. 13.

§. 64.

§. 64. His igitur circa euolutas tam primas quam secundas expeditis finem huic tractationi imponimus, quoniam, si omnes curuae desiderentur, quae suis sint euolutis, vel tertiis, vel quartis, vel altioris ordinis similes, supra praecepta iam dilucide sunt exposita, quorum beneficio pro quouis casu formulae omnes plane solutiones in se continentes assignari poterunt. Ipsae autem hae curuae plerumque tantopere fiunt complicatae, vt vix quicquam notatu dignum occurrat, quod operae pretium foret commemorare.

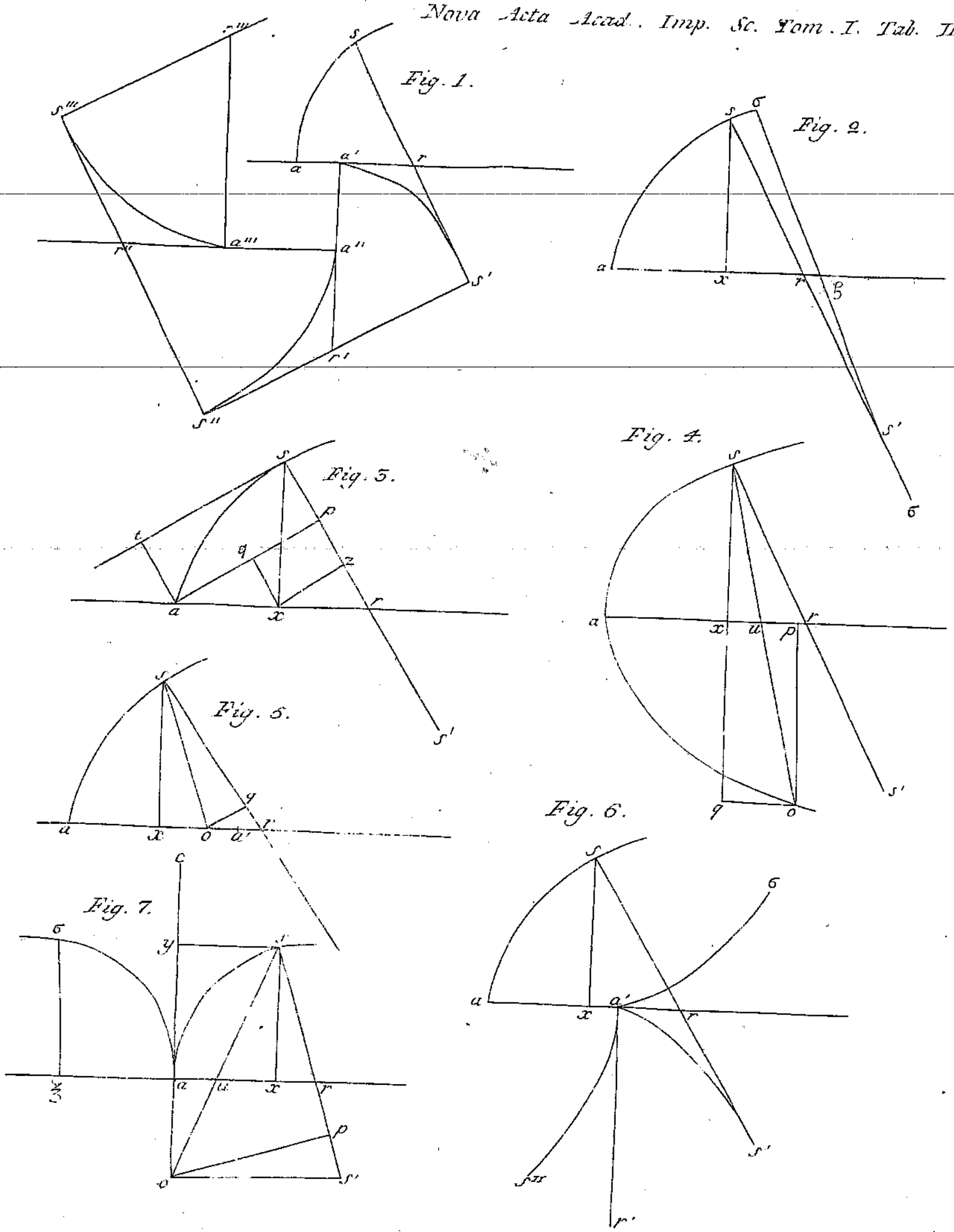
Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4.
Fig. 5.
Fig. 6.
Fig. 7.

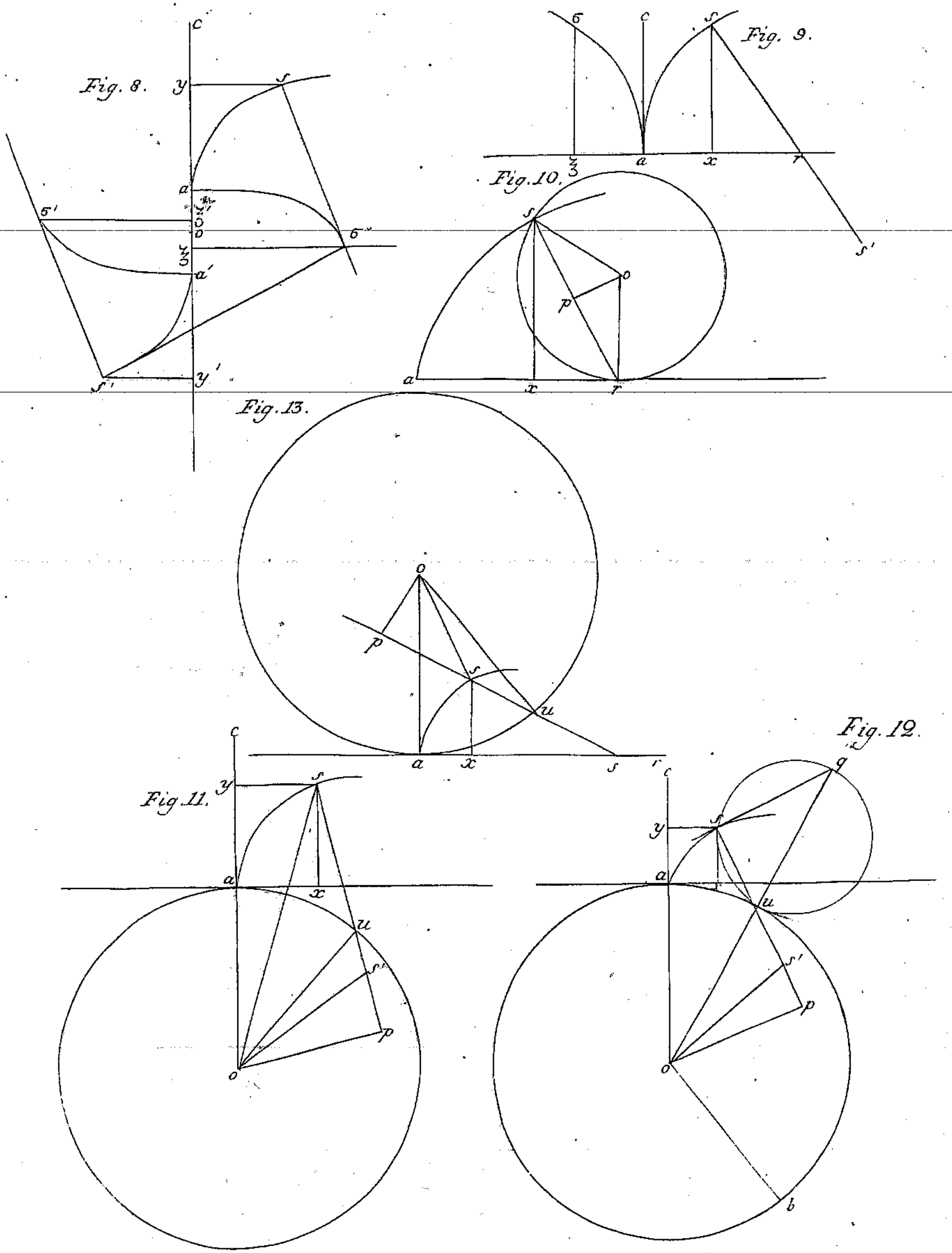
Fig. 8.
Fig. 9.
Fig. 10.
Fig. 13.
Fig. 11.
Fig. 12.