COMMENTATIO
DE
CVRVIS TRACTORIIS.

Auctore

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Conuent. exhib. d. 19. Iun. 1775.

§. 1.

Quae olim a Geometris de curuis tractoriis ſunt inueſtigata, quanquam ad doctrinam motus pertinere videntur, tamen nullo modo ad Mechanicam referri poſſunt: eiusmodi enim hypotheſi innituntur, quae veris principiis motus manifeſto refragatur. Nihilo vero minus, admiſſa iſta hypotheſi, ſi res tantum geometrice conſideretur, quae ſuper hoc argumento ſunt inuenta omni attentione digna ſunt putanda, atque adeo ab experientia vix aberrare ſolent. Quamobrem haud inutile fore arbitror, totum hoc negotium accuratius perſcrutari et ſecundum vera motus principia diiudicare.

§. 2. Conſiderari autem ſolet via, quam corpuſculum ſuper plano horizontali deſcribit, dum ope fili ſecundum lineam ſiue rectam ſiue curuam protrahitur; atque haec quaeſtio ita ad Geometriam reuocari ſolet, vt curua deſcripta perpetuo a directione fili tangatur, atque adeo omnes tangentes iſtius curuae deſcriptae vsque ad lineam, iuxta quam

 filum

filum protrahitur, productae, vbique eiusdem ſint longitudinis. Vt autem talis motus eueniat, auctores probe monuerunt, planum, ſuper quo iſte motus producitur, neutiquam politum, ſed ſatis eſſe debere aſperum; tum vero etiam neceſſe eſſe, vt filum lente promoueatur, quandoquidem, niſi hae conditiones obſeruentur, curua deſcripta plurimum a calculo eſſet diſcrepatura.

Tab. I. Fig. 1. §. 3. Ita ſi corpuſculo C alligatum ſit filum $CA = a$, cuius terminus A iuxta lineam rectam AB protrahitur, corpuſculum in linea quadam curua CY promouebitur, cuius tangentes YT e ſingulis punctis ad rectam AB productae vbique longitudini fili a aequentur; vnde ſi pro puncto Y vocetur abſciſſa $AX = x$ et applicata $XY = y$, elementum vero curuae $Yy = \partial s$, erit $-\partial y : \partial s = y : a$, ideoque $y \partial s = -a \partial y$ et $\partial s = -\frac{a \partial y}{y}$, vnde integrando ſtatim colligitur arcus curuae $Cy = s = -a l y + C$. Quare ſi initio filum CA ad rectam AB fuerit normale, tum erat $y = a$ et $s = 0$, ex quo colligitur $s = a l \frac{a}{y}$. Vt autem aequatio inter coordinatas eruatur, loco ∂s ſcribatur eius valor $\sqrt{(\partial x^2 + \partial y^2)}$, et ſumtis quadratis erit $yy \partial x^2 + yy \partial y^2 = aa \partial y^2$, vnde deducitur $\partial x = -\frac{\partial y \sqrt{(aa - yy)}}{y}$, pro cuius integratione faciamus $\sqrt{(aa - yy)} = v$, eritque $yy = aa - vv$, hinc $\frac{\partial y}{y} = -\frac{v \partial v}{aa - vv}$, ergo

$$\partial x = \frac{vv \partial v}{aa - vv} = -\partial v + \frac{aa \partial v}{aa - vv},$$

conſequenter

$$x = C - v + \tfrac{1}{2} a l \frac{a + v}{a - v} = C - \sqrt{(aa - yy)} + \tfrac{1}{2} a l \frac{a + \sqrt{(aa - yy)}}{a - \sqrt{(aa - yy)}},$$

et quia caſu $x = 0$ fieri debet $y = a$, fiet

$$x = \tfrac{1}{2} a l \frac{a + \sqrt{(aa - yy)}}{a - \sqrt{(aa - yy)}} - \sqrt{(aa - yy)}, \text{ ſiue}$$

$$x = a l \frac{a + \sqrt{(aa - yy)}}{y} - \sqrt{(aa - yy)}.$$

Vnde

Vnde patet, corpufculum non ante ad rectam AB peruenire quam percurfo fpatio infinito.

§. 4. Confideremus nunc quoque cafum, quo filum iuxta lineam curuam quamcunque A T protrahitur. Ita fi Y fit punctum in Tractoria, eiusque tangens vsque ad curuam datam in T ducatur, recta Y T perpetuo aequetur longitudini fili $= a$. Referatur curua data ad axem A B, ad quem ex T demittatur perpendiculum T U, fitque A U $= u$ et U T $= t$, atque ob curuam datam dabitur aequatio inter t et u. Nunc vero ex puncto Tractoriae Y ad eundem axem ducatur normalis YX, fitque AX $= x$ et XY $= y$ et arcus Tractoriae $= s$. Hinc cum Y T curuam tangat, ducta ex T axi normali T S, ob Y T $= a$, erit $\partial s : \partial x = a :$ T S et $\partial s : -\partial y = a :$ Y S, vnde fit T S $= (u - x) = \frac{a \partial x}{\partial s}$ et S Y $= y - t = -\frac{a \partial y}{\partial s}$. Ponamus nunc $\partial y = p \partial x$, erit $\partial s = \partial x \sqrt{(1 + p p)}$, hincque fiet $u - x = \frac{a}{\sqrt{(1 + p p)}}$ et $t - y = \frac{a p}{\sqrt{(1 + p p)}}$. Ex his igitur formulis, fi curua tractoria effet cognita, facile determinaretur curua A T, iuxta quam filum produci debet. Tab. I. Fig. 2.

§. 5. Vt autem ex data aequatione inter t et u inueftigemus aequationem inter x et y, calculus ita inftituatur. Ex binis formulis inuentis: $u = x + \frac{a}{\sqrt{(1 + p p)}}$ et $t = y + \frac{a p}{\sqrt{(1 + p p)}}$, habebimus differentiando

$$\text{I. } \partial u = \partial x - \frac{a p \partial p}{(1 + p p)^{\frac{3}{2}}} \text{ et II. } \partial t = p \partial x + \frac{a \partial p}{(1 + p p)^{\frac{3}{2}}},$$

vnde II $-$ I $\times p$ praebet $\partial t - p \partial u = \frac{a \partial p}{\sqrt{(1 + p p)}}$, ex qua, conceffa aequationum differentialium refolutione, quantitas variabilis p definietur per coordinatas datas t et u; ita vt t fpectari poffit tanquam certa functio ipfius u, quia t per u dari affumitur.

 Porro

Porro haec combinatio: I. + II. p dat $\partial u + p\partial t = \partial x(1+pp)$, vnde colligimus $\partial x = \frac{\partial u + p\partial t}{1+pp}$, hincque porro $\partial y = \frac{p(\partial u + p\partial t)}{1+pp}$, sicque etiam x et y per eandem variabilem u determinabuntur.

§. 6. Hic quidem assumere sumus coacti, resolutionem aequationis differentialis $\frac{a\partial p}{\sqrt{(1+pp)}} + p\partial u = \partial t$ esse in potestate, quod tamen paucissimis tantum casibus exsequi licet. Vicissim igitur, si curuam tractoriam tanquam iam cognitam spectemus, quandoquidem eius descriptio mechanica datur, ipsam hanc aequationem differentialem resoluere licebit. Atque adeo iam olim hoc modo constructionem aequationis *Riccatianae* exhibui.

§. 7. Vt hanc aequationem ab irrationalitate liberemus, faciamus $p = \frac{zz-1}{2z}$, vt fiat $\frac{\partial p}{\sqrt{(1+pp)}} = \frac{\partial z}{z}$, et nostra aequatio differentialis erit $\frac{a\partial z}{z} + \frac{(zz-1)\partial u}{2z} = \partial t$, siue

$$a\partial z + \tfrac{1}{2}(zz-1)\partial u = z\partial t,$$

quam ergo semper per motum tractorium construere licet, qualiscunque functio quantitas t fuerit ipsius u. Inuento valore literae z erit

$$x = \int \frac{4zz\partial u + 2z(zz-1)\partial t}{(1+zz)^2} \text{ et}$$

$$y = \int \frac{(2z\partial u + (zz-1)\partial t)(zz-1)}{(1+zz)^2}.$$

Euidens autem est, in hac aequatione formulam illam *Riccatianam* latissimo sensu acceptam contineri. Si enim statuamus $z = e^{\frac{t}{a}}$, erit $\partial z = e^{\frac{t}{a}}\partial v + e^{\frac{t}{a}}\frac{v\partial t}{a}$, et aequatio nostra hanc induet formam:

$$a e^{\frac{t}{a}}\partial v + \tfrac{1}{2}e^{\frac{2t}{a}}vv\partial u = \tfrac{1}{2}\partial u, \text{ siue}$$

$$a\partial v + \tfrac{1}{2}e^{\frac{t}{a}}vv\partial u = \tfrac{1}{2}e^{-\frac{t}{a}}\partial u,$$

vnde

vnde cum $e^{\frac{f}{a}}$ ſemper ſit certa functio ipſius u, quae ponatur $= U$, conſtrui poterit haec aequatio differentialis:

$$a \partial v + \tfrac{1}{2} v v U \partial u = \frac{\frac{1}{2} \partial u}{U}.$$

§. 8. Hanc igitur ob cauſſam ſi curua, iuxta quam filum protrahitur, pro lubitu accipiatur, determinatio Tractoriae plerumque vires Analyſeos ſuperat. At ſi filum iuxta peripheriam circuli protrahatur, cuius centrum ſit in C, et radius $AC = c$, ſingulari fortuna euenit, vt Tractoria definiri poſſit. Inceperit enim iſte motus, dum corpuſculum erat in B et filum $BA = a$ ad circulum erat normale; nunc autem corpuſculum peruenerit in Z, vbi recta tangens ZT circulo in T occurrat, ita vt ſit $ZT = a$. Iam ducta recta CZ vocetur angulus $ACZ = \omega$ et $CZ = z$, ita vt pro Tractoria inuenienda ſit aequatio inter rectam z et angulum ω, quae quidem inueſtigatio, niſi artificium adhibeatur, in calculos non parum moleſtos induceret. Tab. I. Fig. 3.

§. 9. Ad has difficultates euitandas in calculum introducamus angulum $CZT = \Phi$; ſic enim conſideratio trianguli CZT, cuius latera ſunt $CZ = z$, $ZT = a$ et $CT = c$, ſtatim praebet $cc = aa + zz - 2az \cos \Phi$, vnde deducitur $z = a \cos \Phi \pm \sqrt{(cc - aa \sin \Phi^2)}$, vbi ſignum ambiguum ad ſitum puncti z reſpicit, prouti id fuerit vel extra circulum vel intra circulum. Quia autem in figura punctum z extra circulum ſitum repraeſentatur, valebit ſignum ſuperius, eritque $z = a \cos \Phi + \sqrt{(cc - aa \sin \Phi^2)}$. Praeterea hinc ſimul innoteſcunt anguli ZCT et ZTC; erit enim $\sin ZCT = \frac{a \sin \Phi}{c}$ et $\sin ZTC = \frac{z \sin \Phi}{c}$. Nunc quia recta ZT eſt tangens Tractoriae in Z, ducatur recta proxima $Cz = z + \partial z$, et ex Z deſcripto arculo zs, in triangulo Zzs erit

$Z s = -\partial z$, et ob angulum $Z c z = \partial \omega$ erit $z s = z \partial \omega$, vnde ſtatim colligitur tang. $s Z z$, hoc eſt tang. $\Phi = \frac{z \partial \omega}{-\partial z}$, hincque porro $\frac{\partial z}{z} = -\frac{\partial \omega}{tang. \Phi}$, ſiue $\partial \omega = -\frac{\partial z}{z}$ tang. Φ, ſicque angulus ω per z et Φ definitur. Iam vero relationem inter z et Φ inuenimus. Praeterea vero cum ipſum Tractoriae elementum Zz, quod vocemus $= \partial s$, ſit $\partial s = -\frac{\partial z}{cof. \Phi}$, hinc longitudo Tractoriae concluditur $B Z = s = -\int \frac{\partial z}{cof. \Phi}$.

§. 10. Cum igitur inuenerimus

$$z = a \operatorname{cof.} \Phi + \sqrt{(c c - a a \operatorname{fin.} \Phi^2)}, \text{ erit}$$

$$\partial z = -a \partial \Phi \operatorname{fin.} \Phi - \frac{a a \partial \Phi \operatorname{fin.} \Phi \operatorname{cof.} \Phi}{\sqrt{(c c - a a \operatorname{fin.} \Phi^2}}$$

$$- \frac{a \partial \Phi \operatorname{fin.} \Phi (\sqrt{(c c - a a \operatorname{fin.} \Phi^2)} + a \operatorname{cof.} \Phi)}{\sqrt{(c c - a a \operatorname{fin.} \Phi^2)}},$$

quae manifeſto reducitur ad hanc formam $\frac{-a z \partial \Phi \operatorname{fin.} \Phi}{\sqrt{(c c - a a \operatorname{fin.} \Phi^2)}}$, ita vt ſit $\frac{\partial z}{z} = -\frac{a \partial \Phi \operatorname{fin.} \Phi}{\sqrt{(c c - a a \operatorname{fin.} \Phi^2)}}$. Quamobrem angulus ω ita determinabitur, vt ſit $\partial \omega = \frac{a \partial \Phi \operatorname{fin.} \Phi \operatorname{tang.} \Phi}{\sqrt{(c c - a a \operatorname{fin.} \Phi^2)}}$; tum vero erit etiam

$$\partial s = \frac{a z \partial \Phi \operatorname{tang.} \Phi}{\sqrt{(c c - a a \operatorname{fin.} \Phi^2)}} = \frac{a a \partial \Phi \operatorname{fin.} \Phi}{\sqrt{(c c - a a \operatorname{fin.} \Phi^2)}} + a \partial \Phi \operatorname{tang.} \Phi,$$

vnde integrando prodit

$$s = -a l \operatorname{cof.} \Phi + a a \int \frac{\partial \Phi \operatorname{fin.} \Phi}{\sqrt{(c c - a a \operatorname{fin.} \Phi^2)}}.$$

§. 11. Totum ergo negotium reducitur ad has formulas integrales; $\int \frac{\partial \Phi \operatorname{fin.} \Phi}{\sqrt{(c c - a a \operatorname{fin.} \Phi^2)}}$ et $\int \frac{\partial \Phi \operatorname{fin.} \Phi \operatorname{tang.} \Phi}{\sqrt{(c c - a a \operatorname{fin.} \Phi^2)}}$. Quod ad priorem attinet, quia $-\partial \Phi \operatorname{fin.} \Phi$ eſt differentiale ipſius cof. Φ, ponamus cof. $\Phi = v$, et haec formula transformabitur in hanc:

$$\int \frac{\partial \Phi \operatorname{fin.} \Phi}{\sqrt{(c c - a a \operatorname{fin.} \Phi^2)}} = -\int \frac{\partial v}{\sqrt{(c c - a a (1 - v v))}},$$

cuius integrale eſt

$$-\frac{1}{a} l \left(\frac{a v + \sqrt{(b b + a a v v)}}{b}\right) = -\frac{1}{a} l \frac{a v + \sqrt{(c c - a a + a a v v)}}{\sqrt{(c c - a a)}},$$

vnde

vnde restituto valore cos. Φ loco v reperietur tandem

$$s = C - a\,l\cos.\Phi - a\,l\,[a\cos.\Phi + \sqrt{(cc - aa\,\text{fin.}\,\Phi^2)}].$$

Vbi ad constantem definiendam notetur, initio fuisse tam $s = 0$ quam $\Phi = 0$: erit igitur $C = a\,l\,(a + c)$, hinc fit

$$s = a\,l\,\frac{a + c}{\cos.\Phi\,(a\cos.\Phi + \sqrt{(cc - aa\,\text{fin.}\,\Phi^2)})},$$

vnde patet, rectificationem huius Tractoriae per solos logarithmos expediri.

§. 12. Praecipuum autem negotium versatur in integratione formulae $\omega = a\int\frac{\partial\Phi\,\text{fin.}\,\Phi\,\text{tang.}\,\Phi}{\sqrt{(cc - aa\,\text{fin.}\,\Phi^2)}}$, quae commodissime tractabitur si statuamus $\sqrt{(cc - aa\,\text{fin.}\,\Phi^2)} = x\,\text{fin.}\,\Phi$, vt fiat $\omega = a\int\frac{\partial\Phi\,\text{fin.}\,\Phi}{x\cos.\Phi}$. Verum inde habebitur

$$cc - aa\,\text{fin.}\,\Phi^2 = xx\,\text{fin.}\,\Phi^2, \text{ hincque}$$

$$\text{fin.}\,\Phi^2 = \frac{cc}{aa + xx} \text{ et } \cos.\Phi^2 = \frac{aa - cc + xx}{aa + xx}.$$

Sumtis logarithmis erit

$$2\,l\cos.\Phi = l\,(aa - cc + xx) - l\,(aa + xx),$$

vnde differentiando fiet

$$\frac{\partial\Phi\,\text{fin.}\,\Phi}{\cos.\Phi} = \frac{-x\,\partial x}{aa - cc + xx} + \frac{x\,\partial x}{aa + xx},$$

quo valore substituto prodit

$$\omega = a\int\frac{\partial x}{aa + xx} - a\int\frac{\partial x}{aa - cc + xx},$$

vbi pars prior manifesto fit

$$= \text{A tang.}\,\frac{x}{a} = \text{A tang.}\,\frac{\sqrt{(cc - aa\,\text{fin.}\,\Phi^2)}}{a\,\text{fin.}\,\Phi}.$$

Pro parte autem posteriore tres casus considerari conuenit, prouti fuerit vel $a > c$, vel $a < c$, vel $a = c$, quos singulos igitur percurramus.

Casus I.

$a > c$.

§. 13. Sit igitur primo $a > c$, ponaturque $aa - cc = bb$, eritque

$$\int \frac{a \partial x}{aa - cc + xx} = \int \frac{a \partial x}{bb + xx} = \frac{a}{b} \int \frac{b \partial x}{bb + xx},$$

cuius integrale est

$$\frac{a}{b} \text{ A tang. } \frac{x}{b} = \frac{a}{b} \text{ A tang. } \frac{\sqrt{(cc - aa \sin. \Phi^2)}}{b \sin. \Phi},$$

quocirca pro hoc casu habebimus

$$\omega = \text{A tang. } \frac{\sqrt{(cc - aa \sin. \Phi^2)}}{a \sin. \Phi} - \frac{a}{\sqrt{(aa - cc)}} \text{ A tang. } \frac{\sqrt{(cc - aa \sin. \Phi^2)}}{\sin. \Phi \sqrt{(aa - cc)}} + C.$$

Pro constante C autem determinanda notetur, initio fieri tam $\omega = 0$ quam $\Phi = 0$, vnde concluditur $C = \frac{\pi}{2} \left(\frac{a - \sqrt{(aa - cc)}}{\sqrt{(aa - cc)}} \right)$, quo valore inducto erit

$$\omega = \frac{\pi}{2} \left(\frac{a - \sqrt{(aa - cc)}}{\sqrt{(aa - cc)}} \right) + \text{A tang. } \frac{\sqrt{(cc - aa \sin. \Phi^2)}}{a \sin. \Phi} - \frac{a}{\sqrt{(aa - cc)}} \text{ A tg. } \frac{\sqrt{(cc - aa \sin. \Phi^2)}}{\sin. \Phi \sqrt{(aa - cc)}},$$

qui valor etiam ita referri potest:

$$\omega = \frac{a}{\sqrt{(aa - cc)}} \text{ A tang. } \frac{\sin. \Phi \sqrt{(aa - cc)}}{\sqrt{(cc - aa \sin. \Phi^2)}} - \text{A tang. } \frac{a \sin. \Phi}{\sqrt{(cc - aa \sin. \Phi^2)}}.$$

Hoc igitur casu sin. Φ non vltra terminum $\frac{c}{a}$ augeri potest; quando autem fit sin. $\Phi = \frac{c}{a}$, tum fit angulus

$$\omega = \left(\frac{a}{\sqrt{(aa - cc)}} - 1 \right) 90^\circ$$

et distantia $z = \sqrt{(aa - cc)}$.

§. 14. Hoc igitur casu angulus ω per solos arcus circulares, ideoque etiam per angulos definitur; vnde si modo hi anguli rationem teneant rationalem inter se, id quod euenit quoties $\frac{a}{\sqrt{(aa - cc)}}$ fuerit numerus rationalis, angulum ω geometrice definire licebit, sicque ipsa curua tractoria euadet algebraica, siue eius natura per aequationem algebraicam exprimi poterit. Haec igitur circumstantia vtique meretur, vt exemplo illustretur.

Exem-

Exemplum.

§. 15. Euoluamus igitur caſum quo $\frac{a}{\sqrt{(aa-cc)}} = 2$, ſiue $c = \frac{a\sqrt{3}}{2}$: ſic enim fiet $\sqrt{(aa-cc)} = \frac{1}{2}a$, hincque porro

$$\omega = 2\,\text{A tang.}\,\frac{\text{ſin.}\,\Phi}{\sqrt{(3-4\,\text{ſin.}\,\Phi^2)}} - \text{A tang.}\,\frac{2\,\text{ſin.}\,\Phi}{\sqrt{(3-4\,\text{ſin.}\,\Phi^2)}}.$$

Cum igitur in genere ſit $2\,\text{A tang.}\,t = \text{A tang.}\,\frac{2t}{1-tt}$, noſtro autem caſu ſit $t = \frac{\text{ſin.}\,\Phi}{\sqrt{(3-4\,\text{ſin.}\,\Phi^2)}}$, erit

$$2\,\text{A tang.}\,\frac{\text{ſin.}\,\Phi}{\sqrt{(3-4\,\text{ſin.}\,\Phi^2)}} = \text{A tang.}\,\frac{2\,\text{ſin.}\,\Phi\sqrt{(3-4\,\text{ſin.}\,\Phi^2)}}{3-5\,\text{ſin.}\,\Phi^2},$$

ideoque erit

$$\omega = \text{A tang.}\,\frac{2\,\text{ſin.}\,\Phi\sqrt{(3-4\,\text{ſin.}\,\Phi^2)}}{3-5\,\text{ſin.}\,\Phi^2} - \text{A tang.}\,\frac{2\,\text{ſin.}\,\Phi}{\sqrt{(3-4\,\text{ſin.}\,\Phi^2)}}.$$

Cum porro ſit $\text{A tang.}\,p - \text{A tang.}\,q = \frac{p-q}{1+pq}$, quia noſtro caſu eſt

$$p = \frac{2\,\text{ſin.}\,\Phi\sqrt{(3-4\,\text{ſin.}\,\Phi^2)}}{3-5\,\text{ſin.}\,\Phi^2} \text{ et } q = \frac{2\,\text{ſin.}\,\Phi}{\sqrt{(3-4\,\text{ſin.}\,\Phi^2)}}, \text{ erit}$$

$$p - q = \frac{2\,\text{ſin.}\,\Phi^3}{(3-5\,\text{ſin.}\,\Phi^2)\sqrt{(3-4\,\text{ſin.}\,\Phi^2)}} \text{ et } 1 + pq = \frac{3-\text{ſin.}\,\Phi^2}{3-5\,\text{ſin.}\,\Phi^2},$$

conſequenter obtinebimus

$$\omega = \text{A tang.}\,\frac{2\,\text{ſin.}\,\Phi^3}{(3-\text{ſin.}\,\Phi^2)\sqrt{(3-4\,\text{ſin.}\,\Phi^2)}}, \text{ ideoque}$$

$$\text{tang.}\,\omega = \frac{2\,\text{ſin.}\,\Phi^3}{(3-\text{ſin.}\,\Phi^2)\sqrt{(3-4\,\text{ſin.}\,\Phi^2)}}.$$

Hoc igitur modo ex aſſumto angulo Φ colligitur angulus ω.

§. 16. Porro igitur cum pro hoc exemplo ſit

$$z = a\,\text{coſ.}\,\Phi + \tfrac{1}{2}a\sqrt{(3-4\,\text{ſin.}\,\Phi^2)},$$

ſi ex puncto Z ad rectam CB ducatur normalis ZX, et pro Tractoria vocenter coordinatae $CX = x$ et $XZ = y$, fiet $x = z\,\text{coſ.}\,\omega$ et $y = z\,\text{ſin.}\,\omega$, ſicque tam x quam y per eundem angulum Φ determinabitur. Ex tangente autem anguli ω concluditur

$$\text{ſin.}\,\omega = \frac{2\,\text{ſin.}\,\Phi^3}{3\,\text{coſ.}\,\Phi^2\sqrt{3}} \text{ et } \text{coſ.}\,\omega = \frac{(3-\text{ſin.}\,\Phi^2)\sqrt{(3-4\,\text{ſin.}\,\Phi^2)}}{3\,\text{coſ.}\,\Phi^2\sqrt{3}}.$$

 Quod

Quodsi autem hinc ipsum angulum Φ eliminare vellemus, aequatio inter x et y sine dubio ad plures dimensiones assurgeret. Interim tamen constructio geometrica huius curuae non nimis est prolixa.

§. 17. Ad has formulas simpliciores reddendas statuatur $\sqrt{(3 - 4\,\text{sin.}\,\Phi^2)} = 2u\,\text{sin.}\,\Phi$, vt fiat $z = a\,\text{cos.}\,\Phi + a\,u\,\text{sin.}\,\Phi$, et tang. $\omega = \frac{\text{sin.}\,\Phi^2}{u(3 - \text{sin.}\,\Phi^2)}$; tum autem erit $\text{sin.}\,\Phi^2 = \frac{3}{4(1 + uu)}$, vnde fit tang. $\omega = \frac{1}{3 + 4uu}$. Deinde vero ob $\text{cos.}\,\Phi^2 = \frac{1 + 4uu}{4(1 + uu)}$ fiet $z = \frac{\sqrt{(1 + 4uu)} + u\sqrt{3}}{2\sqrt{(1 + uu)}}$. Ponatur porro $\frac{u\sqrt{3}}{\sqrt{(1 + 4uu)}} = \text{cos.}\,\theta$, erit $\text{sin.}\,\theta = \sqrt{\frac{1 + uu}{1 + 4uu}}$, vnde fit $\frac{z}{a} = \frac{1 + \text{cos.}\,\theta}{2\,\text{sin.}\,\theta} = \frac{1}{2}\,\text{cot.}\,\frac{1}{2}\theta$, deinde vero ob $uu = \frac{\text{cos.}\,\theta^2}{3 - 4\,\text{cos.}\,\theta^2}$ erit tang. $\omega = \frac{3 - 4\,\text{cos.}\,\theta^2}{9 - 8\,\text{cos.}\,\theta^2} = \frac{4\,\text{sin.}\,\theta^2 - 1}{1 + 8\,\text{sin.}\,\theta^2}$.

Casus II.

$a < c$.

§. 18. Sit iam $a < c$, ponaturque $cc = aa + bb$, eritque

$$\omega = \text{A tang.}\,\frac{\sqrt{(cc - aa\,\text{sin.}\,\Phi^2)}}{a\,\text{sin.}\,\Phi} - a\int\frac{\partial x}{xx - bb}.$$

Est vero

$$\int\frac{a\,\partial x}{xx - bb} = \frac{a}{b}\int\frac{b\,\partial x}{xx - bb} = \frac{a}{2b}\,l\,\frac{x - b}{x + b}.$$

Cum igitur sit $x = \frac{\sqrt{(cc - aa\,\text{sin.}\,\Phi^2)}}{\text{sin.}\,\Phi}$ et $b = \sqrt{(cc - aa)}$, hinc colligitur

$$\omega = C + \text{A tang.}\,\frac{\sqrt{(cc - aa\,\text{sin.}\,\Phi^2)}}{a\,\text{sin.}\,\Phi} - \frac{a}{2\sqrt{(cc - aa)}}\,l\,\frac{\sqrt{(cc - aa\,\text{sin.}\,\Phi^2)} - \text{sin.}\,\Phi\sqrt{(cc - aa)}}{\sqrt{(cc - aa\,\text{sin.}\,\Phi^2)} + \text{sin.}\,\Phi\sqrt{(cc - aa)}},$$

vbi quia initio fieri debet tam $\Phi = 0$ quam $\omega = 0$, erit constans $C = -\frac{\pi}{2}$, vnde fit

$$\omega = \frac{a}{2\sqrt{(cc - aa)}}\,l\,\frac{\sqrt{(cc - aa\,\text{sin.}\,\Phi^2)} + \text{sin.}\,\Phi\sqrt{(cc - aa)}}{\sqrt{(cc - aa\,\text{sin.}\,\Phi^2)} - \text{sin.}\,\Phi\sqrt{(cc - aa)}} - \text{A tang.}\,\frac{a\,\text{sin.}\,\Phi}{\sqrt{(cc - aa\,\text{sin.}\,\Phi^2)}}.$$

Manet autem vt ante $z = a\,\text{cos.}\,\Phi + \sqrt{(cc - aa\,\text{sin.}\,\Phi^2)}$, vnde patet, has curuas semper esse transcendentes. Ceterum quia hic

hic $c > a$, euidens eſt, angulum Φ a o vsque ad $90°$ increſcere poſſe, cum primo caſu, vbi erat $c < a$, angulus Φ eo vsque tantum creſcere poterat, quoad fiat ſin. $\Phi = \frac{c}{a}$.

Caſus III.

$c = a$.

§. 19. Poſito autem $c = a$ ſtatim fit $z = 2a$ coſ. Φ et $\omega = a\int\frac{\partial x}{aa+xx} - \int\frac{a\,\partial x}{xx}$, ideoque

$$\omega = \text{A tang.}\,\frac{x}{a} + \frac{a}{x} + C = \text{A tang.}\,\frac{\text{coſ.}\,\Phi}{\text{ſin.}\,\Phi} + \text{tang.}\,\Phi + C.$$

Hoc ergo modo determinata conſtante prodit $\omega = \text{tang.}\,\Phi - \Phi$; vnde intelligitur, ſi angulus Φ increſcat vsque ad $90°$, tum fore angulum $\omega = \infty$, ſcilicet hoc caſu filum per infinitas reuolutiones in circulo protrahi poterit. Tum autem denique fiet $z = 0$; vnde patet, confectis infinitis reuolutionibus corpuſculum tandem in ipſum centrum circuli peruenire, ibique in quiete eſſe permanſurum.

§. 20. Ceterum pro ſecundo caſu ſingulare phaenomenon ſeſe exſerit. Statim enim primae aequationi $aa + zz - 2az$ coſ. $\Phi = cc$ ſatisfieri manifeſtum eſt, ſi fuerit $\Phi = 90°$ et $z = \sqrt{(cc - aa)}$; tum autem angulus ω plane non determinatur; quia fit $\partial\omega = \frac{0}{0}$, et hoc caſu ipſa curua tractoria erit circulus etiam centro C radio $cc - aa$ deſcriptus: huius enim tangentes, ad circulum A C B productae, aequabuntur longitudini fili a; atque ad hunc caſum omnes reliqui motus poſt infinitas reuolutiones reducentur, ita vt hae Tractoriae tandem in circulum abeant. Neque tamen ex hac ſolutione ipſam formam harum Tractoriarum ſatis commode cognoſcere licet, vnde aliam ſolutionem ſubiungamus ad hunc ſcopum magis accommodatam.

Alia methodus
Tractorias ex circulo natas determinandi.

§. 21. Maneant denominationes ante adhibitae, scilicet longitudo fili $BA = ZT = a$, radius circuli $CA = CT = c$, distantia $CZ = z$, angulus $ACZ = \omega$ et angulus $CZT = \Phi$, vnde fit vt ante $\partial\omega = -\frac{\partial z}{z}$ tang. Φ. Nunc autem insuper vocemus angulum $ZCT = \theta$, ad quem omnia elementa curvae reuocemus. Tandem etiam sit angulus $ACT = \omega + \theta = \psi$; quandoquidem hoc modo statim innotescet punctum T, quousque filum iam est protractum.

§. 22. His positis ex T ad rectam CZ agatur normalis TP, et ex triangulo CTP erit $TP = c$ sin. θ et $CP = c$ cos. θ: at ex triangulo ZTP erit $TP = a$ sin. Φ et $ZP = a$ cos. Φ, vnde statim colligitur $z = a$ cos. $\Phi + c$ cos. θ; tum vero c sin. $\theta = a$ sin. Φ, vnde sin. $\Phi = \frac{c}{a}$ sin. θ, cos. $\Phi = \frac{\sqrt{(aa - cc\,\text{sin.}\,\theta^2)}}{a}$ et tang. $\Phi = \frac{c\,\text{sin.}\,\theta}{\sqrt{(aa - cc\,\text{sin.}\,\theta^2)}}$. Differentiemus nunc binas illas aequationes, et prodibit

$$\text{I.} \quad -\partial z = a\,\partial\Phi\,\text{sin.}\,\Phi + c\,\partial\theta\,\text{sin.}\,\theta \text{ et}$$

$$\text{II.} \quad 0 = a\,\partial\Phi\,\text{cos.}\,\Phi - a\,\partial\theta\,\text{cos.}\,\theta,$$

vnde combinatio: I. cos. Φ — II. sin. Φ praebet $-\partial z$ cos. Φ $= c\,\partial\theta$ sin. θ cos. $\Phi + c\,\partial\theta$ cos. θ sin. $\Phi = c\,\partial\theta$ sin. $(\theta + \Phi)$. At vero ex triangulo CAT habetur CT : sin. $\theta = z$: sin. $(\theta + \Phi)$, ideoque sin. $(\theta + \Phi) = \frac{z\,\text{sin.}\,\theta}{a}$: hoc ergo valore adhibito fiet $-\partial z$ cos. $\Phi = \frac{cz\,\partial\theta\,\text{sin.}\,\theta}{a}$, ideoque $-\frac{\partial z}{z} = \frac{c\,\partial\theta\,\text{sin.}\,\theta}{a\,\text{cos.}\,\Phi}$.

§. 23. Ex hoc igitur valore nanciscimur $\partial\omega = \frac{c\,\partial\theta\,\text{sin.}\,\theta\,\text{sin.}\,\Phi}{a\,\text{cos.}\,\Phi^2}$; erat autem sin. $\Phi = \frac{c}{a}$ sin. θ et cos. $\Phi^2 = \frac{aa - cc\,\text{sin.}\,\theta^2}{aa}$, vnde rationationaliter angulum Φ ex calculo elidimus; prodibit enim

$$\partial\omega = \frac{cc\partial\theta\,\text{fin.}\,\theta^2}{aa-cc\,\text{fin.}\,\theta^2} = -\partial\theta + \frac{aa\partial\theta}{aa-cc\,\text{fin.}\,\theta^2},$$

vnde cum fit $\partial\omega + \partial\theta = \partial\psi$, erit

$$\partial\psi = \frac{aa\partial\theta}{aa-cc\,\text{fin.}\,\theta^2} + \frac{aa\partial\theta}{aa\,\text{cof.}\,\theta^2 + (aa-cc)\,\text{fin.}\,\theta^2}.$$

§. 24. Hinc euoluamus primo cafum quo $a > c$, ac ponamus breuitatis gratia $aa-cc=bb$, vt habeamus $\partial\psi = \frac{aa\partial\theta}{aa\,\text{cof.}\,\theta^2 + bb\,\text{fin.}\,\theta^2}$, pro cuius integrali inueniendo ponamus $\frac{b\,\text{fin.}\,\theta}{a\,\text{cof.}\,\theta} = t$, eritque $\partial t = \frac{ab\partial\theta}{aa\,\text{cof.}\,\theta^2}$; tum vero etiam $1+tt = \frac{aa\,\text{cof.}\,\theta^2 + bb\,\text{fin.}\,\theta^2}{aa\,\text{cof.}\,\theta^2}$, ideoque $\frac{\partial t}{1+tt} = \frac{ab\partial\theta}{aa\,\text{cof.}\,\theta^2 + bb\,\text{fin.}\,\theta^2} = \frac{b\partial\psi}{a}$, hinc integrando $\frac{b\psi}{a} = \text{A tang.}\,t$, quamobrem hinc angulus $ACT = \psi$ ita fuccincte exprimitur, vt fit

$$\psi = \frac{a}{b}\,\text{A tang.}\,\frac{b\,\text{fin.}\,\theta}{b\,\text{cof.}\,\theta}.$$

§. 25. Pro hoc ergo cafu, quo $aa-cc=bb$, ex folo angulo θ omnia elementa, quae ad curuam pertinent, fequenti modo fatis concinne exprimuntur: 1.) Pro angulo Φ inuenimus $\text{fin.}\,\Phi = \frac{c}{a}\,\text{fin.}\,\theta$. 2.) Diftantia $CZ = z = a\,\text{cof.}\,\Phi + c\,\text{cof.}\,\theta$, fiue $z = \sqrt{aa-cc\,\text{fin.}\,\theta^2} + \text{cof.}\,\theta$. Pro angulo $ACT = \psi$, prodiit $\psi = \frac{a}{b}\,\text{A tang.}\,\frac{b\,\text{fin.}\,\theta}{a\,\text{cof.}\,\theta}$, fiue $\psi = \frac{a}{b}\,\text{A tang.}\,\frac{b}{a}\,\text{tang.}\,\theta$, ita vt fit $\frac{b\psi}{a} = \text{A tang.}\,\frac{b}{a}\,\text{tang.}\,\theta$ et hinc $\text{tang.}\,\frac{b\psi}{a} = \frac{b}{a}\,\text{tang.}\,\theta$. Nunc igitur facile erit pro angulo θ valores continuo maiores fubftituere, indeque pro fingulis tam diftantiam z quam angulum ψ affignare. Hinc autem ftatim patet, fumto $\theta = 0$ fore 1.) $\Phi = 0$. 2.) $z = a+c$. 3.) $\psi = 0$.

§. 26. Hae igitur formulae imprimis idoneae ſunt ad curuam conſtruendam, ac fere ſufficiet angulos θ continuo per 90° vel ſaltem per 45° creſcentes aſſumere. Quod ſi enim breuitatis gratia angulos α, β, γ ita capiamus, vt ſit ſin. $\alpha = \frac{c}{a\sqrt{2}}$, contang. $\beta = \frac{b}{a}$ et ſin. $\gamma = \frac{c}{a}$, omnes valores ad curuam conſtruendam neceſſarii in ſequenti tabella exhibentur.

θ	Φ	z	ψ
0°	0°	$a + c$	0
45	α	a coſ. $\alpha + \frac{c}{\sqrt{2}}$	$\frac{a}{b}\beta$
90	γ	a coſ. γ	$\frac{a}{b}90^\circ$
135	α	a coſ. $\alpha - \frac{c}{\sqrt{2}}$	$\frac{a}{b}(180 - \beta)$
180	0	$a - c$	$\frac{a}{b}180$
225	$-\alpha$	a coſ. $\alpha - \frac{c}{\sqrt{2}}$	$\frac{a}{b}(180 + \beta)$
270	$-\gamma$	a coſ. γ	$\frac{a}{b}270$
315	$-\alpha$	a coſ. $\alpha + \frac{c}{\sqrt{2}}$	$\frac{a}{b}(360 - \beta)$
360	0	$a + c$	$\frac{a}{b}360$
405	α	a coſ. $\alpha + \frac{c}{\sqrt{2}}$	$\frac{a}{b}(360 + \beta)$
450	γ	a coſ. γ	$\frac{a}{b}450$
495	α	a coſ. $\alpha - \frac{c}{\sqrt{2}}$	$\frac{a}{b}(540 - \beta)$
540	0	$a - c$	$\frac{a}{b}540$
585	$-\alpha$	a coſ. $\alpha - \frac{c}{\sqrt{2}}$	$\frac{a}{b}(540 + \beta)$
630	$-\gamma$	a coſ. γ	$\frac{a}{b}630$
675	$-\alpha$	a coſ. $\alpha + \frac{c}{\sqrt{2}}$	$\frac{a}{b}(720 - \beta)$
720	0	$a + c$	$\frac{a}{b}720$

Vnde patet, quo maior fuerit fractio $\frac{a}{b}$, tum numerum reuolutionum anguli ψ eo magis multiplicari pro iisdem angulis θ;

ac si fuerit $b = 0$, ideoque $a = 0$, qui erat tertius casus, tum numerum reuolutionum anguli ψ iam fieri infinitum, dum angulus θ tantum vsque ad 90° augetur.

§. 27. Sin autem fuerit $aa < cc$, ponamus $cc - aa = bb$, tum erit $\partial\psi = \frac{aa\partial\theta}{aa\,cos.\theta^2 - bb\,sin.\theta^2}$. Ponatur $\frac{b\,sin.\theta}{a\,cos.\theta} = u$, eritque $\partial u = \frac{ab\partial\theta}{aa\,cos.\theta^2}$ et $1 - uu = \frac{aa\,cos.\theta^2 - bb\,sin.\theta^2}{aa\,cos.\theta^2}$, vnde fit

$$\frac{\partial u}{1 - uu} = \frac{ab\partial\theta}{aa\,cos.\theta^2 - bb\,sin.\theta^2} = \frac{b\partial\psi}{a},$$

hincque integrando colligitur $\frac{b\psi}{a} = \frac{1}{2}l\frac{1+u}{1-u}$, ex quo adipiscimur $\psi = \frac{a}{2b}\,l\,\frac{a\,cos.\theta + b\,sin.\theta}{a\,cos.\theta - b\,sin.\theta}$; vbi patet, quia valorem ipsius u non vltra vnitatem augere licet, angulum θ nunquam maiorem euadere posse, quam donec fiat tang. $\theta = \frac{a}{b}$, quippe quo casu angulus ψ iam in infinitum increscit; atque hinc simul intelligitur, si fuerit $b = 0$, siue $a = c$, tum ob $\partial\psi = \frac{\partial\theta}{cos.\theta^2}$, fore $\psi =$ tang. θ, qui erat tertius casus ante commemoratus.

§. 28. Quoniam igitur, si filum corpusculo alligatum per peripheriam circuli circumducitur, Tractoria semper assignari et construi potest, videamus cuiusmodi forma Riccatianae similis huic casui respondeat.

§. 29. Vt igitur hunc casum ad figuram supra consideratam accommodemus, rectae B A C normaliter iungamus rectam C D, in eamque tam ex Z quam ex T, perpendicula ZX et TU demittamus, sitque, vt supra posuimus, $CX = x$ et $XZ = y$; tum vero $CU = u$ et $UT = t$, statuaturque porro $\partial y = p\,\partial x$, quibus positis supra deducti fuimus ad hanc aequationem: $\frac{a\partial p}{\sqrt{(1+pp)}} + p\,\partial u = \partial t$, quae posito $p = \frac{qq-1}{2q}$ trans- Tab. I. Fig. 4.

formatur

formatur in hanc rationalem: $a\partial q + \frac{1}{2}(qq - 1)\partial u = q\partial t$, siue $a\partial q - q\partial t + \frac{1}{2}qq\partial u = \frac{1}{2}\partial u$. Pro praesente autem casu, ob angulum $ACZ = \omega$ et $CZ = z$, fit $x = z$ sin. ω et $y = z$ cos. ω. Deinde ob $CT = t$ et angulum $ACT = \psi$, erit $u = c$ sin. ψ et $t = c$ cos. ψ; praeterea vero habebimus

$$\partial x = \partial z \text{ sin. } \omega + z\partial\omega \text{ cos. } \omega \text{ et}$$

$$\partial y = \partial z \text{ cos. } \omega - z\partial\omega \text{ sin. } \omega, \text{ vnde fit}$$

$$p = \frac{\partial z \text{ cos. } \omega - z\partial\omega \text{ sin. } \omega}{\partial z \text{ sin. } \omega + z\partial\omega \text{ cos. } \omega}.$$

Erat autem $\frac{\partial z}{z} = -\frac{c\partial\theta \text{ sin. } \theta}{a \text{ cos. } \Phi}$, vnde nanciscimur

$$p = \frac{-c\partial\theta \text{ sin. } \theta \text{ cos. } \omega - a\partial\omega \text{ cos. } \Phi \text{ sin. } \omega}{-c\partial\theta \text{ sin. } \theta \text{ sin. } \omega + a\partial\omega \text{ cos. } \Phi \text{ cos. } \omega}.$$

Quia autem repertum est $\partial\omega = \frac{c\partial\theta \text{ sin. } \theta \text{ sin. } \Phi}{a \text{ cos. } \Phi^2}$, erit exclusis differentialibus

$$p = \frac{\text{cos. } \omega \text{ cos. } \Phi + \text{sin. } \omega \text{ sin. } \Phi}{\text{sin. } \omega \text{ cos. } \Phi - \text{cos. } \omega \text{ sin. } \Phi} = \frac{\text{cos. } (\omega - \Phi)}{\text{sin. } (\omega - \Phi)} = \text{cot. } (\omega - \Phi),$$

tum vero, ob $q = p + \sqrt{(1 + pp)}$, erit nunc

$$q = \frac{1 + \text{cos. } (\omega - \Phi)}{\text{sin. } (\omega - \Phi)} = \text{cot. } \tfrac{1}{2}(\omega - \Phi).$$

Hocque modo valor quantitatis q satis simpliciter per angulos ω et Φ exprimitur. Deinde vero ex valoribus pro t et u inventis erit $\partial t = -c\partial\psi$ sin. ψ et $\partial u = c\partial\psi$ cos. ψ, sicque formula nostra Riccatiana ita se habebit:

$$a\partial q + cq\partial\psi \text{ sin. } \psi + \tfrac{1}{2}cqq\partial\psi \text{ cos. } \psi = \tfrac{1}{2}c\partial\psi \text{ cos. } \psi,$$

inuoluens duas tantum variabiles q et angulum ψ.

§. 30. Vicissim igitur, quoties occurrit huiusmodi aequatio differentialis resoluenda:

$$a\partial q + cq\partial\psi \text{ sin. } \psi + \tfrac{1}{2}cqq\partial\psi \text{ cos. } \psi = \tfrac{1}{2}c\partial\psi \text{ cos. } \psi,$$

eius resolutio in nostra erit potestate, quandoquidem nouimus fore $q = \text{cot. } \frac{1}{2}(\omega - \Phi)$; quomodo autem anguli ω et Φ ab angulo ψ pendeant, ex superioribus est manifestum. Primo enim

enim est $\psi = \omega + \theta$; tum vero a fin. $\Phi = c$ fin. θ; denique vero inuenimus $\psi = \int \frac{a a \partial \theta}{a a \operatorname{cof}. \theta^2 + (a a - c c) \operatorname{fin}. \theta^2}$, cuius ope primo ex angulo ψ reperitur angulus θ, hincque porro angulus Φ ex formula fin. $\Phi = \frac{c}{a}$ fin. θ, ac tandem $\omega = \psi - \theta$. Ex his igitur angulus $(\omega - \Phi)$, per quem quantitas q exprimitur, erit $= \psi - \Phi - \theta$. Hunc in finem prolongetur recta Z T in S, et quia angulus C T S $= \theta + \Phi$ et C T U $= \psi$, erit angulus U T S $= \theta + \Phi - \psi$, ita vt iam fit $q = -\operatorname{cot}. \frac{1}{2}$ U T S.

§. 31. Quo hanc formulam Riccatianam fimpliciorem reddamus, ponamus $c = 2 n a$, vt prodeat

$$\partial q + 2 n q \partial \psi \operatorname{fin}. \psi + n q q \partial \psi \operatorname{cof}. \psi = n \partial \psi \operatorname{cof}. \psi,$$

quam vt ab angulis liberemus, ponamus cof. $\psi = s$, ita vt fin. $\psi = \sqrt{(1 - s s)}$, eritque aequatio

$$\partial q - 2 n q \partial s - \frac{n q q s \partial s}{\sqrt{(1 - s s)}} = - \frac{n s \partial s}{\sqrt{(1 - s s)}},$$

vel fi ponamus fin. $\psi = r$, prodibit haec forma:

$$\partial q - \frac{2 n q r \partial r}{\sqrt{(1 - r r)}} + n q q \partial r = n \partial r.$$

Quod fi ponamus $q = v + \frac{r}{\sqrt{(1 - r r)}}$, prodibit ifta aequatio:

$$\partial v + n v v \partial R = n \partial r - \frac{n r r \partial r}{1 - r r} + \frac{2 n r r \partial r}{\sqrt{(1 - r r)}} - \frac{\partial r}{(1 - r r)^{\frac{3}{2}}},$$

cuius ergo refolutionem ope noftrae Tractoriae expedire licet.

§. 32. Reducamus eandem aequationem tantum ad ternos terminos, ponendo $q = e^{-2 n \sqrt{(1 - r r)}} v$, ac peruenietur ad hanc formam:

$$\partial v + n e^{-2 n \sqrt{(1 - r r)}} v v \partial r = n e^{2 n \sqrt{(1 - r r)}} \partial r$$

quae porro, ponendo $\sqrt{(1 - r r)} = s$, induet hanc formam:

$$\partial v - n e^{-2ns} \frac{v v s \partial s}{\sqrt{(1-ss)}} + \frac{n e^{+2ns} s \partial s}{\sqrt{(1-ss)}} = 0.$$

Hae autem formulae ita comparatae videntur, vt per folitas methodos haud facile tractari queant.

Animaduerfiones generales in hunc motum tractorium.

§. 33. In hoc motu tractorio affumitur, corpufculum quouis momento fecundum ipfam fili directionem protrahi, quod quidem per principia mechanica eueniret, fi corpufculum quouis momento quiefceret, vel iam motum fecundum eandem directionem habuiffet, quod pofterius autem locum habere nequit, quandoquidem directionem motus continuo mutari affumimus; vnde patet, iftam defcriptionem per motum tractorium locum plane habere non poffe, nifi quouis momento motus corpufculo impreffus fubito rurfus extinguatur. Quod cum principiis motus directe aduerfetur, manifeftam eft talem motum tractorium in natura neutiquam produci poffe, nifi forte frictio infinite magna ftatuatur.

§. 34. Vulgo quidem talis motus facile obtineri poffe videtur, cum, experentia tefte, omnia corpora, quae in fuperficie plana protrahi folent, eo ipfo momento, quo vis trahens ceffat, fubito ad quietem redigi cernuntur, quemadmodum currus ab equis protracti, fimulac vis trahens ceffat, fubito fubfiftere folent; vnde plures philofophi principiorum motus ignari concludere funt conati, omnia corpora nifu effe praedita fefe ad ftatum quietis accommodandi. Quam abfurda autem fit talis opinio nunc quidem non amplius probatione eget.

§. 35.

§. 35. Interim tamen, experientiam conſulentes, negare non poſſumus, quin corpora, ſuper plano tantillum aſpero producta, quaſi eo ipſo momento omnem motum perdant, quo vis trahens ceſſauerit, quod certe nullo modo euenire poſſet, ſi planum perfecte eſſet politum, vt omnis frictio excluderetur, quippe quo caſu corpus adeo motu ſemel acquiſito perpetuo vniformiter eſſet progreſſurum; ex quo ſtatim intelligitur, phaenomenon allatum nulli cauſſae, praeter frictionem adſcribi poſſe.

§. 36. Neque vero etiam hoc modo omnibus difficultatibus occurri poteſt, dum ex motus principiis certum eſt, nullum plane motum a frictione, quantumuis fuerit magna, ſubito, atque eo ipſo momento, quo vis trahens ceſſat, deſtrui poſſe, ſed ad hoc ſemper aliquod tempus requiri, quantumuis id fuerit exiguum; ita vt certe affirmare debeamus, nullum plane motum frictione ſubito ad quietem redigi poſſe, ac ſi tale tempus ſentiri nequeat, id ita eſſe exiguum, vt obſeruari non poſſit.

§. 37. Quo igitur omnia dubia, quae in hoc negotio Tab. I.
ſe produnt, clarius diluamus, conſideremus corpus, quod ſuper Fig. 5.
plano horizontali acceperit celeritatem $= c$, ac videamus quanto tempore opus ſit, vt iſte motus a frictione penitus extinguatur. Fuerit igitur iſtud corpus eo momento, quo vis ſollicitans ceſſauit, in A, vnde celeritate ſua c vlterius progredi conetur. Peruenerit igitur poſt tempus $= t$ vsque in p, confecto ſpatio $AP = s$, ſitque maſſa corporis $= M$, et vis frictionis $= F$, celeritas autem in P vocetur $= v$, eritque $\partial v = -\frac{2gF}{M}\partial t$, vnde colligitur $v = C - \frac{2gFt}{M}$. Fiat nunc $v = 0$ ac reperietur tempus, quo hoc euenire poteſt, $t = \frac{Mc}{2gF}$, quod in minutis ſecundis exprimetur, ſi g fuerit altitudo, per quam

 grauia

grauia vno minuto fecundo delabuntur, celeritas autem c per fpatium vno minuto fecundo percurrendum exprimatur. Hinc igitur fi frictio, vt vulgo fumi folet, tertiae parti ponderis M aequetur, vt fit $F = \frac{1}{3}M$, erit tempus quo motus penitus extinguitur $= \frac{3c}{2g}$, vnde cum propemodum fit $g = 16$ ped. Londin. et c in iisdem pedibus exprimatur, fiet $t = \frac{3}{32}c$ ped.

§. 38. Plerumque autem in huiusmodi motibus tractoriis celeritas corporibus impreffa c tam exigua effe folet, vt tempufculum ad motus extinctionem requifitum t fenfus noftros effugiat. Si enim celeritati c pes integer tribuatur, tempus iftud tantum erit $\frac{3}{32}$, ideoque nequidem decima pars minuti fecundi, quod nemo facile obferuare poteft. Verum fi quis forte tale tempufculum animaduerti poffe contendat, probe hic perpendendum, nullam vim trahentem ita fubito ceffare poffe, quemadmodum in hoc calculo fuppofuimus, fed potius paullatim ad nihilum redigi; vnde mirum non eft fi hoc tempufculum plane non obferuare licet, quoniam motus extinctio iam ante incepit, quam vis trahens ad nihilum fuit perducta.

§. 39. Ex his iam intelligitur, tales curuas, quales hactenus per calculum funt definitae, produci non poffe, nifi fuper plano horizontali fatis afpero; praeterea vero imprimis neceffe effe vt motus, quo filum protrahitur, fit non folum lentiffimus, fed etiam per interualla temporis quam minima penitus fiftatur et quafi per faltus peragatur. Statim enim ac motus fili fuerit continuus, curua, quam corpufculum defcribet, plurimum aberrabit a Tractoria vulgari: cuiusmodi autem curvam fit defcripturum, fi filum motu continuo protrahatur, quaeftio eft maxime ardua, cui refoluendae Analyfis vix fufficere videtur, ad quod oftendendum cafum faltem fimpliciffimum,

quo

quo filum ſuper plano horizontali iuxta lineam rectam vniformiter protrahitur, euoluamus.

De vera curua tractoria, dum filum per lineam rectam vniformiter protrahitur.

§. 40. Protrahatur igitur filum per lineam rectam AD celeritate $= c$, et elapſo tempore $= t$ perductum ſit vsque in T, dum motus inceperit in puncto A, eritque ſpatium $AT = ct$, corpuſculum autem nunc ſit in Y, ita vt fili longitudo ſit $TY = a$. Vocemus autem angulum $ATY = \theta$, vnde demiſſo ex Y perpendiculo YX erit $TX = a\, \text{coſ.}\, \theta$ et $YX = a\, \text{ſin.}\, \theta$, ita vt poſitis coordinatis $AX = x$ et $XY = y$, ſit Tab. I. Fig. 6.

$$x = Ct - a\, \text{coſ.}\, \theta;\quad \partial x = c\,\partial t + a\,\partial\theta\, \text{ſin.}\, \theta,$$

$$y = a\, \text{ſin.}\, \theta\quad ;\quad \partial y = a\,\partial\theta\, \text{coſ.}\, \theta.$$

Ponamus autem porro $\frac{\partial y}{\partial x} = \text{tang.}\, \Phi$, ita vt Φ denotet angulum, ſub quo elementum curuae deſcriptae Yy ad axem AB inclinatur, ita vt ſit $\text{tang.}\, \Phi = \frac{a\,\partial\theta\, \text{coſ.}\, \theta}{c\,\partial t + a\,\partial\theta\, \text{ſin.}\, \theta}$.

§. 41. Denotet nunc M maſſam ſeu pondus corpuſculi, et ponatur tenſio fili $TY = T$, quae ergo eſt vis, qua corpuſculum a filo protrahitur, quae ſecundum directiones coordinatarum reſoluta praebet vim ſecundum $AX = T\, \text{coſ.}\, \theta$, et vim ſecundum $XY = T\, \text{ſin.}\, \theta$, vbi notandum eſt hanc vim T adhuc eſſe incognitam. Praeterea vero etiam corpuſculum a frictione ſollicitatur, cuius vis ſit $= F$, quae cum ſemper directioni motus ſit contraria, eius directio erit yY, quae ergo reſoluta praebet vim ſecundum $AX = -F\, \text{coſ.}\, \Phi$ et vim ſecundum $XY = -F\, \text{ſin.}\, \Phi$. His igitur viribus colligendis ſumto elemento temporis ∂t conſtante principia motus ſequentes ſuppeditant aequationes:

I.)

I.) $\frac{M \partial \partial x}{2g \partial t^2} = T \text{ cof. } \theta - F \text{ cof. } \phi.$

II.) $\frac{M \partial \partial y}{2g \partial t^2} = - T \text{ fin. } \theta - F \text{ fin. } \phi.$

§. 42. Elidamus hinc ftatim tenfionem fili T, vtpote incognitam, et haec combinatio: I. fin. θ + II. cof. θ dabit hanc aequationem:

$$\frac{M(\partial \partial x \text{ fin. } \theta + \partial \partial y \text{ cof. } \theta)}{2g \partial t^2} = - F(\text{cof. } \phi \text{ fin. } \theta + \text{fin. } \phi \text{ cof. } \theta)$$
$$= - F \text{ fin. } (\phi + \theta).$$

Statuamus nunc breuitatis gratia $\frac{2gF}{M} = b$; vbi notetur, g exprimere altitudinem lapfus grauium pro vno minuto fecundo, et fractionem $\frac{F}{M}$ vulgo aeftimari $= \frac{1}{3}$; ficque tota quaeftio reducta eft ad refolutionem huius aequationis:

$$\frac{\partial \partial x \text{ fin. } \theta + \partial \partial y \text{ cof. } \theta}{\partial t^2} = - b(\text{fin. } \theta \text{ cof. } \phi + \text{cof. } \theta \text{ fin. } \phi).$$

Cum autem fit

$$\partial \partial x = a \partial \partial \theta \text{ fin. } \theta + a \partial \theta^2 \text{ cof. } \theta \text{ et}$$
$$\partial \partial y = a \partial \partial \theta \text{ cof. } \theta - a \partial \theta^2 \text{ fin. } \theta,$$

aequatio refoluenda induet hanc formam:

$$\frac{a \partial \partial \theta}{\partial t^2} + b(\text{fin. } \theta \text{ cof. } \phi + \text{cof. } \theta \text{ fin. } \phi) = 0,$$

ex qua angulus ϕ facile eliminatur per formulas

$$\text{fin. } \phi = \frac{a \partial \theta \text{ cof. } \theta}{\sqrt{(cc \partial t^2 + 2ac \partial t \partial \theta \text{ fin. } \theta + aa \partial \theta^2)}} \text{ et}$$
$$\text{cof. } \phi = \frac{c \partial t + a \partial \theta \text{ fin. } \theta}{\sqrt{(cc \partial t^2 + 2ac \partial t \partial \theta \text{ fin. } \theta + aa \partial \theta^2)}}.$$

His enim valoribus fubftitutis habebimus

$$\frac{a \partial \partial \theta}{\partial t^2} + \frac{b(a \partial \theta + c \partial t \text{ fin. } \theta)}{\sqrt{(cc \partial t^2 + 2ac \partial t \partial \theta \text{ fin. } \theta + aa \partial \theta^2)}} = 0.$$

§. 43. Antequam autem refolutionem huius aequationis fufcipiamus, perpendamus cafum, quo frictio plane euanefcit

cit, ita vt fit $b = 0$, ac motus totus continebitur in hac simplicissima aequatione: $\frac{a\,\partial\,\partial\,\theta}{\partial\,t^2} = 0$, hinc $\frac{a\,\partial\,\theta}{\partial\,t} =$ const. hoc est celeritas angularis erit constans, quae, quoniam angulus θ continuo minuitur, ponatur $\frac{a\,\partial\,\theta}{\partial\,t} = -f$, vnde fit $a\,\theta = k - f\,t$. Hinc si ponamus initio, vbi $t = 0$, filum tenuisse situm A C normalem ad axem, ita vt tum fuerit $\theta = 90°$, erit $k = a \,.\, 90°$, ideoque $\theta = 90 - \frac{f}{a} \,.\, t$. Denotabit ergo $\frac{f}{a}$ certum angulum, qui sit $= \alpha$, ita vt habeamus $\theta = 90° - \alpha\,t$, quo inuento habebimus $x = c\,t - a$ sin. $\alpha\,t$ et $y = a$ cos. $\alpha\,t$, hincque porro $\frac{\partial\,x}{\partial\,t} = c - a\,\alpha$ cos. $\alpha\,t$ et $\frac{\partial\,y}{\partial\,t} = -a\,\alpha$ sin. $\alpha\,t$. Vnde si initio corpusculum in C quieuisse sumamus, tam $\frac{\partial\,x}{\partial\,t}$ quam $\frac{\partial\,y}{\partial\,t}$ ibi euanuisse necesse est, cui conditioni satisfit si sumatur $\alpha = \frac{c}{a}$, ita vt sit $\theta = 90° - \frac{c\,t}{a}$, hincque

$$x = c\,t - a\,\text{sin.}\,\frac{c\,t}{a} \quad\text{et}\quad y = a\,\text{cos.}\,\frac{c\,t}{a}.$$

Ex posteriore fit $\frac{c\,t}{a} = \text{A cos.}\,\frac{y}{a}$, quo valore substituto fiet

$$x = a\,\text{A cos.}\,\frac{y}{a} - \sqrt{(a\,a - y\,y)},$$

vnde patet hanc curuam fore cycloidem inuersam, a circulo, cuius radius $= a$, sub recta C D axi parallela, voluente descriptam, cuius cuspis in ipso puncto C sit sita.

§. 44. Contemplemur etiam casum oppositum, quo frictio esset infinita, ideoque $b = \infty$, et in nostra aequatione primum membrum prae altero euanescet, eritque $a\,\partial\,\theta + c\,\partial\,t$ sin. $\theta = 0$, vnde fit $c\,\partial\,t = -\frac{a\,\partial\,\theta}{\text{sin.}\,\theta}$ et integrando $c\,t = -a\,l$ tang. $\frac{1}{2}\,\theta + C$. Vnde si pro $t = 0$ fuerit $\theta = 90°$, erit $C = 0$ ideoque $c\,t = +a\,l$ cot. $\frac{1}{2}\,\theta$, ideoque $x = a\,l$ cot. $\frac{1}{2}\,\theta - a$ cos. θ, existente $y = a$ sin. θ, ex quibus formulis manifesto deducitur Tractoria vulgaris. Cum enim ob $c\,\partial\,t = -\frac{a\,\partial\,\theta}{\text{sin.}\,\theta}$, fit $\partial\,x = -\frac{a\,\partial\,\theta\,\text{cos.}\,\theta^2}{\text{sin.}\,\theta}$ et $\partial\,y = a\,\partial\,\theta$ cos. θ, erit $\frac{\partial\,y}{\partial\,x} = -$ tang. θ, vnde patet ipsum fiſum YT eſſe tangentem curuae. Ex hoc iam intelligitur, quod ſupra obſeruauimus, Tractorias vulgares tum demum prodire, quando frictio eſt infinite quaſi magna, vel, quod eodem redit, quando vis trahens frictionem quam minime ſuperat.

§. 45. His praemiſſis videamus quomodo aequationem ſupra inuentam tractari conueniat. Ac primo quidem eam ad differentialem primi gradus reduci conueniet, quod fiet ſi ponatur $\partial t = \frac{\partial \theta}{p}$. Quia enim ∂t conſtans eſt aſſumtum, hinc fiet $\partial \partial \theta = \frac{\partial \theta \partial p}{p}$, quibus valoribus ſubſtitutis aequatio noſtra hanc induet formam:

$$\frac{g p \partial p}{\partial \theta} + \frac{b (a p + c \sin. \theta)}{\sqrt{(c c + 2 a c p \sin. \theta + a a p p)}} = 0,$$

quae autem quomodo ad integrabilitatem perduci queat nullo modo patet.

§. 46. Eam quidem ab irrationalitate liberare haud eſt difficile. Ponatur enim $\frac{a p + c \sin \theta}{c \cos. \theta} = \text{tang.}\, \omega$, ita vt ſit

$$p = \frac{c \cos. \theta \,\text{tang.}\, \omega - c \sin. \theta}{a}, \text{ vnde ſit}$$

$$a p = \frac{c \sin. (\omega - \theta)}{\cos. \omega} \text{ et}$$

$$\partial p = -\tfrac{1}{a} \left(c \partial \theta \sin. \theta \,\text{tang.}\, \omega - \frac{c \partial \omega \cos. \theta}{\cos. \omega^2} + c \partial \theta \cos. \theta\right)$$

$$= + \frac{c \partial \omega \cos. \theta}{a \cos. \omega^2} - \frac{c \partial \theta}{a} \frac{\cos. (\theta - \omega)}{\cos. \omega},$$

formula autem irrationalis ſequentem induet formam: $\frac{c \cos. \theta}{\cos. \theta}$. Subſtituantur igitur iſti valores atque emerget ſequens aequatio:

$$\frac{a c \partial \omega \cos. \theta^2}{a \cos. \omega^3} - \frac{c c \partial \theta \cos. \theta \cos. (\theta - \omega)}{a \cos. \omega^2} + b \partial \theta + \frac{b \partial \theta \sin. \theta \cos. \omega}{\sin. (\omega - \theta)} = 0,$$

quae porro transformatur in hanc:

$$c c \partial \omega \cos. \theta - c c \partial \theta \cos. (\omega - \theta) \cos. \omega + \frac{a b \partial \theta \cos. \omega^3 \sin. \omega}{\sin. (\omega - \theta)} = 0.$$

Statu-

Statuatur porro $\frac{ab}{cc} = n$, eritque

$$\partial \omega \operatorname{cof.} \theta - \partial \theta \operatorname{cof.} \omega \operatorname{cof.} (\omega - \theta) + \frac{n \partial \theta \operatorname{cof.} \omega^2 \operatorname{fin.} \omega}{\operatorname{fin.} (\omega - \theta)} = C.$$

Quanquam autem haec aequatio ſatis prodiit concinna tamen haud patet quomodo eam vlterius reſoluere liceat; vnde haec quaeſtio vires analyſeos ſuperare videtur. Multo minus tales quaeſtiones ſuſcipi poterunt, ſi filum per lineam curuam vel etiam motu non vniformi protrahatur. Quamobrem tales quaeſtiones prorſus relinquere cogimur.