

ANALYSIS  
FACILIS ET PLANA AD EAS SERIES MAXIME  
ABSTRVSAS PERDVCENS,  
QVIBVS OMNIVM  
AEQVATIONVM ALGEBRAICARVM  
NON SOLVM RADICES IPSAE,  
SED ETIAM QVAEVIS EARVM POTESTATES  
EXPRIMI POSSVNT.

Auctore  
L. E V L E R O.

Conuent. exhib. die 15 April. 1776.

Problema.

Proposita aequatione algebraica tribus terminis constante, quam semper hac forma repraesentare licet:  $x = \frac{A}{x^\alpha} + \frac{B}{x^\beta}$ , inuenire seriem, quae exprimat valorem ipsius  $x^n$ .

Solutio.

§. 1. Haec aequatio, ponendo  $x = A^{\frac{1}{\alpha}} Z$ , semper ad hanc formam simpliciorem reuocari potest:  $x = \frac{1}{Z^\alpha} + \frac{B}{A^\alpha Z^\beta}$ ,

vnde ponendo  $B = A^{\frac{\beta}{\alpha}} C$ , erit  $x = \frac{1}{Z^\alpha} + \frac{C}{Z^\beta}$ , quae per  $Z^n$  mul-

multiplicata praebet  $Z^n = Z^{n-\alpha} + CZ^{n-\beta}$ ; hinc igitur quae oportet valorem potestatis  $Z^n$ , quandoquidem hinc erit  $x^n = A^\alpha Z^n$ . Manifestum autem est valorem ipsius  $Z^n$  exprimi debere per seriem, in quam exponens  $n$  ingrediatur, quam ergo spectare licebit tanquam functionem ipsius  $n$ , et quia haec series ex infinitis terminis constabit, eam ita repraesentemus:

$Z^n = f^0 : n + f' : n + f'' : n + f''' : n + f'''' : n + \text{etc.}$   
 quae ergo forma ita debet esse comparata, vt posito  $n = 0$  fiat  $Z^n = 1$ ; unde patet statui debere  $f^0 : n = 1$ , reliquos vero terminos factorem habere debere  $n$ , vt euaneant posito  $n = 0$ , prodeatque  $x^0 = 1$ .

§. 2. Constituta hac serie, si loco  $n$  scribamus  $n - \alpha$ , habebimus:

$$Z^{n-\alpha} = f^0 : (n-\alpha) + f' : (n-\alpha) + f'' : (n-\alpha) + f''' : (n-\alpha) + \text{etc.}$$

similique modo erit

$$Z^{n-\beta} = f^0 : (n-\beta) + f' : (n-\beta) + f'' : (n-\beta) + \text{etc.}$$

vbi iterum notetur esse  $f^0 : (n-\alpha) = 1$  et  $f^0 : (n-\beta) = 1$ . Cum iam nostra aequatio sit  $Z^n - Z^{n-\alpha} = CZ^{n-\beta}$ , scribamus loco potestatum ipsius  $Z$  series assumtas sequenti modo:

$$+ Z^n = + f^0 : n + f' : n + f'' : n + f''' : n + \text{etc.}$$

$$- Z^{n-\alpha} = - f^0 : (n-\alpha) - f' : (n-\alpha) - f'' : (n-\alpha) - f''' : (n-\alpha) - \text{etc.}$$

$$\underline{\underline{= CZ^{n-\beta} = Cf^0 : (n-\beta) + Cf' : (n-\beta) + Cf'' : (n-\beta) + Cf''' : (n-\beta) + \text{etc.}}}$$

nunc functiones istae indefinitae ita determinentur, vt fiat

$$\text{I. } f' : n - f' : (n - \alpha) = Cf^0 : (n - \beta) = C,$$

$$\text{II. } f'': n - f'' : (n - \alpha) = Cf' : (n - \beta);$$

III.

III.  $f''' : n - f''' : (n - \alpha) = C f'' : (n - \beta);$

IV.  $f'''' : n - f'''' : (n - \alpha) = C f''' : (n - \beta).$   
etc. etc.

§. 3. Ope harum aequationum ergo primo quaeri debet natura functionis  $f' : n$ , vt primae aequationi satisfiat; qua inuenta innotescet functio  $f' : (n - \beta)$ , ex eaque per secundam aequationem quaeri debet indeles functionis  $f'' : n$ , vnde innotescet functio  $f'' : (n - \beta)$ , hincque porro simili modo ex aequatione tertia deducetur indeles functionis  $f''' : n$ , et ita porro, donec lex pateat, qua singulae hae functiones ulterius progradientur: vnde patet resolutionem omnium harum aequationum renocari ad hanc quaestionem, qua proposita functione ipsius  $n$  queritur alia functio, veluti  $\Phi : n$ , vt fiat  $\Phi : n - \Phi : (n - \alpha) = N$ , quem in finem sequentia Lemmata euoluamus.

### Lemma I.

§. 4. Si fuerit  $\Phi : n = \Delta n$ , erit  $\Phi : (n - \alpha) = \Delta(n - \alpha)$ , ideoque  $\Phi : n - \Phi : (n - \alpha) = \Delta \alpha$ ; vnde vicissim, si ponatur  $\Delta \alpha = k$ , vt fieri debeat  $\Phi : n - \Phi : (n - \alpha) = k$ , reperietur  $\Phi : n = \frac{k}{\alpha} n$ . Quare cum ex prima aequatione esse debeat  $f' : n - f' : (n - \alpha) = C$ , necesse est vt sit  $f' : n = \frac{c}{\alpha} n$ , vnde pro secunda aequatione fiet  $f' : (n - \beta) = \frac{c}{\alpha} (n - \beta)$ .

### Lemma II.

§. 5. Si fuerit  $\Phi : n = \Delta n (n + \alpha - v)$ , erit

$\Phi : (n - \alpha) = \Delta (n - \alpha) (n - v)$ , vnde colligitur

$\Phi : n - \Phi : (n - \alpha) = 2 \Delta \alpha (n - \frac{1}{2} v)$ .

Quodsi ergo prodire debeat

$\Phi : n - \Phi : (n - \alpha) = k (n - \lambda)$ ,

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ob  $\Delta = \frac{k}{2\alpha}$  et  $v = 2\lambda$ , erit

$$\Phi : n = \frac{k\pi}{2\alpha} (n + \alpha - 2\lambda).$$

Quare cum aequatio secunda iam sit

$$f'' : n - f' : (n - \alpha) = C f' : (n - \beta) = \frac{cc}{\alpha} (n - \beta),$$

ob  $k = \frac{cc}{\alpha}$  et  $\lambda = \beta$  erit

$$f'' : n = \frac{cc}{2\alpha\alpha} n (n + \alpha - 2\beta),$$

vnnde pro tertia aequatione fiet

$$f'' : (n - \beta) = \frac{cc}{2\alpha\alpha} (n - \beta) (n + \alpha - 3\beta).$$

### Lemma III.

§. 6. Si fuerit

$$\Phi : n = \Delta n (n + \alpha - v) (n + 2\alpha - v), \text{ erit}$$

$$\Phi : (n - \alpha) = \Delta (n - \alpha) (n - v) (n + \alpha - v),$$

hinc ergo fit

$$\Phi : n - \Phi : (n - \alpha) = 3\Delta \alpha (n + \alpha - v) (n - \frac{1}{3}v),$$

vnnde vicissim, posito  $3\Delta \alpha = k$  et  $\frac{1}{3}v = \lambda$ , vt prodeat

$$k (n + \alpha - 3\lambda) (n - \lambda), \text{ sumi debet}$$

$$\Phi : n = \frac{k\pi}{3\alpha} (n + \alpha - 3\lambda) (n + 2\alpha - 3\lambda).$$

Quia nunc pro nostra aequatione tertia fieri debet

$$f''' : n - f''' : (n - \alpha) = \frac{c^3}{2\alpha\alpha} (n - \beta) (n + \alpha - 3\beta),$$

facta applicatione fiet  $k = \frac{c^3}{2\alpha\alpha}$  et  $\lambda = \beta$ , hincque concluditur fore

$$f''' : n = \frac{c^3}{6\alpha^3} n (n + \alpha - 3\beta) (n + 2\alpha - 3\beta),$$

vnnde pro aequatione sequente habebimus:

$$f''' : (n - \beta) = \frac{c^3}{6\alpha^3} (n - \beta) (n + \alpha + 4\beta) (n + 2\alpha - 4\beta).$$

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### Lemma IV.

§. 7. Si fuerit

$$\Phi : n = \Delta n(n+\alpha-v)(n+2\alpha-v)(n+3\alpha-v), \text{ erit}$$

$$\Phi : (n-\alpha) = \Delta (n-\alpha)(n-v)(n+\alpha-v)(n+2\alpha-v),$$

hincque

$$\Phi : n - \Phi : (n-\alpha) = 4\Delta \alpha (n+\alpha-v)(n+2\alpha-v)(n-\frac{1}{4}v).$$

Quare si debeat esse

$$\Phi : n - \Phi : (n-\alpha) = k(n-\lambda)(n+\alpha-4\lambda)(n+2\alpha-4\lambda),$$

sumi debet  $\Delta = \frac{k}{4\alpha}$  et  $v = 4\lambda$ , hincque fiet

$$\Phi : n = \frac{k}{4\alpha}(n+\alpha-4\lambda)(n+2\alpha-4\lambda)(n+3\alpha-4\lambda).$$

Quare cum aequatio nostra quarta sit

$$f''''' : n - f''''' : (n-\alpha) = \frac{c^4}{6\alpha^3}(n-\beta)(n+\alpha-4\beta)(n+2\alpha-4\beta),$$

facta applicatione fiet  $k = \frac{c^4}{6\alpha^3}$  et  $\lambda = \beta$ , hincque concluditur fore

$$f''''' : n = \frac{c^4}{24\alpha^4}n(n+\alpha-4\beta)(n+2\alpha-4\beta)(n+3\alpha-4\beta),$$

nde pro quinta aequatione nanciscemur:

$$f''''' : (n-\beta) = \frac{c^4}{24\alpha^4}(n-\beta)(n+\alpha-5\beta)(n+2\alpha-5\beta)(n+3\alpha-5\beta).$$

### Lemma V.

§. 8. Si fuerit

$$\Phi : n = \Delta n(n+\alpha-v)(n+2\alpha-v)(n+3\alpha-v)(n+4\alpha-v),$$

erit

$$\Phi : (n-\alpha) = \Delta (n-\alpha)(n-v)(n+\alpha-v)(n+2\alpha-v)(n+3\alpha-v),$$

hincque

$$\Phi : n - \Phi : (n-\alpha) = 5\Delta \alpha (n+\alpha-v)(n+2\alpha-v)(n+3\alpha-v)(n-\frac{1}{5}v).$$

Quare si debeat esse

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$\Phi : n$

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$$\Phi : n - \Phi : (n - \alpha) = k(n - \lambda)(n + \alpha - 5\lambda)(n + 2\alpha - 5\lambda)(n + 3\alpha - 5\lambda),$$

sumi debet  $\Delta = \frac{k}{5\alpha}$  et  $5\lambda = v$ , tum vero erit

$$\Phi : n = \frac{k}{5\alpha} n(n + \alpha - 5\lambda)(n + 2\alpha - 5\lambda)(n + 3\alpha - 5\lambda)(n + 4\alpha - 5\lambda).$$

Aequatio autem quinta cum ita se habeat:

$$f''''' : n - f''''' : (n - \alpha) = \frac{c^5}{24\alpha^4} (n - \beta)(n + \alpha - 5\beta)(n + 2\alpha - 5\beta)(n + 3\alpha - 5\beta),$$

hic sumi debet  $k = \frac{c^5}{24\alpha^4}$  et  $\lambda = \beta$ , vnde concluditur

$$f''''' : n = \frac{c^5}{120\alpha^5} n(n + \alpha - 5\beta)(n + 2\alpha - 5\beta)(n + 3\alpha - 5\beta)(n + 4\alpha - 5\beta).$$

Hinc iam sine ulteriore calculo concludere licet fore

$$f^{VI} : n = \frac{c^6}{720\alpha^6} n(n + \alpha - 6\beta)(n + 2\alpha - 6\beta)(n + 3\alpha - 6\beta) \times$$
$$\times (n + 4\alpha - 6\beta)(n + 5\alpha - 6\beta) \text{ et}$$

$$f^{VII} : n = \frac{c^7}{5040\alpha^7} n(n + \alpha - 7\beta)(n + 2\alpha - 7\beta)(n + 3\alpha - 7\beta) \times$$
$$\times (n + 4\alpha - 7\beta)(n + 5\alpha - 7\beta)(n + 6\alpha - 7\beta).$$

### Conclusio finalis.

§. 9. His igitur colligendis si aequatio proposita fuerit  $x = \frac{1}{Z^\alpha} + \frac{C}{Z^\beta}$ , tum pro potestate quacunque ipsius  $Z$  sequens resultat series:

$$Z^n = x + \frac{c}{\alpha} n + \frac{c c}{2\alpha\alpha} n(n + \alpha - 2\beta) + \frac{c^2}{6\alpha^3} n(n + \alpha - 3\beta)(n + 2\alpha - 3\beta)$$
$$+ \frac{c^4}{24\alpha^4} n(n + \alpha - 4\beta)(n + 2\alpha - 4\beta)(n + 3\alpha - 4\beta)$$
$$+ \frac{c^5}{120\alpha^5} n(n + \alpha - 5\beta)(n + 2\alpha - 5\beta)(n + 3\beta - 5\beta) \times$$
$$\times (n + 4\alpha - 5\beta) + \text{etc.}$$

### Scholion.

§. 10. Haec series, quam eruimus, eo magis est notata digna, quod nulla alia via patet eam inueniendi. Quin etiam

iam Analysis nostra ita est comparata, vt veritas solutionis non solum ad omnes exponentes integros  $n$ , sed etiam ad quosvis valores fractos, atque adeo negatiuos extenditur. Praeterea vero etiam ex nostra serie generali logarithmus Hyperbolicus ipsius  $Z$  exprimi potest. Cum enim semper, casu  $n=0$ , sit

$$\frac{Z^n - 1}{n} = LZ,$$

erit nostro casu

$$LZ = \frac{c}{\alpha} + \frac{cc}{2\alpha^2} (\alpha - 2\beta) + \frac{c^3}{6\alpha^3} (\alpha - 3\beta)(2\alpha - 3\beta) \\ + \frac{c^4}{24\alpha^4} (\alpha - 4\beta)(2\alpha - 4\beta)(3\alpha - 4\beta) \\ + \frac{c^5}{120\alpha^5} (\alpha - 5\beta)(2\alpha - 5\beta)(3\alpha - 5\beta)(4\alpha - 5\beta) + \text{etc.}$$

Vnde si tota haec series designetur littera  $\Delta$ , vt sit  $LZ = \Delta$ , erit  $Z = e^\Delta$ , ideoque  $Z^n = e^{n\Delta}$ , quae ergo quantitas aequalis erit seriei supra inuentae pro  $Z^n$ . At vero ista expressio  $e^{n\Delta}$  in seriem euoluta praebet

$$Z^n = 1 + n\Delta + \frac{1}{2}nn\Delta^2 + \frac{1}{6}n^3\Delta^3 + \frac{1}{24}n^4\Delta^4 + \frac{1}{120}n^5\Delta^5 + \text{etc.}$$

quae ergo series seriei supra inuentae necessario erit aequalis, id quod etiam comprobabitur, dum saltem priores termini evoluentur. Cum enim sit

$$\Delta = \frac{c}{\alpha} + \frac{cc}{2\alpha^2} (\alpha - 2\beta) + \text{etc. erit}$$

$$\Delta^2 = \frac{cc}{\alpha^2} + \frac{c^3}{\alpha^3} (\alpha - 2\beta)$$

$$\Delta^3 = \frac{c^3}{\alpha^3},$$

vnde deducimus

$$Z^n = 1 + \frac{c}{\alpha}n + \frac{cc}{2\alpha^2}n(\alpha - 2\beta) + \frac{c^3}{6\alpha^3}n(\alpha - 3\beta)(2\alpha - 3\beta) + \text{etc.} \\ \frac{cc}{2\alpha^2}nn + \frac{c^3}{2\alpha^3}n(\alpha - 2\beta) + \text{etc.} \\ + \frac{c^3}{6\alpha^3}n^3 + \text{etc.}$$

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$Z^n =$

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$$Z^n = 1 + \frac{c}{\alpha} n + \frac{cc}{\alpha^2} n(n+\alpha-2\beta) + \frac{c^3}{\alpha^3} (n(\alpha-3\beta)(2\alpha-3\beta) + nn(n+3\alpha-6\beta),$$

quod cum serie pro  $Z^n$  supra inuenta perfecte congruit.

### Theorema generale.

§. 11. Quodsi ergo proposita fuerit aequatio initio commemorata:  $1 = \frac{A}{x^\alpha} + \frac{B}{x^\beta}$ , quoniam posuimus  $Z = \frac{x}{A^\alpha}$  et

$$C = \frac{B}{A^\alpha}, \text{ erit}$$

$$\begin{aligned} \frac{x}{A^\alpha} &= 1 + \frac{B}{A^\alpha} \cdot \frac{n}{\alpha} + \frac{B^2}{A^\alpha} \cdot \frac{n}{\alpha} \cdot \frac{(n+\alpha-2\beta)}{2\alpha} \\ &\quad + \frac{B^3}{A^\alpha} \cdot \frac{n}{\alpha} \cdot \frac{(n+\alpha-3\beta)(n+2\alpha-3\beta)}{2\alpha \cdot 3\alpha} \\ &\quad + \frac{B^4}{A^\alpha} \cdot \frac{n}{\alpha} \cdot \frac{(n+\alpha-4\beta)(n+2\alpha-4\beta)(n+3\alpha-4\beta)}{2\alpha \cdot 3\alpha \cdot 4\alpha} \text{ etc.} \end{aligned}$$

fiue

$$\begin{aligned} X^n &= A^{\frac{n}{\alpha}} + A^{\frac{n-\beta}{\alpha}} B \frac{n}{\alpha} + A^{\frac{n-2\beta}{\alpha}} B^2 \frac{n}{\alpha} \frac{(n+\alpha-2\beta)}{2\alpha} \\ &\quad + A^{\frac{n-3\beta}{\alpha}} B^3 \frac{n}{\alpha} \frac{(n+\alpha-3\beta)(n+2\alpha-3\beta)}{2\alpha \cdot 3\alpha} \\ &\quad + A^{\frac{n-4\beta}{\alpha}} B^4 \cdot \frac{n}{\alpha} \frac{(n+\alpha-4\beta)(n+2\alpha-4\beta)(n+3\alpha-4\beta)}{2\alpha \cdot 3\alpha \cdot 4\alpha} \\ &\quad + A^{\frac{n-5\beta}{\alpha}} B^5 \cdot \frac{n}{\alpha} \frac{(n+\alpha-5\beta)(n+2\alpha-5\beta)}{2\alpha \cdot 3\alpha} \times \\ &\quad \times \frac{(n+3\alpha-5\beta)(n+4\alpha-5\beta)}{4\alpha \cdot 5\alpha} + \text{etc.} \end{aligned}$$

Vicis-

Vicissim igitur proposita hac serie, eius summa erit  $= x^n$ , existente  $x$  radice huius aequationis:  $x = \frac{A}{x^\alpha} + \frac{B}{x^\beta}$ , id quod aliquot exemplis illustrare liceat.

### Exemplum 1.

§. 12. Statuamus  $\alpha = 1$  et  $\beta = 1$ , vt proposita sit ista aequatio:  $x = \frac{A}{x} + \frac{B}{x}$ , vnde fit  $x = A + B$ , consequenter  $x^n = (A + B)^n$ , series autem inuenta hoc casu nobis dat

$$(A + B)^n = A^n + \frac{n}{1} A^{n-1} B + \frac{n(n-1)}{1 \cdot 2} A^{n-2} B^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} A^{n-3} B^3 + \text{etc.}$$

quae est ipsa euolutio Binomii Newtoniana, quam nunc patet veram esse, quicunque numerus pro exponente  $n$  accipiatur, siue integer, siue fractus, siue positius, siue negatius, siue etiam surdus; cum in Algebra communi, vbi haec euolutio est tractata, exponens  $n$  necessario sit integer positius.

### Exemplum 2.

§. 13. Ponamus, vt ante,  $\alpha = 1$ , at sumatur  $\beta = 0$ , ita vt fit  $x = \frac{A}{x} + B$ , vnde fit  $x = \frac{A}{1-B}$ , consequenter

$$x^n = \frac{A^n}{(1-B)^n} = A^n (1-B)^{-n},$$

series autem, ad quam sumus perducti, hoc casu erit

$$A^n (1-B)^n = A^n + \frac{n}{1} A^n B + \frac{n(n+1)}{1 \cdot 2} A^n B^2 + \text{etc.}$$

siue

$$(1-B)^n = 1 + \frac{n}{1} B + \frac{n(n+1)}{1 \cdot 2} B^2 + \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3} B^3 + \text{etc.}$$

quae est demonstratio eiusdem theoromatis Newtoniani pro exponentibus negatiis.

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### Exemplum 3.

§. 14. Sumamus  $A = 2a$  et  $B = b$ , sitque porro  $a = 1$  et  $\beta = 2$ , vt nostra aequatio fiat  $x = \frac{a}{x} + \frac{b}{xx}$ , siue  $xx = 2ax + b$ , vnde fit  $x = a + \sqrt{(aa + b)}$ , quo valore substituto series ante inuenta praebet

$$(a + \sqrt{aa + b})^n = 2^n a^n + \frac{n}{1} 2^{n-2} a^{n-2} b + \frac{n(n-3)}{1 \cdot 2} 2^{n-4} a^{n-4} bb \\ + \frac{n(n-5)(n-4)}{1 \cdot 2 \cdot 3} 2^{n-6} a^{n-6} b^3 + \frac{n(n-7)(n-6)(n-5)}{1 \cdot 2 \cdot 3 \cdot 4} 2^{n-8} a^{n-8} b^4 \\ + \text{etc.}$$

cuius veritas pro casu, quo  $n = 1$ , ex euolutione vulgari confirmari potest. Sumto enim  $n = 1$  erit

$$a + \sqrt{aa + b} = 2a + \frac{b}{2a} - \frac{bb}{2^3 a^3} + \frac{3b^3}{2^4 a^5} - \frac{5b^4}{2^7 a^7} + \frac{7b^5}{2^8 a^9} - \text{etc.}$$

nouimus autem ex resolutione vulgari esse

$$\sqrt{(aa + b)} = a + \frac{b}{2a} - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4} \frac{bb}{a^3} + \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{b^3}{a^5} - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} \frac{b^4}{a^7} + \text{etc.}$$

cui si addatur  $a$ , ipsa illa series prodit.

### Exemplum 4.

§. 15. Sumamus  $\alpha = 2$  et  $\beta = 1$ , vt fit  $x = \frac{a}{xx} + \frac{b}{x}$ , siue  $xx = A + Bx$ , ideoque  $x = \frac{B + \sqrt{(BB + 4A)}}{2}$ , loco A autem scribamus  $aa$  et  $2b$  loco B, vt fit  $x = b + \sqrt{(bb + aa)}$ , quocirca series inuenta nobis dabit

$$(b + \sqrt{bb + aa})^n = a^n + \frac{n}{1} a^{n-1} \cdot 2b + \frac{n}{2} \cdot \frac{n}{4} a^{n-2} \cdot 4bb \\ + \frac{n(n-1)(n+1)}{2 \cdot 4 \cdot 6} a^{n-3} \cdot 8b^3 + \frac{n}{2} \cdot \frac{(n-2)n(n+2)}{4} \frac{n}{3} a^{n-4} \cdot 16b^4 + \text{etc.}$$

quae reducitur ad hanc formam:

$$(b + \sqrt{bb + aa})^n = a^n + \frac{n}{1} a^{n-1} b + \frac{n}{1} \cdot \frac{n}{2} a^{n-2} bb \\ + \frac{n}{1} \cdot \frac{(n-1)(n+1)}{2 \cdot 3} a^{n-3} b^3 + \frac{n}{1} \frac{(n-2)}{2} \frac{n}{3} \cdot \frac{(n+2)}{4} a^{n-4} b^4 \text{ etc.}$$

haec autem forma vltius reducetur ad hanc:

( $b +$

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$$(b + \sqrt{bb+aa})^n = a^n + \frac{n}{1} a^{n-1} b + \frac{\frac{n}{1} \cdot \frac{n}{2}}{1.2} a^{n-2} b^2 \\ + \frac{\frac{n(n-1)}{1.2} \cdot \frac{n(n-2)}{3.4}}{1.2.3.4} a^{n-3} b^3 + \frac{\frac{n(n-1)(n-2)}{1.2.3.4} \cdot \frac{n(n-3)}{5}}{1.2.3.4.5} a^{n-4} b^4 + \dots \\ + \frac{\frac{n(n-1)(n-2)(n-3)}{1.2.3.4.5} \cdot \frac{n(n-4)}{6}}{1.2.3.4.5.6} a^{n-5} b^5 + \frac{\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5.6} \cdot \frac{n(n-5)}{7}}{1.2.3.4.5.6.7} a^{n-6} b^6 + \dots \\ + \text{etc.}$$

Ita si sumamus  $n = 1$ , habebimus

$$b + \sqrt{bb+aa} = a + b + \frac{\frac{1}{1} \cdot \frac{b \cdot b}{a}}{1.2} + \frac{\frac{3}{1} \cdot \frac{b^4}{a^3}}{1.2.3.4} + \frac{\frac{3.15}{1} \cdot \frac{b^6}{a^5}}{1.2.3.4.5} + \text{etc.}$$

nouimus autem esse

$$\sqrt{bb+aa} = a + \frac{\frac{1}{1} \cdot \frac{b \cdot b}{a}}{1.2} + \frac{\frac{1.1.3}{2.4} \cdot \frac{b^4}{a^3}}{2.4.6} + \text{etc.}$$

cui si addatur  $b$ , ipsa illa series prodit.

### Exemplum 5.

§. 16. Sumamus  $\alpha = 1$  et  $\beta = -1$ , vt nostra aequatio sit  $1 = \frac{A}{x} + Bx$ , vnde fit  $x = \frac{1 + \sqrt{1 - 4AB}}{2B}$ , hinc ergo prodit

$$\left(\frac{1 + \sqrt{1 - 4AB}}{2B}\right)^n = A^n + \frac{n}{1} A^{n+1} B + \frac{\frac{n(n+3)}{1.2}}{1.2} A^{n+2} B^2 \\ + \frac{\frac{n(n+4)(n+5)}{2.3}}{1.2.3.4} A^{n+3} B^3 + \frac{\frac{n(n+5)(n+6)(n+7)}{2.3.4.5}}{1.2.3.4.5} A^{n+4} B^4 \\ + \frac{\frac{n(n+6)(n+7)(n+8)(n+9)}{2.3.4.5.6}}{1.2.3.4.5.6} A^{n+5} B^5 + \text{etc.}$$

Hinc ergo si sumamus  $n = 1$ , erit

$$\frac{1 + \sqrt{1 - 4AB}}{2B} = A + A^2 B + \frac{1}{2} A^3 B^2 + \frac{5.6}{2.3} A^4 B^3 \\ + \frac{6.7.8}{2.3.4} A^5 B^4 + \frac{7.8.9.10}{2.3.4.5} A^6 B^5 + \text{etc.}$$

Est vero

$$\sqrt{1 - 4AB} = 1 - 2AB - 2A^2 B^2 - 4A^3 B^3 \\ - 2.5A^4 B^4 - \text{etc.}$$

quae series ab unitate subtrahita et per  $2B$  diuisa praebet series inuentam.

### Scholion.

§. 17. Series autem generalis, quam supra elicuimus, primum ab acutissimo *Lamberto* ex principiis maxime diuersis est inuenta, quam idcirco *Lambertinam* appellare liceat, properea quod inter egregia huius viri inuenta merito est referenda. Methodus autem, qua hic vsi sumus, ad aequationes multo generaliores extendi potest, quando scilicet aequatio proposita quatuor pluresue terminos continet; id quod pro casu quatuor terminorum ostendisse operae erit pretium.

### Problema generalius.

*Si proposita fuerit aequatio algebraica huius formae:*

$$1 - \frac{1}{Z^\alpha} = \frac{B}{Z^\beta} + \frac{C}{Z^\gamma},$$

*inuenire seriem, quae valorem potestatis cuiuscunque ipsius  $Z$ , puta  $Z^n$  exprimat.*

### Solutio.

§. 18. Multiplicetur aequatio proposita per  $Z^n$ , vt habeatur  $Z^n - Z^{n-\alpha} = B Z^{n-\beta} + C Z^{n-\gamma}$ ; et potestatem quae sitam  $Z^n$  vt ante tanquam functionem ipsius  $n$  spectare licebit, quae per partes continuo procedentes ita repraesentetur, vt sit

$$Z^n = f^0 : n + f^1 : n + f^2 : n + f^3 : n + f^4 : n + \text{etc.}$$

vbi cum sumto  $n = 0$  fieri debeat  $Z^n = 1$ , sit perpetuo  $f^0 : n = 1$ , tum vero vt reliquae partes casu  $n = 0$  euanescent, singulas factorem  $n$  habere necesse est. Hinc ergo erit

$$Z^{n-\alpha} = f^0 : (n - \alpha) + f^1 : (n - \alpha) + f^2 : (n - \alpha) + \text{etc.}$$

$$Z^{n-\beta} = f^0 : (n - \beta) + f^1 : (n - \beta) + f^2 : (n - \beta) + \text{etc.}$$

$$Z^{n-\gamma} = f^0 : (n - \gamma) + f^1 : (n - \gamma) + f^2 : (n - \gamma) + \text{etc.}$$

Iam

Iam istae series loco harum potestatum in nostra aequatione substituantur, et cum partes in membro sinistro sponte se tollant, reliquae partes sinistrae partibus antecedentibus in dextro membro aequari debebunt, vnde sequentes aequationes resul-  
tabunt.

- I.  $f' : n - f' : (n - \alpha) = B f^o : (n - \beta) + C f^o : (n - \gamma) = B + C$
- II.  $f'' : n - f'' : (n - \alpha) = B f' : (n - \beta) + C f' : (n - \gamma)$
- III.  $f''' : n - f''' : (n - \alpha) = B f'' : (n - \beta) + C f'' : (n - \gamma)$
- IV.  $f'''' : n - f'''' : (n - \alpha) = B f''' : (n - \beta) + C f''' : (n - \gamma)$   
etc. etc.

§. 19. In subsidium iam vocemus lemmata supra al-  
lata, ex quibus constat fore vt sequitur:

- I. Si fuerit  $\Phi : n - \Phi : (n - \alpha) = k$ , erit  $\Phi : n = \frac{k n}{\alpha}$ ,
  - II. Si fuerit  $\Phi : n - \Phi : (n - \alpha) = k(n - \lambda)$ , erit  
 $\Phi : n = \frac{k n}{\alpha} (n + \alpha - 2\lambda)$ .
  - III. Si fuerit  $\Phi : n - \Phi : (n - \alpha) = k(n - \lambda)(n + \alpha - 3\lambda)$ , erit  
 $\Phi : n = \frac{k n}{3\alpha} (n + \alpha - 3\lambda)(n + 2\alpha - 3\lambda)$ .
  - IV. Si fuerit  $\Phi : n - \Phi : (n - \alpha) = k(n - \lambda)(n + \alpha - 4\lambda)(n + 2\alpha - 4\lambda)$ ,  
erit  
 $\Phi : n = \frac{k n}{4\alpha} (n + \alpha - 4\lambda)(n + 2\alpha - 4\lambda)(n + 3\alpha - 4\lambda)$ .
  - V. Si fuerit  $\Phi : n - \Phi : (n - \alpha) = k(n - \lambda)(n + \alpha - 5\lambda)(n + 2\alpha - 5\lambda)(n + 3\alpha - 5\lambda)$ ,  
erit  
 $\Phi : n = \frac{k n}{5\alpha} (n + \alpha - 5\lambda)(n + 2\alpha - 5\lambda)(n + 3\alpha - 5\lambda)(n + 4\alpha - 5\lambda)$ .
- et ita porro.

— (68) —

§. 20. Harum lemmatum ope ex prima aequatione,  
vbi pro lemmate primo est  $k = B + C$ , elicimus  $f' : n = \frac{Bn}{\alpha} + \frac{Cn}{\alpha}$ ,  
hinc igitur pro secunda aequatione erit

$$Bf' : (n - \beta) = B^2 \frac{(n - \beta)}{\alpha} + BC \frac{(n - \beta)}{\alpha} \text{ et}$$

$$Cf' : (n - \gamma) = C^2 \frac{(n - \gamma)}{\alpha} + BC \frac{(n - \gamma)}{\alpha},$$

---

$$\text{Summa } = \frac{BB}{\alpha} (n - \beta) + \frac{zBC}{\alpha} (n - \frac{\beta - \gamma}{2}) + \frac{CC}{\alpha} (n - \gamma),$$

quae formula quia ex tribus constat partibus, singulas cum  
lemmate secundo conferri oportet, ac pro prima parte erit  
 $k = \frac{BB}{\alpha}$  et  $\lambda = \beta$ , pro secunda parte est  $k = \frac{zBC}{\alpha}$  et  
 $\lambda = \frac{\beta + \gamma}{2}$ , pro tertia parte est  $k = \frac{CC}{\alpha}$  et  $\lambda = \gamma$ , vnde ex  
omnibus simul sumtis colligitur:

$$f'' : n = \frac{BB}{2\alpha^2} (n + \alpha - 2\beta) + \frac{zBC}{2\alpha^2} n (n + \alpha - \beta - \gamma)$$
$$+ \frac{CC}{2\alpha^2} n (n + \alpha - 2\gamma).$$

§. 21. Progrediamur iam ad aequationem tertiam, ac  
pro eius membro dextro habebimus:

$$Bf'' : (n - \beta) = \frac{B^3}{2\alpha^2} (n - \beta) (n + \alpha - 3\beta)$$

$$+ \frac{zBB^2}{2\alpha^2} (n - \beta) (n + \alpha - 2\beta - \gamma) + \frac{B^2C}{2\alpha^2} (n - \beta) (n + \alpha - \beta - 2\gamma),$$

$$Cf'' : (n - \gamma) = \frac{C^3}{2\alpha^2} (n - \gamma) (n + \alpha - 3\gamma)$$

$$+ \frac{B^2BC}{2\alpha^2} (n - \gamma) (n + \alpha - \beta - 2\gamma)$$

$$+ \frac{zB^2C}{2\alpha^2} (n - \gamma) (n + \alpha - \beta - 2\gamma)$$

---

$$\text{Summa } = \frac{B^3}{2\alpha^2} (n - \beta) (n + \alpha - 3\beta)$$

$$+ \frac{3BB^2C}{2\alpha^2} (n + \alpha - 2\beta - \gamma) (n - \frac{2\beta - \gamma}{3})$$

$$+ \frac{3B^2C}{2\alpha^2} (n + \alpha - \beta - 2\gamma) (n - \frac{\beta - 2\gamma}{3})$$

$$+ \frac{C^3}{2\alpha^2} (n - \gamma) (n + \alpha - 3\gamma),$$

quae

quac quia quatuor constat partibus, cum lemmate III. conferendis, pro prima parte erit  $k = \frac{B^3}{2\alpha^2}$  et  $\lambda = \beta$ , pro secunda parte erit  $k = \frac{3BBC}{2\alpha^2}$  et  $\lambda = \frac{2\beta + \gamma}{3}$ ; pro tertia vero parte est  $k = \frac{3BC^2}{2\alpha^2}$  et  $\lambda = \frac{\beta + 2\gamma}{3}$ , denique pro quarta parte est  $k = \frac{C^3}{2\alpha^2}$  et  $\lambda = \gamma$ , quibus obseruatis functio quaesita  $f'''$  itidem ex quatuor partibus constabit, quae sunt:

$$f''' : n = \left\{ \begin{array}{l} + \frac{B^3}{6\alpha^3} n(n+\alpha-3\beta)(n+2\alpha-3\beta) \\ + \frac{3BBC}{6\alpha^3} n(n+\alpha-2\beta-\gamma)(n+2\alpha-2\beta-\gamma) \\ + \frac{3BC^2}{6\alpha^3} n(n+\alpha-\beta-2\gamma)(n+2\alpha-\beta-2\gamma) \\ + \frac{C^3}{6\alpha^3} n(n+\alpha-3\gamma)(n+2\alpha-3\gamma). \end{array} \right\}$$

§. 22. Tractemus simili modo aequationem quartam, atque ex valore  $f''' : n$  inuento habebimus:

$$\begin{aligned} Bf''' : (n-\beta) &= \left\{ \begin{array}{l} + \frac{B^4}{6\alpha^3} (n-\beta)(n+\alpha-4\beta)(n+2\alpha-4\beta) \\ + \frac{3B^3C}{6\alpha^3} (n-\beta)(n+\alpha-3\beta-\gamma)(n+2\alpha-3\beta-\gamma) \\ + \frac{3B^2C^2}{6\alpha^3} (n-\beta)(n+\alpha-2\beta-2\gamma)(n+2\alpha-2\beta-2\gamma) \\ + \frac{BC^3}{6\alpha^3} (n-\beta)(n+\alpha-\beta-3\gamma)(n+2\alpha-\beta-3\gamma) \end{array} \right\}. \\ Cf''' : (n-\gamma) &= \left\{ \begin{array}{l} + \frac{3B^3C}{6\alpha^3} (n-\gamma)(n+\alpha-\beta-3\gamma)(n+2\alpha-\beta-3\gamma) \\ + \frac{B^3C}{6\alpha^3} (n-\gamma)(n+\alpha-3\beta-\gamma)(n+2\alpha-3\beta-\gamma) \\ + \frac{3B^2C^2}{6\alpha^3} (n-\gamma)(n+\alpha-2\beta-2\gamma)(n+2\alpha-2\beta-2\gamma) \\ + \frac{C^4}{6\alpha^3} (n-\gamma)(n+\alpha-4\gamma)(n+2\alpha-4\gamma) \end{array} \right\}. \end{aligned}$$

His iam terminis collectis valor formulae

$$Bf''' : (n-\beta) + Cf''' : (n-\gamma),$$

constabit sequentibus quinque partibus:

==== (70) ====

$$\begin{aligned}
 & \frac{B^4}{6\alpha^3} (n - \beta)(n + \alpha - 4\beta) \\
 & + \frac{B^3 C}{6\alpha^3} \left(n - \frac{3\beta - \gamma}{4}\right) (n + \alpha - 3\beta - \gamma) (n + 2\alpha - 3\beta - \gamma) \\
 & + \frac{6B^2 C^2}{6\alpha^3} \left(n - \frac{\beta - \gamma}{2}\right) (n + \alpha - 2\beta - 2\gamma) (n + 2\alpha - 2\beta - 2\gamma) \\
 & + \frac{4B C^3}{6\alpha^3} \left(n - \frac{\beta - 3\gamma}{4}\right) (n + \alpha - \beta - 3\gamma) (n + 2\alpha - \beta - 3\gamma) \\
 & + \frac{C^4}{6\alpha^3} (n - \gamma) (n + \alpha - 4\gamma) (n + 2\alpha - 4\gamma)
 \end{aligned}$$

§. 23. Quoniam igitur functio quæsita  $f^{IV}: n$  ex quinque partibus componitur, singulas cum lemmate quarto comparari oportebit, ac pro parte prima erit  $k = \frac{B^4}{6\alpha^3}$  et  $\lambda = \beta$ ; pro parte secunda est  $k = \frac{4B^3 C}{6\alpha^3}$  et  $\lambda = \frac{3\beta + \gamma}{4}$ ; pro parte tertia  $k = \frac{6B^2 C^2}{6\alpha^3}$  et  $\lambda = \frac{\beta + \gamma}{2}$ ; pro parte quarta est  $k = \frac{4B C^3}{6\alpha^3}$  et  $\lambda = \frac{\beta + 3\gamma}{4}$ ; denique pro parte quinta erit  $k = \frac{C^4}{6\alpha^3}$  et  $\lambda = \gamma$ ; unde collectis omnibus terminis reperietur

$$f^{IV}: n = \left\{ \begin{array}{l} \frac{B^4}{24\alpha^4} n (n + \alpha - 4\beta) (n + 2\alpha - 4\beta) (n + 3\alpha - 4\beta) \\ + \frac{4B^3 C}{24\alpha^4} n (n + \alpha - 3\beta - \gamma) (n + 2\alpha - 3\beta - \gamma) (n + 3\alpha - 3\beta - \gamma) \\ + \frac{6B^2 C^2}{24\alpha^4} n (n + \alpha - 2\beta - 2\gamma) (n + 2\alpha - 2\beta - 2\gamma) (n + 3\alpha - 2\beta - 2\gamma) \\ + \frac{4B C^3}{24\alpha^4} n (n + \alpha - \beta - 3\gamma) (n + 2\alpha - \beta - 3\gamma) (n + 3\alpha - \beta - 3\gamma) \\ + \frac{C^4}{24\alpha^4} n (n + \alpha - 4\gamma) (n + 2\alpha - 4\gamma) (n + 3\alpha - 4\gamma). \end{array} \right\}$$

§. 24. Superfluum foret hos calculos ulterius prosequi; quandoquidem ex allatis iam tuto concludere licet, sequentem functionem  $f^V: n$  hunc habituram esse valorem:

(71)

$$f^r : n = \left\{ \begin{array}{l} + \frac{B^5}{120 \alpha^5} n(n+\alpha-5\beta)(n+2\alpha-5\beta)(n+3\alpha-5\beta)(n+4\alpha-5\beta) \\ + \frac{5B^4C}{120 \alpha^5} n(n+\alpha-4\beta-\gamma)(n+2\alpha-4\beta-\gamma)(n+3\alpha-4\beta-\gamma) \times \\ \quad \times (n+4\alpha-4\beta-\gamma) \\ + \frac{10B^3C^2}{120 \alpha^5} n(n+\alpha-3\beta-2\gamma)(n+2\alpha-3\beta-2\gamma)(n+3\alpha-3\beta-2\gamma) \times \\ \quad \times (n+4\alpha-3\beta-2\gamma) \\ + \frac{10B^2C^3}{120 \alpha^5} n(n+\alpha-2\beta-3\gamma)(n+2\alpha-2\beta-\gamma)(n+3\alpha-2\beta-3\gamma) \times \\ \quad \times (n+4\alpha-2\beta-3\gamma) \\ + \frac{5BC^4}{120 \alpha^5} n(n+\alpha-\beta-4\gamma)(n+2\alpha-\beta-4\gamma)(n+3\alpha-\beta-4\gamma) \times \\ \quad \times (n+4\alpha-\beta-4\gamma) \\ + \frac{C^5}{120 \alpha^5} n(n+\alpha-5\beta)(n+2\alpha-4\beta)(n+3\alpha-5\beta)(n+4\alpha-5\beta) \end{array} \right\}$$

vnde formatio omnium sequentium functionum satis dilucide perspicitur.

§. 25. Quodsi iam omnes isti valores, quos pro functionibus  $f' : n$ ;  $f'' : n$ ;  $f''' : n$ ; etc. elicuimus, in unam sumam colligantur, et ob  $f^0 : n = 1$  unitas praefigatur, obtinetur series desiderata, quae scilicet valorem potestatis  $Z^n$  exprimit, neque ergo opus est omnes istas functiones hic denuo collectas referre.

### Corollarium.

Hae ergo series infinitis constant terminis, in quibus omnes possibles binarum litterarum B et C combinationes occurunt. Quin etiam pro combinatione quacunquae, quae sit  $B^b C^c$ , in genere terminus eam inuoluens assignari poterit. Primo enim ista forma multiplicetur per numerum omnium combinationum, qui posito breuitatis gratia  $b+c=i$ , si indicetur littera N, erit vti constat  $N = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots i}{1 \cdot 2 \cdot 3 \cdots b \cdots z \cdots c}$ , deinde si ponamus  $b\beta + c\gamma = 0$ , erit terminus huic formae respondens:

$N B^b$

(72)

$$\frac{N B^b C^c}{I. 2. 3. \dots i \alpha^i} n(n+\alpha-\theta)(n+2\alpha-\theta) \times \\ \times (n+3\alpha-\theta) \dots [n+(i-1)\alpha-\theta].$$

Veluti si forma proposita fuerit  $B^3 C^2$ , erit  $i = 5$ , hincque  
 $N = \frac{1. 2. 3. 4. 5}{1. 2. 3. 1. 2} = 10$ , deinde vero erit  $\theta = 3\beta + 2\gamma$ , siveque  
 ipse terminus huius formae erit

$$\frac{10 B^3 C^2}{120 \alpha^5} n(n+\alpha-3\beta-2\gamma)(n+2\alpha-3\beta-2\gamma) \times \\ \times (n+3\alpha-3\beta-2\gamma)(n+4\alpha-3\beta-2\gamma),$$

prorsus ut supra est exhibitus.

### Scholion.

§. 26. Hinc iam abunde patet, si aequatio proposita pluribus adhuc constet terminis, habeatque hanc formam:

$$I - \frac{1}{Z^\alpha} = \frac{B}{Z^\beta} + \frac{C}{Z^\gamma} + \frac{D}{Z^\delta} + \frac{E}{Z^\epsilon} + \text{etc.}$$

tum ope eiusdem methodi seriem infinitam inuestigari posse,  
 quae valorem potestatis  $Z^n$  exprimat; ista enim series incipiens  
 ab unitate infinitos inuoluet terminos ex omnibus plane com-  
 binationibus litterarum B, C, D, E, etc. formatos. Si enim  
 in genere proponatur haec combinatio:  $B^b. C^c. D^d. E^e$ . etc. sta-  
 tuatur primo  $b+c+d+e = i$ , et quaeratur numerus N,  
 ut fit

$$N = \frac{1. 2. 3. 4. \dots i}{1. 2. 3. \dots b. 1. 2. 3. \dots c. 1. 2. 3. \dots d. 1. 2. 3. \dots e},$$

siveque breuitatis gratia  $I. 2. 3. 4. \dots i = I$ , factor prior  
 huius termini erit  $\frac{N B^b. C^c. D^d. E^e}{I. \alpha^i}$ , praeterea vero adiungendi  
 sunt factores exponentem  $n$  inuoluentes, pro quibus inueni-  
 endis statuatur:

$$b\beta + c\gamma + d\delta + e\epsilon = \theta,$$

erunt-

eruntque isti factores numero  $i$  isti:

$$n(n+\alpha-\theta)(n+2\alpha-\theta)(n+3\alpha-\theta)\dots[n+(i-1)\alpha-\theta],$$

ita vt totus terminus fit

$$\frac{N \cdot B^b \cdot C^c \cdot D^d \cdot E^e}{1 \cdot \alpha^i} n(n+\alpha-\theta)(n+2\alpha-\theta) \times \\ \times(n+3\alpha-\theta)\dots[n+(i-1)\alpha-\theta].$$

Quamobrem super indole omnium harum serierum maxime memorabilium, quas olim in Tomo XV. Nouorum Commentariorum fusius, sed sufficiente demonstratione descripsi, nihil amplius desiderari posse videtur.

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