DE INNVMERIS CVRVIS ALGEBRAICIS, QVARVM LONGITVDINEM PER ARCVS ELLIPTICOS METIRI LICET.

Auctore

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§. 1.

Pro Ellipsi, cuius singuli arcus nobis mensuram curuarum quaesitarum suppeditare debent, sit abscissa $= v$, applicata vero $= n\sqrt{(1-vv)}$, vnde elementum arcus colligitur $= \frac{\partial v \sqrt{[1+(nn-1)vv]}}{\sqrt{(1-vv)}}$; quamobrem sequens nobis propositum sit problema.

Problema.

Pro coordinatis x et y eiusmodi functiones algebraicas ipsius v inuestigare, vt fiat

$$\sqrt{(\partial x^2 + \partial y^2)} = \frac{\partial v \sqrt{[1+(nn-1)vv]}}{\sqrt{(1-vv)}}.$$

Solutio.

§. 2. Vt formulae $\sqrt{(\partial x^2 + \partial y^2)}$ formam praescriptam conciliemus, quoniam denominator $\sqrt{(1-vv)}$ duos habet factores $\sqrt{(1+v)}$ et $\sqrt{(1-v)}$, statuamus $\partial x = \frac{(p+q)\partial v}{\sqrt{2(1+v)}}$

∂y

$\partial y = \frac{(p-q)\partial v}{\sqrt{2(1-v)}}$, hinc autem fiet

$$\sqrt{(\partial x^2 + \partial y^2)} = \partial v \sqrt{(pp + qq - 2pqv)},$$

vnde patet pro p et q eiusmodi quantitates quaeri debere, vt prodeat $pp + qq - 2pqv = 1 + (nn - 1)vv$.

§. 3. Ante omnia autem hic euidens eſt, ſi modo pro litteris p et q functiones rationales integrae ipſius v aſſignari queant, ambas formulas pro ∂x et ∂y aſſumtas ſemper integrationem eſſe admiſſuras, propterea quod ambae iſtae formulae $\frac{v^i \partial v}{\sqrt{(1+v)}}$ et $\frac{v^i \partial v}{\sqrt{(1-v)}}$ ſemper ſunt integrabiles, ſi modo exponens i fuerit integer poſitiuus. Ad hoc igitur negotium abſoluendum ſequentes caſus euoluamus.

I. Caſus
quo $p = 1$ et $q = \alpha v$.

§. 4. Hic igitur erit $pp + qq = 1 + \alpha\alpha vv$ et $2pqv = 2\alpha vv$, quamobrem effici oportet

$$1 + \alpha\alpha vv - 2\alpha vv = 1 + (nn - 1)vv$$

vnde patet ſumi debere $\alpha = 1 + n$, ita vt noſtra elementa hoc caſu fiant

$$\partial x = \frac{[1 + (n+1)v]\partial v}{\sqrt{2(1+v)}} \text{ et } \partial y = \frac{[1 - (n+1)v]\partial v}{\sqrt{2(1-v)}}$$

vbi integralibus ſumtis reperitur

$$x = \tfrac{1}{3}[1 - 2n + (n+1)v]\sqrt{2(1+v)} \text{ et}$$
$$y = \tfrac{1}{3}[2n - 1 + (n+1)v]\sqrt{2(1-v)}.$$

§. 5. Vt hinc quantitatem v eliminemus, addamus ambo quadrata, et obtinebimus

$$\frac{9(xx + yy)}{4} = (2n - 1)^2 - 3(nn - 1)vv,$$

ex

ex qua aequatione v facile per x et y determinatur; inde enim fit $vv = \frac{(2n-1)^2}{3(nn-1)} - \frac{3(xx+yy)}{4(nn-1)}$. Quo iam hunc valorem loco vv facilius ſubſtituere queamus, ſumamus productum noſtrarum formularum

$$\frac{9xy}{2} = [(n+1)^2 vv - (2n-1)^2] \sqrt{(1-vv)}$$

quae aequatio ſi quadretur, vbique tantum pares dimenſiones ipſius v occurrent, ac loco vv valore ſubſtituto aequatio inter x et y ad ſextum ordinem aſcendet.

II. Caſus

quo $p = 1 + \beta vv$ et $q = \alpha v$.

§. 6. Hic ergo erit

$$pp + qq = 1 + (\alpha\alpha + 2\beta) vv + \beta\beta v^4 \text{ et}$$
$$2pqv = 1 + 2\alpha vv + 2\alpha\beta v^4,$$

vnde conditio adimplenda erit

$$1 + (\alpha\alpha + 2\beta - 2\alpha) vv$$
$$+ (\beta\beta - 2\alpha\beta) v^4 = 1 + (nn-1) vv.$$

Hic igitur ante omnia eſſe oportet $\beta\beta - 2\alpha\beta = 0$, ideoque $\beta = 2\alpha$, atque nunc ſupereſt vt fiat

$$\alpha\alpha + 2\beta - 2\alpha = \alpha\alpha + 2\alpha = nn - 1,$$

ſicque capi debet $\alpha = n - 1$ et $\beta = 2(n-1)$.

§. 7. Pro curua igitur definienda habebimus

$$p = 1 + 2(n-1) vv \text{ et } q = (n-1) v,$$

ſicque nunc erit

$$\partial x = \frac{1 + (n-1)v + 2(n-1)vv}{\sqrt{2(1+v)}} \partial v \text{ et}$$
$$\partial y = \frac{1 - (n-1)v + 2(n-1)vv}{\sqrt{2(1-v)}} \partial v$$

quarum integratio nulla amplius laborat difficultate, vnde hoc labore merito ſuperſedemus.

III. Caſus

quo $p = 1 + \beta v v$ et $q = \alpha v + \gamma v^3$.

§. 8. Hic igitur erit

$$pp + qq = 1 + (\alpha\alpha + 2\beta) v v + (\beta\beta + 2\alpha\gamma) v^4 + \gamma\gamma v^6 \text{ et}$$
$$pq = \alpha v + (\alpha\beta + \gamma) v^3 + \beta\gamma v^5,$$

vnde conficitur

$$pp + qq - 2pqv = 1 + (\alpha\alpha + 2\beta - 2\alpha) v v$$
$$+ (\beta\beta + 2\alpha\gamma - 2\alpha\beta - 2\gamma) v^4 + (\gamma\gamma - 2\beta\gamma) v^6,$$

quae quantitas aequari debet $1 + (nn - 1) v v$. Hic igitur primo poteſtas v^6 tolli debet, quod fit ponendo $\gamma\gamma - 2\beta\gamma = 0$, ideoque $\gamma = 2\beta$; deinde vero etiam poteſtatem quartam tolli oportet, vnde fit

$$\beta\beta + 2\alpha\gamma - 2\alpha\beta - 2\gamma = 0, \text{ ſiue}$$
$$\beta\beta + 2\alpha\beta - 4\beta = 0, \text{ ideoque}$$
$$\beta = 4 - 2\alpha \text{ et } \gamma = 8 - 4\alpha.$$

Iam vero coëfficiens ipſius vv erit

$$\alpha\alpha + 2\beta - 2\alpha = \alpha\alpha + 8 - 6\alpha$$

quem aequari oportet ipſi $nn - 1$, vnde colligitur $\alpha - 3 = n$, ſiue $\alpha = n + 3$, tum vero

$$\beta = -2(n + 1) \text{ et } \gamma = -4(n + 1).$$

§. 9. His igitur valoribus inuentis noſtrae formulae integrandae erunt

$$\partial x = \frac{1 + (n+3)v - 2(n+1)vv - 4(n+1)v^3}{\sqrt{2(1+v)}} \partial v \text{ et}$$
$$\partial y = \frac{1 - (n+3)v - 2(n+1)vv + 4(n+1)v^3}{\sqrt{2(1-v)}} \partial v,$$

qua-

quarum integrationi iterum non immorabimur. Vnicum tantum adhuc talem casum attingamus.

IV. Casus

quo $p = 1 + \beta vv + \delta v^4$ et $q = \alpha v + \gamma v^3$.

§. 10. Hic igitur erit

$$pp + qq = 1 + (\alpha\alpha + 2\beta)vv + (\beta\beta + 2\delta + 2\alpha\gamma)v^4 + (2\beta\delta + \gamma\gamma)v^6 + \delta\delta v^8 \text{ et}$$

$$pq = \alpha v + (\alpha\beta + \gamma)v^3 + (\alpha\delta + \beta\gamma)v^5 + \gamma\delta v^7,$$

ex quibus conficitur formula

$$pp + qq - 2pqv = 1 + (\alpha\alpha + 2\beta - 2\alpha)vv + (\beta\beta + 2\alpha\gamma + 2\delta - 2\alpha\beta - 2\gamma)v^4 + (\gamma\gamma + 2\beta\delta - 2\alpha\delta - 2\beta\gamma)v^6 + (\delta\delta - 2\gamma\delta)v^8,$$

quae formula cum aequari debeat huic: $1 + (nn - 1)vv$, primo tollatur potestas octaua, vnde fit $\delta\delta - 2\gamma\delta = 0$, ideoque $\delta = 2\gamma$. Iam potestas sexta afficitur hac forma:

$$\gamma\gamma + 2\beta\delta - 2\alpha\delta - 2\beta\gamma = \gamma\gamma + 2\beta\gamma - 4\alpha\gamma,$$

quae nihilo aequata praebet $\gamma = 4\alpha - 2\beta$, hincque $\delta = 8\alpha - 4\beta$. Porro autem potestatis quartae coëfficiens est

$$\beta\beta + 2\alpha\gamma + 2\delta - 2\alpha\beta - 2\gamma =$$
$$\beta\beta - 6\alpha\beta - 4\beta + 8\alpha\alpha + 8\alpha = 0$$

quae aequatio diuisa per $\beta - 2\alpha$ praebet $\beta - 4\alpha - 4$, ita vt pro β geminos nanciscamur valores, alterum $\beta = 2\alpha$, alterum vero $\beta = 4\alpha + 4$, quorum vtrumque seorsim euoluamus.

§. 11. Sit igitur $\beta = 2\alpha$, eritque $\gamma = 0$ et $\delta = 0$, quo ergo casu res ad casum secundum reuoluitur. Sit igitur $\beta = 4\alpha + 4$, et nunc fiet $\gamma = -4(\alpha + 2)$ et $\delta = -8(\alpha + 2)$.

Verum hinc fiet poteſtatis vv coëfficiens $\alpha\alpha + 2\beta - 2\alpha$, ipſi $nn - 1$ aequandus, vnde fit $\alpha + 3 = n$, ſiue $\alpha = n - 3$, hincque porro fiet

$$\beta = 4(n-2);\ \gamma = -4(n-1) \text{ et } \delta = -8(n-1);$$

ex quibus ergo conficitur

$$p = 1 + 4(n-2)vv - 8(n-1)v^4 \text{ et}$$
$$q = (n-3)v - 4(n-1)v^3,$$

vnde tandem colligitur

$$x = \int \frac{(p+q)\partial v}{\sqrt{2(1+v)}} \text{ et } y = \int \frac{(p-q)\partial v}{\sqrt{2(1-v)}}$$

quem integrationis laborem ſuſcipere foret ſuperfluum.

Digreſſio
pro caſu $n = \pm 1$.

§. 12. Ex euolutione caſuum ſuperiorum manifeſtum eſt, curuas continuo ad altiores gradus aſcendere; hinc autem perpetuo excipi oportet caſum, quo foret $n = \pm 1$, quandoquidem arcus ellipticus $\int \frac{\partial v \sqrt{1 + (nn-1)vv}}{\sqrt{(1-vv)}}$ abiret in $\int \frac{\partial v}{\sqrt{(1-vv)}}$, hoc eſt in arcum circularem. Cum igitur praeter circulum nullae aliae dentur tales curuae, neceſſe eſt, vt curuae, ad quos caſus praecedentes nos deducunt, quando fuerit $n = \pm 1$, circulum exhibeant.

§. 13. Pro caſu autem primo, vbi integralia iam euolvimus, quando fit $n = \pm 1$ ideoque $nn - 1 = 0$, aequatio penultima abit in hanc: $\frac{9}{4}(xx + yy) = (2n-1)^2$, hoc eſt aequabitur vel $= 1$ vel $= 9$, ita vt vtroque caſu curua manifeſto ſit circulus, cum tamen pro aliis omnibus valoribus ipſius n aequatio ad ſextum ordinem aſſurgere ſit obſeruata.

§. 14.

§. 14. Pro casu secundo faciamus primo $n = +1$, eritque $\partial x = \frac{\partial v}{\sqrt{2(1+v)}}$ et $\partial y = \frac{\partial v}{\sqrt{2(1-v)}}$, vnde integrando fit $x = \sqrt{2(1+v)}$ et $y = -\sqrt{2(1-v)}$, ex quibus manifesto colligitur $xx + yy = 4$, quae vtique est aequatio ad circulum. Sin autem sumamus $n = -1$, reperitur

$$\partial x = \frac{1-2v-4vv}{\sqrt{2(1+v)}}\partial v \text{ et}$$
$$\partial y = \frac{1+2v-4vv}{\sqrt{2(1-v)}}\partial v,$$

hinc autem circulum enasci sequenti modo facillime ostendetur.

§. 15. Hunc in finem statuatur $v = \cos. 2\phi$, eritque $\partial v = -2\partial\phi \sin. 2\phi$ et $\sqrt{2(1+v)} = 2\cos.\phi$, similique modo $\sqrt{2(1-v)} = 2\sin.\phi$. Ergo pro priore formula erit $\frac{\partial v}{\sqrt{2(1+v)}} = -2\partial\phi \sin.\phi$, alter vero factor fiet

$$= 1 - 2\cos. 2\phi - 4\cos. 2\phi^2$$
$$= -1 - 2\cos. 2\phi - 2\cos. 4\phi$$

quamobrem habebimus

$$\partial x = 2\partial\phi \sin.\phi\,(1 + 2\cos. 2\phi + 2\cos. 4\phi).$$

Constat autem esse

$$2\sin.\phi\cos. 2\phi = \sin. 3\phi - \sin.\phi \text{ et}$$
$$2\sin.\phi\cos. 4\phi = \sin. 5\phi - \sin. 3\phi,$$

quibus substitutis obtinebitur $\partial x = 2\partial\phi \sin. 5\phi$, cuius integrale est $x = -\frac{2}{5}\cos. 5\phi$. Simili modo pro altera formula prodit $\frac{\partial v}{\sqrt{2(1-v)}} = -2\partial\phi\cos.\phi$, alter vero factor erit

$$1 + 2\cos. 2\phi - 4\cos. 2\phi^2 = -1 + 2\cos. 2\phi - 2\cos. 4\phi$$

sicque fiet

$$\partial y = 2\partial\phi\cos.\phi\,(1 - 2\cos. 2\phi + 2\cos. 4\phi).$$

Constat autem esse

$$2\cos.\phi\cos. 2\phi = \cos. 3\phi + \cos.\phi \text{ et}$$
$$2\cos.\phi\cos. 4\phi = \cos. 5\phi + \cos. 3\phi,$$

 quo-

quocirca proueniet $\partial y = 2\,\partial\Phi \cos.\, 5\,\Phi$, ideoque $y = \frac{2}{5} \sin.\, 5\,\Phi$, consequenter hic casus nobis suppeditat $xx + yy = \frac{4}{25}$, quae iterum manifesto est aequatio ad circulum.

§. 16. Tractemus simili modo casum tertium, ponendo primo $n = +1$, et habebimus

$$\partial x = \frac{1 + 4v - 4vv - 8v^3}{\sqrt{2(1+v)}}\,\partial v \text{ et}$$
$$\partial y = \frac{1 - 4v - 4vv + 8v^3}{\sqrt{2(1-v)}}\,\partial v.$$

Statuamus iterum $v = \cos.\, 2\,\Phi$ fietque

$$\partial x = -2\,\partial\Phi \sin.\,\Phi\,(1 + 4\cos.\, 2\,\Phi - 4\cos.\, 2\,\Phi^2 - 8\cos.\, 2\,\Phi^3) \text{ et}$$
$$\partial y = -2\,\partial\Phi \cos.\,\Phi\,(1 - 4\cos.\, 2\,\Phi - 4\cos.\, 2\,\Phi^2 + 8\cos.\, 2\,\Phi^3),$$

vbi notetur esse

$$4\cos.\, 2\,\Phi^2 = 2 + 2\cos.\, 4\,\Phi \text{ et}$$
$$8\cos.\, 2\,\Phi^3 = 6\cos.\, 2\,\Phi + 2\cos.\, 6\,\Phi,$$

vnde habebimus

$$\partial x = 2\,\partial\Phi \sin.\,\Phi\,(1 + 2\cos.\, 2\,\Phi + 2\cos.\, 4\,\Phi + 2\cos.\, 6\,\Phi) \text{ et}$$
$$\partial y = 2\,\partial\Phi \cos.\,\Phi\,(1 - 2\cos.\, 2\,\Phi + 2\cos.\, 4\,\Phi - 2\cos.\, 6\,\Phi).$$

Quodsi iam has reductiones vlterius prosequamur, nanciscemur tandem

$$\partial x = 2\,\partial\Phi \sin.\, 7\,\Phi, \text{ ideoque } x = -\tfrac{2}{7}\cos.\, 7\,\Phi,$$

eodemque modo

$$\partial y = 2\,\partial\Phi \cos.\, 7\,\Phi, \text{ ergo } y = \tfrac{2}{7}\sin.\, 7\,\Phi,$$

vnde iterum colligitur $xx + yy = \frac{4}{49}$, ideoque pro circulo.

§. 17. Si pro eodem casu tertio ponatur $n = -1$, fiet

$$\partial x = \frac{1 + 2v}{\sqrt{2(1+v)}}\,\partial v \text{ et } \partial y = \frac{1 - 2v}{\sqrt{2(1-v)}}\,\partial v.$$

Statuamus igitur $v = \cos.\, 2\,\Phi$, eritque $\sqrt{2(1+v)} = 2\cos.\,\Phi$ et $\sqrt{2(1-v)} = 2\sin.\,\Phi$ et $\partial v = -2\,\partial\Phi \sin.\, 2\,\Phi$; quamobrem

obrem habebitur

$$\partial x = -\frac{(1 + 2\,cos.\,2\,\Phi)}{2\,cos.\,\Phi}\partial\Phi\,sin.\,2\,\Phi = -2\,\partial\Phi\,sin.\,\Phi(1 + 2\,cos.\,2\,\Phi) \text{ et}$$

$$\partial y = -\frac{(1 - 2\,cos.\,2\,\Phi)}{2\,sin.\,\Phi}\partial\Phi\,sin.\,2\,\Phi = -2\,\partial\Phi\,cos.\,\Phi(1 - 2\,cos.\,2\,\Phi),$$

quae formulae porro reducuntur ad has:

$$\partial x = -2\,\partial\Phi\,sin.\,3\,\Phi \text{ et } \partial y = -2\,\partial\Phi\,cos.\,3\,\Phi,$$

hincque integrando fiet

$$x = \tfrac{2}{3}\,cos.\,3\,\Phi \text{ et } y = -\tfrac{2}{3}\,sin.\,3\,\Phi,$$

vnde colligitur $xx + yy = \frac{4}{9}$, quae est aequatio pro circulo, cuius radius $= \frac{2}{3}$.

§. 18. Quodsi quis simili modo casum quartum euolvere voluerit, ponendo siue $n = +1$ siue $n = -1$, itidem reperiet curuas satisfacientes pariter ad circulum reduci. Hinc igitur ansam arripimus problema nostrum alio modo resoluendi, dum scilicet in formulam, qua arcus curuae exprimi debet, statim sinum cosinumue cuiuspiam anguli introducemus.

Alia problematis solutio ex calculo angulorum petita.

§. 19. Cum elementum arcus curuarum quaesitarum debeat esse

$$\partial s = \frac{\partial v\sqrt{[1 + (nn - 1)vv]}}{\sqrt{(1 - vv)}},$$

ponamus statim $v = sin.\,\Phi$, vt fiat $\frac{\partial v}{\sqrt{(1 - vv)}} = \partial\Phi$, eritque

$$\partial s = \partial\Phi\sqrt{(cos.\,\Phi^2 + nn\,sin.\,\Phi^2)},$$

vnde statim manifestum est capi posse

$$\partial x = \partial\Phi\,cos.\,\Phi \text{ et } \partial y = n\,\partial\Phi\,sin.\,\Phi,$$

vnde fit $x = sin.\,\Phi$ et $y = -n\,cos.\,\Phi$, ideoque $nx = n\,sin.\,\Phi$, vnde colligitur $nnxx + yy = nn$, quae est ipsa aequatio pro Ellipsi,

Ellipſi, cuius arcus menſuram reliquarum curuarum conſtituere debent.

§. 20. Ex hac autem ſolutione infinitas alias curuas quaeſito pariter ſatisfacientes deriuare poſſumus, ponendo

$$\partial x = \partial \Phi \text{ coſ. } \Phi \text{ coſ. } \omega - n \partial \Phi \text{ ſin. } \Phi \text{ ſin. } \omega \text{ et}$$
$$\partial y = \partial \Phi \text{ coſ. } \Phi \text{ ſin. } \omega + n \partial \Phi \text{ ſin. } \Phi \text{ coſ. } \omega,$$

ſic enim euadet

$$\partial x + \partial y^2 = \partial \Phi^2 \text{ coſ. } \Phi^2 + n n \partial \Phi^2 \text{ ſin. } \Phi^2 = \partial s^2.$$

Tantum igitur ſupereſt vt iſtae duae formulae differentiales integrabiles reddantur, quod manifeſto in genere eueniet, ſumendo $\omega = \lambda \Phi$, tum enim per reductiones ſatis cognitas nanciſcemur:

$$\frac{2\partial x}{\partial \Phi} = (n+1) \text{ coſ. } (\lambda+1)\Phi - (n-1) \text{ coſ. } (\lambda-1)\Phi \text{ et}$$
$$\frac{2\partial y}{\partial \Phi} = (n+1) \text{ ſin. } (\lambda+1)\Phi - (n-1) \text{ ſin. } (\lambda-1)\Phi,$$

atque hinc integrando impetrabimus:

$$2x = \frac{n+1}{\lambda+1} \text{ ſin. } (\lambda+1)\Phi - \frac{(n-1)}{(\lambda-1)} \text{ ſin. } (\lambda-1)\Phi,$$
$$2y = \frac{n+1}{\lambda+1} \text{ coſ. } (\lambda+1)\Phi - \frac{n-1}{\lambda-1} \text{ coſ. } (\lambda-1)\Phi,$$

quae ergo ambae formulae ſemper ſunt algebraicae, ſolo caſu excepto vbi $\lambda = \pm 1$. Caeterum quando $n = \pm 1$, curuae reſultantes manifeſto abeunt in circulum, quicunque valor ipſi λ tribuatur.

§. 21. Haec ſolutio non ſolum eſt admodum ſuccincta, ſed etiam multo latius patet quam praecedens, quandoquidem praecedentes caſus ex hac ſolutione deducuntur, ſumendo $\lambda = \pm \frac{1}{2}$; vel $\lambda = \pm \frac{3}{2}$; vel $\lambda = \pm \frac{5}{2}$; vel $\lambda = \pm \frac{7}{2}$. Quatenus igitur hic pro λ numeros integros accipere licet, vel

vel etiam quascunque alias fractiones, eatenus haec solutio longe alias suppeditat lineas curuas, quae ex priori solutione nullo modo deduci possunt. Euoluamus igitur aliquot exempla.

Exemplum 1.

§. 22. Quia pro λ vnitatem assumere non licet, ponamus statim $\lambda = 2$, atque habebimus

$$x = +\frac{n+1}{6}\,\text{sin.}\,3\,\phi - \frac{(n-1)}{2}\,\text{sin.}\,\phi \text{ et}$$

$$y = -\frac{(n+1)}{6}\,\text{cos.}\,3\,\phi + \frac{(n-1)}{2}\,\text{cos.}\,\phi,$$

hinc iam colligimus

$$xx + yy = \frac{(n+1)^2}{36} + \frac{(n-1)^2}{4} - \frac{(nn-1)}{6}\,\text{cos.}\,2\,\phi,$$

ex qua aequatione angulus ϕ haud difficulter per x et y determinatur, qui deinceps in alterutra substitutus praebebit aequationem inter x et y.

Exempum 2.

Sumamus etiam $\lambda = \frac{1}{2}$, erit

$$x = +\frac{(n+1)}{3}\,\text{sin.}\,\tfrac{3}{2}\,\phi + (n-1)\,\text{sin.}\,\tfrac{1}{2}\,\phi \text{ et}$$

$$y = -\frac{(n+1)}{3}\,\text{cos.}\,\tfrac{3}{2}\,\phi + (n-1)\,\text{cos.}\,\tfrac{1}{2}\,\phi.$$

Facile autem patet hoc exemplum cum casu supra §. 4. tractato congruere.

Scholion.

§. 23. Haec igitur solutio praecedentem maxime supereminet, cum non solum infinities plures curuas in se complectatur, sed etiam valores pro coordinatis x et y inuenti tam simpliciter exprimantur, vt duobus tantum terminis constent, cum sit

$$2x = \frac{n+1}{\lambda+1} \text{ fin. } (\lambda+1)\Phi - \frac{(n-1)}{\lambda-1} \text{ fin. } (\lambda-1)\Phi \text{ et}$$
$$-2y = \frac{n+1}{\lambda+1} \text{ cof. } (\lambda+1)\Phi - \frac{(n-1)}{\lambda-1} \text{ cof. } (\lambda-1)\Phi.$$

Ex qua forma coordinatarum colligitur istas curuas omnes affines esse Epicycloidibus, et generari posse ex prouolutione circuli super circulo, dum scilicet stylus describens non in ipsa peripheria circuli mobilis assumitur. Interim tamen ne haec quidem solutio pro generali haberi potest: namque innumerabiles alias curuas satisfacientes assignare licet, quae ne in hac solutione continentur, quam inuentionem hic subiungamus.

Adhuc alia solutio
problematis propositi.

§. 24. Maneat vt ante $v = \text{fin. } \Phi$, et cum hinc fiat

$$\partial s = \partial\Phi \sqrt{(\text{cof. } \Phi^2 + nn \text{ fin. } \Phi^2)},$$

scribamus $1 - \text{fin. } \Phi^2$ loco $\text{cof. } \Phi^2$, eritque

$$\partial s = \partial\Phi \sqrt{(1 + (nn-1) \text{ fin. } \Phi^2)}.$$

Faciamus autem breuitatis gratia $nn - 1 = mm$, atque huic conditioni statim satisfieret, sumendo $\partial x = \partial\Phi$ et $\partial y = m\partial\Phi \text{ fin. } \Phi$; hinc autem ob $x = \Phi$ prodiret curua transcendens, quod tamen non impedit, quo minus infinitae curuae algebraicae hinc deduci queant. Statuamus enim

$$\partial x = \partial\Phi \text{ cof. } \lambda\Phi - m\partial\Phi \text{ fin. } \Phi \text{ fin. } \lambda\Phi \text{ et}$$
$$\partial y = \partial\Phi \text{ fin. } \lambda\Phi + m\partial\Phi \text{ fin. } \Phi \text{ cof. } \lambda\Phi,$$

atque hinc prodit

$$\partial x^2 + \partial y^2 = \partial\Phi^2 (1 + mm \text{ fin. } \Phi^2).$$

Nunc igitur membra posteriora more solito euoluantur et obtinebitur

$2\partial x$

$$\frac{2\partial x}{\partial \Phi} = 2 \text{ cos.} \lambda \Phi - m \text{ cos.} (\lambda - 1) \Phi + m \text{ cos.} (\lambda + 1) \Phi \text{ et}$$
$$\frac{2\partial y}{\partial \Phi} = 2 \text{ sin.} \lambda \Phi + m \text{ sin.} (\lambda + 1) \Phi - m \text{ sin.} (\lambda - 1) \Phi,$$

vnde sumtis integralibus erit

$$2x = \frac{2}{\lambda} \text{ sin.} \lambda \Phi - \frac{m}{\lambda - 1} \text{ sin.} (\lambda - 1) \Phi + \frac{m}{\lambda + 1} \text{ sin.} (\lambda + 1) \Phi \text{ et}$$
$$-2y = \frac{2}{\lambda} \text{ cos.} \lambda \Phi - \frac{m}{\lambda - 1} \text{ cos.} (\lambda - 1) \Phi + \frac{m}{\lambda + 1} \text{ cos.} (\lambda + 1) \Phi,$$

quae ergo formulae etiam sunt algebraicae, solo casu $\lambda = \pm 1$ exceptò. Perspicuum autem est, hos valores penitus esse diuersos a praecedentibus, propterea quod terna membra involuunt.

§. 25. Praeterea vero hic manifesto assumsimus esse $nn > 1$, ita vt haec solutio extendi nequeat ad casus quibus $nn < 1$, cum prior solutio pro omnibus valoribus numeri n valeat; interim tamen etiam haec solutio adaptari potest ad casus quibus $nn < 1$, ita vt sit

$$\partial s = \partial \Phi \sqrt{(1 - (1 - nn) \text{ sin.} \Phi)},$$

quae expressio, posito $\text{sin.} \Phi^2 = 1 - \text{cos.} \Phi^2$, abit in hanc:

$$\partial s = \partial \Phi \sqrt{(nn + (1 - nn) \text{ cos.} \Phi^2)},$$

et posito breuitatis gratia $1 - nn = kk$, fiet

$$\partial s = \partial \Phi \sqrt{(nn + kk \text{ cos.} \Phi^2)},$$

vbi notetur esse $nn + kk = 1$.

§. 26. Huic ergo formulae statim satisfiet ponendo

$$\partial x = n \partial \Phi \text{ et } \partial y = k \partial \Phi \text{ cos.} \Phi,$$

vnde autem curua resultaret transcendens, quare vt curuas algebraicas eruamus, statuamus vt ante

$$\partial x = n\,\partial\Phi \text{ sin. } \lambda\Phi + k\,\partial\Phi \text{ cos. } \Phi \text{ cos. } \lambda\Phi \text{ et}$$
$$\partial y = n\,\partial\Phi \text{ cos. } \lambda\Phi - k\,\partial\Phi \text{ cos. } \Phi \text{ sin. } \lambda\Phi,$$

vnde ambo valores prodibunt algebraici, dum ne sit $\lambda = \pm 1$.

§. 27. Reductione igitur solita in vsum vocata nanciscemur has formulas:

$$\frac{2\,\partial x}{\partial\Phi} = 2n \text{ sin. } \lambda\Phi + k \text{ cos. } (\lambda+1)\Phi + k \text{ cos. } (\lambda-1)\Phi \text{ et}$$
$$\frac{2\,\partial y}{\partial\Phi} = 2n \text{ cos. } \lambda\Phi - k \text{ sin. } (\lambda+1)\Phi - k \text{ sin. } (\lambda-1)\Phi,$$

vnde integrando deducimus:

$$2x = -\frac{2n}{\lambda} \text{ cos. } \lambda\Phi + \frac{k}{\lambda+1} \text{ sin. } (\lambda+1)\Phi + \frac{k}{\lambda-1} \text{ sin. } (\lambda-1)\Phi \text{ et}$$
$$2y = +\frac{2n}{\lambda} \text{ sin. } \lambda\Phi + \frac{k}{\lambda+1} \text{ cos. } (\lambda+1)\Phi + \frac{k}{\lambda-1} \text{ cos. } (\lambda-1)\Phi,$$

quae curuae itidem maxime discrepant a praecedente solutione.

Scholion.

§. 28. Quamuis autem hae solutiones infinities infinitas suppeditent lineas curuas algebraicas problemati nostro satisfacientes, tamen vix affirmari posse videtur, in his formulis omnes plane solutiones contineri: tam parum enim adhuc istud argumentum est elaboratum, vt vix quicquam certi in hoc negotio statui posse videatur; sed potius quaestio generalis, qua curuae algebraicae desiderantur, quarum longitudo per datam formulam integralem $\int V\,\partial v$ exprimatur, vbi V denotet functionem quamcunque ipsius v, tantopere etiamnunc tenebris obuoluta deprehenditur, vt solutionem pauciſſimis tantum casibus euoluere liceat, quemadmodum nobis solutio successit pro arcubus Parabolicis et Ellipticis. Si enim talis quaestio circa arcus Hyperbolicos proponatur, fateri cogor, nullo adhuc modo me vel vnicam saltem curuam algebraicam eruere potuisse, cuius singuli arcus per formulam:

$\int$

$$\int \frac{\partial v}{vv} \sqrt{(1 + v^4)},$$

exprimerentur. Si enim v denotet abſciſſam Hyperbolae aequilaterae inter aſymptotos, applicata erit $y = \frac{1}{v}$, ideoque $\partial y = -\frac{\partial v}{vv}$, vnde elementum arcus reſultat

$$\partial s = \frac{\partial v}{vv} \sqrt{(1 + v^4)}.$$

Sin autem aequationem generalem pro Hyperbola aſſumere velimus, qua eſt $y = n \sqrt{(1 + vv)}$, elementum arcus inde nascitur $\partial s = \frac{\partial v \sqrt{(1 + (nn + 1) vv)}}{\sqrt{(1 + vv)}}$, quae formula ita comparata eſt, vt omnia artificia, quae quidem mihi detegere licuit, penitus fruſtretur. Quin etiam hic nullo modo calculus angulorum cum vllo ſucceſſu in ſubſidium vocari poteſt. Neutiquam autem etiamnunc aſſeuerare auſim, praeter Hyperbolam nullas alias dari curuas algebraicas, quarum longitudinem per arcus Hyperbolicos metiri liceat, quemadmodum hoc de circulo audacter pronunciare non dubitaui. Hac igitur ſpeculatione ampliſſimus campus aperitur, in quo Geometrae non ſine inſigni fructu et Analyſeos vlteriori perfectione elaborare poterunt.