OBSERUATIONES GENERALES

CIRCA SERIES, QVARVM TERMINI SECVNDVM SINVS VEL COSINVS ANGVLORVM MVLTIPLORVM PROGREDIVNTVR.

Auctore

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§. 1.

Quodsi huius seriei $A + Bx + Cxx + Dx^3 +$ etc. summatio fuerit cognita, ita vt, quicunque valor litterae x tribuatur, eius summa assignari queat, tum etiam semper tam summa huius seriei:

$$A + B \cos. \phi + C \cos. 2\phi + D \cos. 3\phi + \text{etc.}$$

quam huius:

$$B \sin. \phi + C \sin. 2\phi + D \sin. 3\phi + E \sin. 4\phi + \text{etc.}$$

exhiberi poterit. Cum enim summa primae seriei exprimatur per certam quandam functionem ipsius x, quam hoc charactere $\Delta : x$ designemus, ita vt sit

$$\Delta : x = A + Bx + Cxx + Dx^3 + \text{etc.}$$

si loco x scribamus tam

$$\cos. \phi + \sqrt{-1} \sin. \phi, \text{ quam } \cos. \phi - \sqrt{-1} \sin. \phi$$

sum-

summa seriesrum inde resultantium erit

$$2A + 2B\,\text{cos}.\,\Phi + 2C\,\text{cos}.\,2\Phi + 2D\,\text{cos}.\,3\Phi + 2E\,\text{cos}.\,4\Phi + \text{etc.}$$

cuius ergo summa erit

$$\Delta : (\text{cos}.\,\Phi + \sqrt{-1}\,\text{sin}.\,\Phi) + \Delta : (\text{cos}.\,\Phi - \sqrt{-1}\,\text{sin}.\,\Phi);$$

sin autem posteriorem a priore subtrahamus, prodibit ista series:

$$2B\sqrt{-1}\,\text{sin}.\,\Phi + 2C\sqrt{-1}\,\text{sin}.\,2\Phi + 2D\sqrt{-1}\,\text{sin}.\,3\Phi$$
$$+ 2E\sqrt{-1}\,\text{sin}.\,4\Phi\ \text{etc.}$$

cuius ergo summa erit

$$\Delta : (\text{cos}.\,\Phi + \sqrt{-1}\,\text{sin}.\,\Phi) - \Delta : (\text{cos}.\,\Phi - \sqrt{-1}\,\text{sin}.\,\Phi).$$

§. 2. Quo expressiones has commodiores reddamus, statuamus breuitatis gratia

$$\text{cos}.\,\Phi + \sqrt{-1}\,\text{sin}.\,\Phi = p;\ \text{cos}.\,\Phi - \sqrt{-1}\,\text{sin}.\,\Phi = q$$

eritque, vti in vulgus notum est, $pq = 1$ ideoque $q = \frac{1}{p}$; tum vero erit

$$\text{cos}.\,\Phi = \frac{p+q}{2};\ \text{cos}.\,2\Phi = \frac{pp+qq}{2};\ \text{cos}.\,3\Phi = \frac{p^3+q^3}{2};$$
$$\text{cos}.\,4\Phi = \frac{p^4+q^4}{2};\ \text{etc.}$$

Praeterea vero pro sinibus habebitur

$$\text{sin}.\,\Phi = \frac{p-q}{2\sqrt{-1}};\ \text{sin}.\,2\Phi = \frac{pp-qq}{2\sqrt{-1}};\ \text{sin}.\,3\Phi = \frac{p^3-q^3}{2\sqrt{-1}};\ \text{etc.}$$

quibus constitutis nanciscimur has duas summationes:

$$A\,\text{cos}.\,0\Phi + B\,\text{cos}.\,\Phi + C\,\text{cos}.\,2\Phi + D\,\text{cos}.\,3\Phi + E\,\text{cos}.\,4\Phi + \text{etc.}$$
$$= \frac{\Delta:p + \Delta:q}{2};\ \text{et}$$

$$A\,\text{sin}.\,0\Phi + B\,\text{sin}.\,\Phi + C\,\text{sin}.\,2\Phi + D\,\text{sin}.\,3\Phi + \text{etc.} = \frac{\Delta:p - \Delta:q}{2\sqrt{-1}}$$

§. 3. Sumamus nunc pro serie principali potestatem quamcunque Binomii euolutam, quae est

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1.\ 2}xx + \frac{n(n-1)(n-2)}{1.\ 2.\ 3}x^3 + \text{etc.}$$

ita

ita vt hoc casu sit $\Delta : x = (1 + x)^n$; tum vero, vt hanc expressionem contrahamus, designemus singulos coefficientes, vt iam aliquoties fecimus, his characteribus: $\binom{n}{0}$, $\binom{n}{1}$, $\binom{n}{2}$, $\binom{n}{3}$, $\binom{n}{4}$, ita vt sit

$$\binom{n}{0} = 1,$$
$$\binom{n}{1} = n,$$
$$\binom{n}{2} = \frac{n}{1} \cdot \frac{n-1}{2},$$
$$\binom{n}{3} = \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3},$$
etc.

vbi obseruasse iuuabit esse in genere $\binom{n}{i} = \binom{n}{n-i}$, ideoque $\binom{n}{n} = \binom{n}{0} = 1$. Praeterea vero euidens est, quoties fuerit i vel numerus negatiuus, vel positiuus, maior quam n, tum semper esse $\binom{n}{i} = 0$, siquidem n fuerit numerus integer. His ergo obseruatis habebimus hanc summationem principalem:

$$(1 + x)^n = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \binom{n}{3} x^3 + \binom{n}{4} x^4 + \text{etc.}$$

vnde ergo per praecepta modo tradita deriuabimus binas sequentes summationes:

$$\binom{n}{0} \operatorname{cos.} 0\Phi + \binom{n}{1} \operatorname{cos.} \Phi + \binom{n}{2} \operatorname{cos.} 2\Phi + \binom{n}{3} \operatorname{cos.} 3\Phi + \text{etc.} = \frac{(1+p)^n + (1+q)^n}{2} \text{ et}$$

$$\binom{n}{0} \operatorname{sin.} 0\Phi + \binom{n}{1} \operatorname{sin.} \Phi + \binom{n}{2} \operatorname{sin.} 2\Phi + \binom{n}{3} \operatorname{sin.} 3\Phi + \text{etc.} = \frac{(1+p)^n - (1+q)^n}{2\sqrt{-1}}.$$

Quouis autem casu, quamquam formulae pro p et q assumptae sunt imaginariae, tamen semper istas formulas ad valores reales reuocare licebit, quemadmodum in sequentibus binis problematibus sumus ostensuri.

Problema 1.

Proposita hac serie cosinuum:

$$1 + \frac{n}{1} \operatorname{cos.} \Phi + \frac{n}{1} \cdot \frac{n-1}{2} \operatorname{cos.} 2\Phi + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \operatorname{cos.} 3\Phi + \text{etc.} = s,$$

 ita

ita vt per characteres ſtabilitos ſit

$$s = \binom{n}{0}\operatorname{cof.} 0\,\Phi + \binom{n}{1}\operatorname{cof.}\Phi + \binom{n}{2}\operatorname{cof.} 2\,\Phi + \binom{n}{3}\operatorname{cof.} 3\,\Phi + \text{etc.}$$

eius ſummam realiter exprimere.

Solutio.

§. 4. Cum igitur ſit, vt modo vidimus

$$s = \frac{(1+p)^n + (1+q)^n}{2},$$

exiſtente $p = \operatorname{cof.}\Phi + \sqrt{-1}\,\operatorname{fin.}\Phi$ et $q = \operatorname{cof.}\Phi - \sqrt{-1}\,\operatorname{fin.}\Phi$, totum negotium huc redit, vt iſta expreſſio pro s exhibita ab imaginariis liberetur; euidens enim eſt, ſi formulae $(1+p)^n$ et $(1+q)^n$ actu euoluantur, tum imaginaria ſe ſponte eſſe deſtructura, quandoquidem hinc ipſa ſeries ſummanda exoritur; quam ob rem in aliam reſolutionem nobis erit inquirendum, vt ſine euolutione adhibita imaginaria e medio tollantur, id quod ſequenti modo fieri poterit.

§. 5. Cum ſit $p\,q = 1$, formula $1+p$ ita exprimi poterit, vt ſit $1+p = (\sqrt{p} + \sqrt{q})\sqrt{p}$, ſimilique modo erit $1+q = (\sqrt{p} + \sqrt{q})\sqrt{q}$, hisque valoribus introductis prodibit noſtra ſumma

$$s = \tfrac{1}{2}(\sqrt{p} + \sqrt{q})^n (p^{\frac{n}{2}} + q^{\frac{n}{2}}).$$

Cum iam in genere ſit $p^\alpha + q^\alpha = 2\operatorname{cof.}\alpha\,\Phi$, erit

$$p^{\frac{1}{2}} + q^{\frac{1}{2}} = 2\operatorname{cof.}\tfrac{1}{2}\Phi \text{ et } p^{\frac{n}{2}} + q^{\frac{n}{2}} = 2\operatorname{cof.}\tfrac{1}{2}n\,\Phi,$$

quibus valoribus ſubſtitutis ſumma quaeſita iam realiter ſequenti modo exprimetur: $s = 2^n \operatorname{cof.}\tfrac{1}{2}\Phi^n \operatorname{cof.}\tfrac{1}{2}n\,\Phi$.

§. 6. Hoc igitur pacto ſummationem maxime memorabilem ſumus adepti, quae ita ſe habet, vt ſemper ſit

1 +

$$1 + \frac{n}{1} \operatorname{cos}. \Phi + \frac{n}{1} \cdot \frac{n-1}{2} \operatorname{cos}. 2\Phi + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \operatorname{cos}. 3\Phi + \text{etc.}$$
$$= 2^n \operatorname{cos}. \tfrac{1}{2}\Phi^n \operatorname{cos}. \tfrac{1}{2} n \Phi,$$

quae ſemper veritati eſt conſentanea, quicunque numeri pro n ſubſtituantur, ſiue integri ſiue fracti, ſiue etiam negatiui. Operae ergo pretium erit ex quouis genere caſus ſimpliciores ob oculos exponere.

Euolutio caſuum, quibus exponens n eſt numerus integer poſitiuus.

§. 7. Conſideremus caſus ſequentes:

1°. Sit $n = 0$, et ipſa ſeries in vnitatem coaleſcit, ſumma autem erit $= 1$.

2°. Sit $n = 1$, et ſeries abibit in $1 + \operatorname{cos}. \Phi$; ſumma autem inuenta praebet $2 \operatorname{cos}. \tfrac{1}{2}\Phi^2$. Conſtat autem eſſe $2 \operatorname{cos}. \tfrac{1}{2}\Phi^2 = 1 + \operatorname{cos}. \Phi$.

3°. Sit $n = 2$, et ſeries abit in $1 + 2 \operatorname{cos}. \Phi + \operatorname{cos}. 2\Phi$; ſumma autem oritur $= 4 \operatorname{cos}. \tfrac{1}{2}\Phi^2 \operatorname{cos}. \Phi$. Modo autem vidimus eſſe $2 \operatorname{cos}. \tfrac{1}{2}\Phi^2 = 1 + \operatorname{cos}. \Phi$, quae forma per $2 \operatorname{cos}. \Phi$ multiplicata producit

$$2 \operatorname{cos}. \Phi \times 2 \operatorname{cos}. \tfrac{1}{2}\Phi^2 = 1 + 2 \operatorname{cos}. \Phi + \operatorname{cos}. 2\Phi.$$

4°. Sit nunc $n = 3$, et ſeries naſcitur haec: $1 + 3 \operatorname{cos}. \Phi + 3 \operatorname{cos}. 2\Phi + \operatorname{cos}. 3\Phi$, cuius ſumma eſt $= 8 \operatorname{cos}. \tfrac{1}{2}\Phi^3 \operatorname{cos}. \tfrac{3}{2}\Phi$, quae formula per reductiones ſatis cognitas ipſam ſeriem producit.

5°. Sit nunc $n = 4$, et ſeries abit in

$$1 + 4 \operatorname{cos}. \Phi + 6 \operatorname{cos}. 2\Phi + 4 \operatorname{cos}. 3\Phi + \operatorname{cos}. 4\Phi$$

cuius ergo ſumma erit $2^4 \operatorname{cos}. \tfrac{1}{2}\Phi^4 \operatorname{cos}. 2\Phi$, cuius veritas etiam non difficulter oſtenditur. Sicque ſemper veritatem per reductiones cognitas oſtendere licebit.

Euolutio casuum,

quibus pro n numerus integer negatiuus accipitur.

§. 8. Statuamus 1°. $n = -1$, atque hinc nascetur sequens series infinita:

$$1 - \cos.\phi + \cos.2\phi - \cos.3\phi + \cos.4\phi - \cos.5\phi \text{ etc.}$$

in infinitum, cuius ergo summa per nostram seriem generalem erit $\frac{\cos.\frac{1}{2}\phi}{2\cos.\frac{1}{2}\phi} = \frac{1}{2}$, quod quidem iam dudum a Geometris est obseruatum. Quodsi enim haec series, cuius summa tantisper ponatur $= s$, ducatur in $2\cos.\frac{1}{2}\phi$, reperietur per reductiones notissimas

$$2s\cos.\tfrac{1}{2}\phi = \left\{\begin{matrix} 2\cos.\frac{1}{2}\phi - \cos.\frac{3}{2}\phi + \cos.\frac{5}{2}\phi - \cos.\frac{7}{2}\phi + \cos.\frac{9}{2}\phi \\ -\cos.\frac{1}{2}\phi + \cos.\frac{3}{2}\phi - \cos.\frac{5}{2}\phi + \cos.\frac{7}{2}\phi - \cos.\frac{9}{2}\phi \end{matrix}\right\} \text{etc.}$$

quod manifesto redit ad $2s\cos.\frac{1}{2}\phi = \cos.\frac{1}{2}\phi$ ideoque $s = \frac{1}{2}$.

§. 9. Statuamus nunc $n = -2$, et series orietur sequens:

$$1 - 2\cos.\phi + 3\cos.2\phi - 4\cos.3\phi + 5\cos.4\phi - 6\cos.5\phi \text{ etc.}$$

cuius ergo summa erit $= \frac{\cos.\phi}{4\cos.\frac{1}{2}\phi^2}$, cuius veritas etiamnunc sequenti modo ostendi potest. Posita seriei summa $= s$, erit

$$2s\cos.\tfrac{1}{2}\phi = \left\{\begin{matrix} 2\cos.\frac{1}{2}\phi - 2\cos.\frac{3}{2}\phi + 3\cos.\frac{5}{2}\phi - 4\cos.\frac{7}{2}\phi \\ -2\cos.\frac{1}{2}\phi + 3\cos.\frac{3}{2}\phi - 4\cos.\frac{5}{2}\phi + 5\cos.\frac{7}{2}\phi \end{matrix}\right\} \text{etc.}$$

qui valor coalescit in sequentem seriem:

$$2s\cos.\tfrac{1}{2}\phi = \cos.\tfrac{3}{2}\phi - \cos.\tfrac{5}{2}\phi + \cos.\tfrac{7}{2}\phi - \cos.\tfrac{9}{2}\phi \text{ etc.}$$

Multiplicetur denuo per $2\cos.\frac{1}{2}\phi$, ac prodibit

$$4s\cos.\tfrac{1}{2}\phi^2 = \left\{\begin{matrix} \cos.\phi + \cos.2\phi - \cos.3\phi + \cos.4\phi + \cos.5\phi \\ -\cos.2\phi + \cos.3\phi - \cos.4\phi - \cos.5\phi \end{matrix}\right\} \text{etc.} = \cos.\phi$$

ideo-

ideoque $s = \frac{\cos. \Phi}{4 \cos. \frac{1}{2} \Phi^2}$, vti inuenimus, siue erit etiam

$$s = \frac{\cos. \Phi}{2(1 + \cos. \Phi)}.$$

§. 10. Sit nunc $n = -3$, orieturque haec series infinita:

$$1 - 3 \cos. \Phi + 6 \cos. 2\Phi - 10 \cos. 3\Phi + 15 \cos. 4\Phi - 21 \cos. 5\Phi \text{ etc.}$$

cuius ergo summa erit $= \frac{\cos. \frac{3}{2} \Phi}{8 \cos. \frac{1}{2} \Phi^3}$. Haec autem expressio porro reducitur ad hanc

$$s = \tfrac{1}{2} - \frac{3}{8 \cos. \frac{1}{2} \Phi^2} = \tfrac{1}{2} - \frac{3}{4(1 + \cos. \Phi)}$$

ita vt quoque sit $s = \frac{-1 + 2 \cos. \Phi}{4(1 + \cos. \Phi)}$.

§. 11. Simili modo sequentes adipiscemur summationes:

$$1 - 4 \cos. \Phi + 10 \cos. 2\Phi - 20 \cos. 3\Phi \text{ etc.} = \frac{\cos. 2\Phi}{16 \cos. \frac{1}{2} \Phi^4},$$

$$1 - 5 \cos. \Phi + 15 \cos. 2\Phi - 35 \cos. 3\Phi + 70 \cos. 4\Phi \text{ etc.} = \frac{\cos. \frac{5}{2} \Phi}{32 \cos. \frac{1}{2} \Phi^5},$$

$$1 - 6 \cos. \Phi + 21 \cos. 2\Phi - 56 \cos. 3\Phi + 126 \cos. 4\Phi \text{ etc.} = \frac{\cos. 3\Phi}{64 \cos. \frac{1}{2} \Phi^6},$$

$$1 - 7 \cos. \Phi + 28 \cos. 2\Phi - 84 \cos. 3\Phi + 210 \cos. 4\Phi \text{ etc.} = \frac{\cos. \frac{7}{2} \Phi}{128 \cos. \frac{1}{2} \Phi^7}.$$

Euolutio casus
quo $n = \frac{1}{2}$.

§. 12. Hinc ergo formabitur sequens series infinita:

$$1 + \tfrac{1}{2} \cos. \Phi - \tfrac{1.1}{2.4} \cos. 2\Phi + \tfrac{1.1.3}{2.4.6} \cos. 3\Phi - \tfrac{1.1.3.5}{2.4.6.8} \cos. 4\Phi \text{ etc.}$$

 cuius

cuius ergo summa erit $= \cos.\frac{1}{4}\Phi\sqrt{2\cos.\frac{1}{2}\Phi}$, cuius veritatem haud facile erit aliunde comprobare; certis autem casibus manifesto in oculos incurrit. Veluti si fuerit $\Phi = 0$, habebitur

$$1 + \frac{1}{2} - \frac{1.1}{2.4} + \frac{1.1.3}{2.4.6} - \frac{1.1.3.5}{2.4.6.8}\text{ etc.} = \sqrt{2};$$

series enim manifesto oritur ex euolutione $(1+1)^{\frac{1}{2}} = \sqrt{2}$. Faciamus nunc $\Phi = 180^\circ$, vt sit $\frac{1}{2}\Phi = 90^\circ$, et series erit

$$1 - \frac{1}{2} - \frac{1.1}{2.4} - \frac{1.1.3}{2.4.6} - \frac{1.1.3.5}{2.4.6.8}\text{ etc.} = 0$$

quod etiam inde euidens est, quia haec series nascitur ex forma $(1-1)^{\frac{1}{2}}$. Sit etiam $\Phi = 90^\circ$ et series inde nata erit

$$1 + \frac{1.1}{2.4} - \frac{1.1.3.5}{2.4.6.8} + \frac{1.1.3.5.7.9}{2.4.6.8.10.12}\text{ etc.} = \cos.\,22^\circ, 30'\sqrt[4]{2}.$$

Est vero

$$\cos.\,22^\circ, 30^\circ = \sqrt{\frac{(1+\cos.45^\circ)}{2}} = \sqrt{\frac{1+\sqrt{2}}{2\sqrt{2}}}$$

vnde summa concluditur $= \sqrt{\frac{(1+\sqrt{2})}{2}}$, sicque habetur haec summatio maxime memoratu digna:

$$1 + \frac{1.1}{2.4} - \frac{1.1.3.5}{2.4.6.8} + \frac{1.1.3.5.7.9}{2.4.6.8.10.12}\text{ etc.} = \sqrt{\frac{1+\sqrt{2}}{2}}.$$

Sumamus etiam $\Phi = 60^\circ$, et orietur ista series:

$$1 + \frac{1}{2}.\frac{1}{2} + \frac{1.1}{2.4}.\frac{1}{2} - \frac{1.1.3}{2.4.6}1 + \frac{1.1.3.5}{2.4.6.8}.\frac{1}{2} + \frac{1.1.3.5.7}{2.4.6.8.10}.\frac{1}{2} + \frac{1.1.3.5.7.9}{2.4.6.8.10.12}.1\text{ etc.}$$

cuius seriei ergo summa erit $\cos.\,15^\circ\sqrt[4]{3}$. Cum igitur sit

$$\cos.\,15^\circ = \sqrt{\frac{1+\cos.30^\circ}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}},$$

seriei summa erit $\frac{1}{2}\sqrt{(3+2\sqrt{3})}$.

Euolutio casus
quo $n = -\frac{1}{2}$.

§. 13. Hinc ergo sequens formabitur series infinita:

$$1 - \frac{1}{2}\cos.\Phi + \frac{1.3}{2.4}\cos.2\Phi - \frac{1.3.5}{2.4.6}\cos.3\Phi + \frac{1.3.5.7}{2.4.6.8}\cos.4\Phi\text{ etc.}$$

cuius

cuius ergo summa erit $= \frac{\cos.\frac{1}{4}\Phi}{\sqrt{2}\cos.\frac{1}{2}\Phi}$. Hinc si fuerit $\Phi = 0$, oritur haec summatio:

$$1 - \frac{1}{2} + \frac{1.3}{2.4} - \frac{1.3.5}{2.4.6} + \frac{1.3.5.7}{2.4.6.8} \text{ etc.} = \frac{1}{\sqrt{2}}.$$

Nascitur enim haec series ex forma $(1+1)^{-\frac{1}{2}}$. Sit nunc $\Phi = 180°$, et series resultans erit

$$1 + \frac{1}{2} + \frac{1.3}{2.4} + \frac{1.3.5}{2.4.6} + \frac{1.3.5.7}{2.4.6.8} + \text{etc.} = \infty.$$

Oritur enim haec series ex euolutione $(1-1)^{-\frac{1}{2}}$. Sumamus etiam $\Phi = 90°$ et series erit

$$1 - \frac{1.3}{2.4} + \frac{1.3.5.7}{2.4.6.8} - \frac{1.3.5.7.9.11}{2.4.6.8.10.12} \text{ etc.} = \frac{\cos.\ 22°,\ 30'}{\sqrt[4]{2}}.$$

Ante autem vidimus esse $\cos.\ 22°,\ 30' = \sqrt{\frac{1+\sqrt{2}}{2\sqrt{2}}}$, vnde ista summa erit $= \frac{1}{2}\sqrt{(1+\sqrt{2})}$.

§. 14. In genere etiam pro exponentibus quibuscunque n operae pretium erit angulo Φ certos tribuere valores, ac primo quidem sumpto $\Phi = 0$ habebimus

$$1 + (\tfrac{n}{1}) + (\tfrac{n}{2}) + (\tfrac{n}{3}) + (\tfrac{n}{4}) + \text{etc.} = 2^n;$$

haec scilicet series est ipsa formula $(1+1)^n$ euoluta. Sumamus nunc $\Phi = 180°$ et orietur haec series:

$$1 - (\tfrac{n}{1}) + (\tfrac{n}{2}) - (\tfrac{n}{3}) + (\tfrac{n}{4}) - \text{etc.} = 0,$$

scilicet haec series est $= (1-1)^n$. Sit etiam $\Phi = 90°$ et series hinc nata erit

$$1 - (\tfrac{n}{2}) + (\tfrac{n}{4}) - (\tfrac{n}{6}) + (\tfrac{n}{8}) - (\tfrac{n}{10}) \text{ etc.}$$

cuius ergo summa erit

$$2^n \cos.\ 45^n \cos.\ n\ 45° = 2^{\frac{1}{2}n} \cos.\ n\ 45°.$$

§. 15.

§. 15. Haec postrema series eo maiori attentione digna videtur, quod eius veritas non parum est abscondita; vnde haud abs re erit casus aliquot speciales contemplari, ac primo quidem pro numeris integris positiuis:

1°. Si $n=0$ erit $1=1$,
2°. Si $n=1$ erit $1=\cos.45°\sqrt{2}=1$,
3°. Si $n=2$ erit $1-1=2\cos.90°=0$,
4°. Si $n=3$ erit $1-3=2^{\frac{3}{2}}\cos.3.45°=-2$,
5°. Si $n=4$ erit $1-6+1=2^2\cos.4.45°=-4$,
6°. Si $n=5$ erit $1-10+5=2^{\frac{5}{2}}\cos.5.45°=-4$,
7°. Si $n=6$ erit $1-15+15-1=2^3\cos.6.45°=0$,
8°. Si $n=7$ erit $1-21+35-7=2^{\frac{7}{2}}\cos.7.45°=2^3$,
9°. Si $n=8$ erit $1-28+70-28+1=2^4\cos.8.45°=2^4$,
etc. etc.

§. 16. Maiorem attentionem merentur casus, quibus pro n numeri negatiui accipiuntur, quippe quibus series infinitae proueniunt.

1°. Si $n=-1$ erit $1-1+1-1+1-1+1-1+1-1$ etc. $=\frac{\cos.45°}{\sqrt{2}}=\frac{1}{2}$,

2°. Si $n=-2$ erit $1-3+5-7+9-11+13-15+17$ etc. $=2\cos.2.45°=0$,

3°. Si $n=-3$ erit $1-6+15-28+45-66+91$ etc. $=\frac{\cos.3.45°}{\sqrt{8}}=-\frac{1}{4}$,

4°. Si $n=-4$ erit $1-10+35-84+165-286+455$ etc. $=\frac{\cos.4.45°}{4}=-\frac{1}{4}$,

5°.

5°. Si $n=-5$ erit $1-15+70-210+495-1001$ etc.

$$=\frac{\text{cof. } 5.\, 45^\circ}{2^{\frac{5}{2}}}=-\tfrac{1}{8},$$

6°. Si $n=-6$ erit $1-21+126-462+1287-3003$ etc.

$$=\frac{\text{cof. } 6.\, 45^\circ}{8}=0,$$

etc. etc.

In genere autem pro his cafibus erit

$$1-\frac{\lambda(\lambda+1)}{1.\ 2}+\frac{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}{1.\ 2.\ 3.\ 4}-\frac{\lambda(\lambda+1)\ \ldots\ \ldots\ (\lambda+5)}{1.\ 2.\ 3.\ 4.\ 5.\ 6}$$
$$+\frac{\lambda(\lambda+1)\ \ldots\ \ldots\ (\lambda+7)}{1.\ 2.\ \ldots\ \ldots\ 8}-\frac{\lambda(\lambda+1)\ \ldots\ \ldots\ (\lambda+9)}{1.\ 2.\ \ldots\ \ldots\ 10}\text{ etc.}$$

cuius ergo feriei fumma erit

$$=\frac{\text{cof.}\lambda.\ 45^\circ}{2^{\frac{1}{2}\lambda}}.$$

Problema 2.

Propofita hac ferie finuum:

$$\tfrac{n}{1}\text{ fin. }\Phi+\tfrac{n}{1}.\tfrac{n-1}{2}\text{ fin. }2\Phi+\tfrac{n}{1}.\tfrac{n-1}{2}.\tfrac{n-2}{3}\text{ fin. }3\Phi+\text{etc.}=s,$$

ita vt per characteres fupra adhibitos fit

$$s=(\tfrac{n}{0})\text{ fin. }0\Phi+(\tfrac{n}{1})\text{ fin. }\Phi+(\tfrac{n}{2})\text{ fin. }2\Phi+(\tfrac{n}{3})\text{ fin. }3\Phi+\text{etc.}$$

valorem huius fummae s realiter exprimere.

Solutio.

§. 17. Quodfi hic iterum introducamus litteras

$$p=\text{cof. }\Phi+\sqrt{-1}\text{ fin. }\Phi\text{ et}$$
$$q=\text{cof. }\Phi-\sqrt{-1}\text{ fin. }\Phi,$$

quoniam eft $p^n-q^n=2\sqrt{-1}\text{ fin. }n\Phi$, feries propofita in duas fequentes difcerpetur

$$2 s \sqrt{-1} = \left\{ \begin{array}{l} + (\tfrac{n}{1}) p + (\tfrac{n}{2}) pp + (\tfrac{n}{3}) p^3 + (\tfrac{n}{4}) p^4 \\ - (\tfrac{n}{1}) q - (\tfrac{n}{2}) qq - (\tfrac{n}{3}) q^3 - (\tfrac{n}{4}) q^4 \end{array} \right\} \text{ etc.}$$

vnde manifesto erit

$$2 s \sqrt{-1} = (1 + p)^n - (1 + q)^n.$$

§. 18. Hic iam iterum obferuaffe iuuabit effe

$$1 + p = (\sqrt{p} + \sqrt{q}) \sqrt{p} \text{ et}$$
$$1 + q = (\sqrt{p} + \sqrt{q}) \sqrt{q}$$

quibus valoribus adhibitis erit

$$2 s \sqrt{-1} = (\sqrt{p} + \sqrt{q})^n (p^{\frac{1}{2}n} - q^{\frac{1}{2}n}).$$

Quoniam igitur eft

$$p^{\frac{1}{2}n} - q^{\frac{1}{2}n} = 2 \sqrt{-1} \text{ fin. } \tfrac{1}{2} n \Phi \text{ et}$$
$$\sqrt{p} + \sqrt{q} = 2 \text{ cof. } \tfrac{1}{2} \Phi,$$

hinc per $2 \sqrt{-1}$ diuidendo prodibit fumma quaefita realiter expreffa $s = 2^n \text{ cof. } \tfrac{1}{2} \Phi^n \text{ fin. } \tfrac{1}{2} n \Phi$.

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