DE VERO VALORE FORMVLAE INTEGRALIS $\int \partial x \, (l \frac{1}{x})^n$ A TERMINO $x = 0$ VSQVE AD TERMINVM $x = 1$ EXTENSAE.

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§. 1.

Cum haec formula exprimat aream curuae transcendentis, cuius abscissae x respondet applicata $= (l \frac{1}{x})^n$; quaestio huc redit, vt eadem area, quatenus abscissae $x = 1$ conuenit, vel per numeros absolutos, vel saltem per quadraturas curuarum algebraicarum exhibeatur. Ac primo quidem manifestum est, quoties exponens n fuerit numerus integer, hanc formulam integralem sistere terminum generalem progressionis hypergeometricae, cum sit

$$\int \partial x \, (l \tfrac{1}{x})^0 = 1,$$
$$\int \partial x \, (l \tfrac{1}{x})^1 = 1$$
$$\int \partial x \, (l \tfrac{1}{x})^2 = 1 . 2,$$
$$\int \partial x \, (l \tfrac{1}{x})^3 = 1 . 2 . 3$$
$$\int \partial x \, (l \tfrac{1}{x})^4 = 1 . 2 . 3 . 4,$$

atque adeo in genere

$$\int \partial x \, (l \tfrac{1}{x})^n = 1 . 2 . 3 . 4 . \ldots\ldots n$$

quem autem valorem cognoscere non datur, nisi exponens n fuerit

fuerit numerus integer positiuus; praeterea vero si exponens n fuerit numerus integer negatiuus, ex indole seriei hyper-geometricae facile perspicitur valores nostrae formulae omnes fieri infinite magnos. Quaestio igitur hic potissimum complectetur casus, quibus exponens n est numerus fractus, quibus vtique valor nostrae formulae neutiquam per numeros absolutos assignari potest, sed potius quadraturas curuarum algebraicarum eo altiorum ordinum postulat, quo maior fuerit denominator fractionis pro n assumtae, quemadmodum iam olim fusius monstraui. Nuper autem se mihi obtulit alia methodus eosdem valores transcendentes inuestigandi, quam ergo hic explicare constitui; cum inde haud contemnenda incrementa in Analysin redundare videantur.

§. 2. Ante omnia igitur statuamus breuitatis gratia $l\frac{1}{x} = u$, vt sit $\partial u = -\frac{\partial x}{x}$, ideoque $\partial x = -x\,\partial u$. Hinc statim insignes reductiones ope lemmatis vulgatissimi, quo est $\int P\,\partial Q = PQ - \int Q\,\partial P$, deriuari possunt. Sumto enim $P = u^n$ et $\partial Q = \partial x$, ob $\partial P = n\,u^{n-1}\,\partial u = -\frac{n\,u^{n-1}\,\partial x}{x}$ et $Q = x$ hoc Lemma nobis praebet

$$\int u^n\,\partial x = x\,u^n + n\int u^{n-1}\,\partial x.$$

Deinde cum sit $u^n\,\partial x = -x^n\,\partial u$, si hic capiatur $P = -x^n$ et $\partial Q = u^n\,\partial u$, ob $\partial P = -\partial x$ et $Q = \frac{1}{n+1}u^{n+1}$, habebimus

$$\int u^n\,\partial x = -\frac{1}{n+1}x\,u^{n+1} + \frac{1}{n+1}\int u^{n+1}\,\partial x.$$

Quare, quoniam haec integralia ita capi debent, vt euanescant posito $x = 0$, tum vero statui debet $x = 1$, notum est tum membra absoluta in his reductionibus in nihilum abire, ita vt pro hoc casu, de quo hic vnice agitur, sit $\int u^n\,\partial x = n\int u^{n-1}\,\partial x$, tum vero etiam $\int u^n\,\partial x = \frac{1}{n+1}\int u^{n+1}\,\partial x$, quae quidem posterior reductio sponte ex priori fluit.

§. 3.

§. 3. Quod autem formula $x u^n$ casu $x = 0$ semper euanescat, vulgo non satis directe demonstrari solet, atque adeo dubium videri queat, propterea quod posito $x = 0$ fiat $u^n = \infty$; at vero haec veritas sequenti modo rigorose ostendi potest: Namque pro casu $x = 0$ statuamus $x u^n = v$, ita vt valor huius litterae v nobis sit explorandus, quem ergo ita per fractionem repraesentemus: $v = \frac{x}{u^{-n}}$, cuius tam numerator quam denominator casu $x = 0$ euanescit, vnde per regulam communem tam loco numeratoris quam denominatoris eorum differentialia scribantur, et quia valor ipsius v idem prodire debet, erit quoque

$$v = \frac{\partial x}{-n u^{-n-1} \partial u} = \frac{+x}{n u^{-n-1}} \left(\text{ob } \partial u = -\frac{\partial x}{x}\right).$$

Cum igitur ex priore valore sit $v = x u^n$, ex posteriori vero $v = \frac{1}{n} x u^{n+1}$, inde fiet $v^{n+1} = x^{n+1} u^{n(n+1)}$, hinc vero

$$v^n = \left(\frac{x}{n}\right)^n u^{n(n+1)},$$

quorum valorum ille per hunc diuisus dabit $v = n^n x$, haecque expressio pariter verum valorem ipsius v pro casu $x = 0$ exhibere debet, hic autem posito $x = 0$ manifesto fit $v = 0$.

§. 4. Quoniam nostra inuestigatio hic potissimum ad casus, quibus exponens n est fractio, restringitur, ope reductionis $\int u^n \partial x = n \int u^{n-1} \partial x$ omnes fractiones loco n assumtae, quantumuis fuerint magnae, continuo vnitate diminui ideoque tandem adeo infra vnitatem deprimi poterunt, ita vt intra limites 0 et 1 contineantur. Deinde vero ope alterius reductionis: $\int u^n \partial x = \frac{1}{n+1} \int u^{n+1} \partial x$, si forte exponens n fuerit fractio negatiua, tandem eius valor pariter ad fractionem inter limites 0 et 1 redigi poterit; vnde nobis hoc loco sufficiet

eos tantum casus euoluisse, quibus fractiones pro n assumtae intra limites 0 et 1 consistunt; haeque fractiones commode in varias classes distribuuntur, prouti denominatores harum fractionum fuerint vel 2, vel 3, vel 4, vel 5, etc.

§. 5. Cum nuper series, quae ex vnciis potestatum Binomii formantur, essem contemplatus, ostendi, si ponatur $(1+z)^m = 1 + Az + Bz^2 + Cz^3$, etc. tum vero etiam $(1+z)^n = 1 + \alpha z + \beta z^2 + \gamma z^3$, etc. tum summam huius seriei $1 + A\alpha + B\beta + C\gamma +$ etc. $= s$, ita exprimi posse, vt sit $s = \int \frac{u^{m+n}\partial x}{\int u^m \partial x . \int u^n \partial x}$, quae ergo summa per valorem formulae integralis propositae definitur; deinde vero etiam monstraui, eandem summam quoque hoc modo exprimi posse:

$$s = \frac{m+n}{mn \int x^{m-1} \partial x (1-x)^{n-1}},$$

vnde ergo sequitur semper fore

$$\frac{m+n}{mn} \int u^m \partial x . \int u^n \partial x = \int u^{m+n} \partial x . \int x^{m-1} \partial x (1-x)^{n-1},$$

siquidem singula haec integralia a termino $x = 0$ vsque ad terminum $x = 1$ extendantur.

§. 6. Quoniam autem praesens nostrum institutum circa fractiones, easque vnitate minores, versatur, ponamus in genere $m = \frac{\mu}{\lambda}$ et $n = \frac{\nu}{\lambda}$, ita vt sit

$$\frac{\lambda(\mu+\nu)}{\mu\nu} \int u^{\frac{\mu}{\lambda}} \partial x . \int u^{\frac{\nu}{\lambda}} \partial x = \int u^{\frac{\mu+\nu}{\lambda}} \partial x \int x^{\frac{\mu-\lambda}{\lambda}} \partial x (1-x)^{\frac{\nu-\lambda}{\lambda}}.$$

Nunc vero vt postremam formulam integralem ab exponentibus fractis liberemus, statuamus $x = z^\lambda$, et ob $\partial x = \lambda z^{\lambda-1} \partial z$ erit

$$\int x^{\frac{\mu-\lambda}{\lambda}} \partial x (1-x)^{\frac{\nu-\lambda}{\lambda}} = \lambda \int z^{\mu-1} \partial z (1-z^\lambda)^{\frac{\nu-\lambda}{\lambda}},$$

quae

quae formula, ob $\nu - \lambda$, ita referri potest: $\lambda \int \frac{z^{\mu-1}\partial z}{\sqrt[\lambda]{(1-z^\lambda)^{\lambda-\nu}}}$, quod integrale pariter a $z = 0$ vsque ad $z = 1$ est extendendum. Facta igitur hac substitutione aequatio nostra principalis ita se habebit:

$$\frac{\mu+\nu}{\mu\nu}\int \partial x \sqrt[\lambda]{u^\mu} \cdot \int \partial x \sqrt[\lambda]{u^\nu} = \int \partial x \sqrt[\lambda]{u^{\mu+\nu}} \int \frac{z^{\mu-1}\partial z}{\sqrt[\lambda]{(1-z^\lambda)^{\lambda-\nu}}};$$

vbi ambo numeri μ et ν perpetuo nobis erunt positiui et minores quam λ. Imprimis autem hic obseruari meretur, casu quo $\mu + \nu = \lambda$ postremum integrale semper ad quadraturam circuli ita reduci posse, vt sit

$$\int \frac{z^{\mu-1}\partial z}{\sqrt[\lambda]{(1-z^\lambda)^\mu}} = \frac{\pi}{\lambda \sin. \frac{\mu\pi}{\lambda}}.$$

§. 7. Ex hac iam aequatione principali haud difficulter valores formulae integralis propositae pro singulis denominatoribus λ elicientur, si modo litteris μ et ν quouis casu omnes numeri denominatore λ minores successiue tribuantur, tum enim plures formabuntur aequationes, ex quibus valores formularum $\int \partial x \sqrt[\lambda]{u^\mu}$ et $\int \partial x \sqrt[\lambda]{u^\nu}$ definiri poterunt. Quod autem ad formulam $\int \partial x \sqrt[\lambda]{u^{\mu+\nu}}$ attinet, quae nata est ex $\int u^{m+n}\partial x$, quando fuerit $m + n > 1$, siue $\mu + \nu > \lambda$, quoniam vidimus esse $\int u^{m+n}\partial x = (m+n)\int u^{m+n-1}\partial x$, erit

$$\int \partial x \sqrt[\lambda]{u^{\mu+\nu}} = \frac{\mu+\nu}{\lambda}\int \partial x \sqrt[\lambda]{u^{\mu+\nu-\lambda}},$$

quae ergo formula valebit, quando $\mu + \nu > \lambda$. Denique vero omnes valores, qui ex postrema formula integrali

$\int$

$$\int \frac{z^{\mu+1}\partial z}{\sqrt[\alpha]{(1-z^{\lambda})^{\lambda-\nu}}}$$

nascuntur, tanquam cogniti spectari poterunt, vnde eos litteris A, B, C, D, etc. indicabimus. His igitur praenotatis pro denominatore λ ordine numeros 2, 3, 4, 5, etc. accipiamus, ideoque sequentes casus euoluamus, pro quibus in genere observasse iuuabit, numeros μ et ν semper inter se permutari posse, ita vt sit

$$\int \frac{z^{\mu-1}\partial z}{\sqrt[\lambda]{(1-z^{\lambda})^{\lambda-\nu}}} = \int \frac{z^{\nu-1}\partial z}{\sqrt[\lambda]{(1-z^{\lambda})^{\lambda-\mu}}}.$$

I. Euolutio casus, quo $\lambda = 2$.

§. 8. Pro hoc ergo casu aequatio nostra principalis erit

$$\frac{\mu+\nu}{\mu\nu}\int \partial x \sqrt{u^{\mu}} . \int \partial x \sqrt{u^{\nu}} = \int \partial x \sqrt{u^{\mu+\nu}} \int \frac{z^{\mu-1}\partial z}{\sqrt{(1-zz)^{2-}}}$$

vbi cum loco μ et ν alios numeros praeter vnitatem accipere non liceat, posito $\mu = 1$ et $\nu = 1$ pro formula postrema vnica species oritur $\int \frac{\partial z}{\sqrt{(1-zz)}}$, cuius valor, vti constat, est $= \frac{\pi}{2}$, quem autem ob analogiam sequentium casuum littera A designabimus. Hinc igitur cum sit $\mu + \nu = 2$, erit

$$\int \partial x \sqrt{uu} = \int u \partial x = 1,$$

aequatio autem principalis induet hanc formam:

$$2\int \partial x \sqrt{u} . \int \partial x \sqrt{u} = \frac{\pi}{2} = A,$$

vnde fit

$$\int \partial x \sqrt{u} = \sqrt{\frac{A}{2}} = \frac{1}{2}\sqrt{\pi}.$$

§. 9. Quoniam igitur iuuenimus esse $\int u^{\frac{1}{2}} \partial x = \frac{1}{2}\sqrt{\pi}$, si exponentem ipsius u continuo vnitate augeamus, per reductionem

tionem ſupra oſtenſam, $\int u^n \partial x = n \int u^{n-1} \partial x$, impetrabimus ſequentes valores:

$$\int u^{\frac{3}{2}} \partial x = \frac{1.3}{2.2} \sqrt{\pi},$$

$$\int u^{\frac{5}{2}} \partial x = \frac{1.3.5}{2.2.2} \sqrt{\pi},$$

$$\int u^{\frac{7}{2}} \partial x = \frac{1.3.5.7}{2.2.2.2} \sqrt{\pi},$$

et ita porro. Deinde vero regrediendo per alteram reductionem $\int u^n \partial x = \frac{1}{n+1} \int u^{n+1} \partial x$, reperiemus,

$$\int u^{-\frac{1}{2}} \partial x = \sqrt{\pi};$$ hincque porro

$$\int u^{-\frac{3}{2}} \partial x = -2\sqrt{\pi},$$

$$\int u^{-\frac{5}{2}} \partial x = +\frac{2.2}{3} \sqrt{\pi},$$

$$\int u^{-\frac{7}{2}} \partial x = -\frac{2.2.2}{3.5} \sqrt{\pi},$$

$$\int u^{-\frac{9}{2}} \partial x = \frac{2.2.2.2}{3.5.7} \sqrt{\pi},$$

ſicque valores noſtrae formulae inuenimus pro omnibus fractionibus, quarum denominator eſt $= 2$.

Euolutio caſus, quo $\lambda = 3$.

§. 10. Quoniam hic litterae μ et ν binos valores recipere poſſunt, ſcilicet 1 et 2, formula integralis poſtrema quatuor nobis ſuppeditat valores, quos ſequenti modo indicemus:

$$\int \frac{\partial z}{\sqrt[3]{(1-z^3)}} = A, \quad \int \frac{z \partial z}{\sqrt[3]{(1-z^3)}} = B,$$

$$\int \frac{\partial z}{\sqrt[3]{(1-z^3)^2}} = A', \quad \int \frac{z \partial z}{\sqrt[3]{(1-z^3)^2}} = B'.$$

In harum formularum prima et quarta eſt $\mu + \nu = \lambda = 3$, vnde per quadraturam circuli habebimus

$$A = \frac{\pi}{3 \sin. \frac{1}{3}\pi} = \frac{2\pi}{3\sqrt{3}} \text{ et } B' = \frac{\pi}{3 \sin. \frac{2}{3}\pi} = \frac{2\pi}{3\sqrt{3}},$$

vnde patet eſſe $B' = A$, quod etiam inde ſequitur, quod literae μ et ν ſunt permutabiles. Praeterea vero notetur caſu $\mu + \nu = 3$ fore

$$\int \partial x \sqrt[3]{u^{\mu+\nu}} = \int u \, \partial x = 1,$$

at vero caſu $\mu + \nu = 4$, erit

$$\int \partial x \sqrt[3]{u^4} = \int u^{\frac{4}{3}} \partial x = \tfrac{4}{3} \int \partial x \sqrt[3]{u}.$$

§. 11. His praemonitis omnes caſus aequationis noſtrae principalis ordine euoluamus ſequenti modo:

I. Si $\mu = 1$ et $\nu = 2$, erit $\frac{3}{2} \int \partial x \sqrt[3]{u} . \int \partial x \sqrt[3]{u^2} = A$.

II. Si $\mu = 2$ et $\nu = 2$, erit $\frac{4}{4} \int \partial x \sqrt[3]{u^2} . \int \partial x \sqrt[3]{u^2} = \frac{4}{3} B \int \partial x \sqrt[3]{u}$

III. Si $\mu = 1$ et $\nu = 1$, erit $2 \int \partial x \sqrt[3]{u} . \int \partial x \sqrt[3]{u} = A' \int \partial x \sqrt[3]{u}$

IV. Si $\mu = 2$ et $\nu = 1$, erit $\frac{3}{2} \int \partial x \sqrt[3]{u^2} . \int \partial x \sqrt[3]{u} = B'$.

Sicque quatuor nacti ſumus aequationes pro determinandis binis valoribus incognitis, ſcilicet $\int \partial x \sqrt[3]{u}$ et $\int \partial x \sqrt[3]{u^2}$, quos ergo pluribus modis definire licebit, quandoquidem ad hoc duae tantum aequationes ſufficiunt.

§. 12. Quo autem hic calculus facilior reddatur, ſtatuamus breuitatis gratia $\int \partial x \sqrt[3]{u} = p$ et $\int \partial x \sqrt[3]{u^2} = q$, et combinemus primo aequationem I et II, quae erunt

$$\tfrac{3}{2} p q = A \text{ et } q q = \tfrac{4}{3} B p,$$

quarum poſterior dat $p = \frac{3qq}{4B}$, qui valor in priore ſubſtitutus dat

dat $\frac{2q^3}{9B} = A$, vnde reperitur $q = 2\sqrt[3]{\frac{AB}{9}}$, ex quo porro colligitur $p = \frac{3}{B}\sqrt[3]{\frac{A^2B^2}{81}}$, siue etiam $p = \sqrt[3]{\frac{AA}{3B}}$, sicque pro p et q restitutis valoribus iam nacti sumus has determinationes:

$$\int \partial x \sqrt[3]{u} = \sqrt[3]{\frac{AA}{3B}} \text{ et } \int \partial x \sqrt[3]{u^2} = 2\sqrt[3]{\frac{AB}{9}}.$$

§. 13. Combinemus nunc primam aequationem cum tertia, et habebimus $\frac{3}{2}pq = A$ et $2pp = A'q$. Ex posteriore fit $q = \frac{2pp}{A'}$, qui valor in priore substitutus dat $\frac{3p^3}{A'} = A$, vnde reperitur $p = \sqrt[3]{\frac{AA'}{3}}$, hincque

$$q = \frac{2}{A'}\sqrt[3]{\frac{A^2A'A'}{9}} = 2\sqrt[3]{\frac{A^2}{9A'}},$$

sicque haec combinatio nos perducit ad hos valores:

$$\int \partial x \sqrt{u} = \sqrt[3]{\frac{AA'}{3}} \text{ et } \int \partial x \sqrt[3]{u^2} = 2\sqrt[3]{\frac{A^2}{9A'}}.$$

§. 14. Combinemus nunc quoque primam aequationem cum quarta, et habebimus $\frac{3}{2}pq = A$ et $\frac{3}{2}pq = B'$, vnde nihil aliud sequitur, nisi $B' = A$, vti ante inuenimus. Combinemus igitur secundam cum tertia, et habebimus $qq = \frac{4}{3}Bp$ et $2pp = A'q$, ex quarum posteriore fit $q = \frac{2pp}{A'}$, qui valor in prima substitutus dat $\frac{4p^3}{A'A'} = \frac{4}{3}B$, vnde reperitur

$$p = \sqrt[3]{\frac{A'A'B}{3}}, \text{ ex quo fit}$$

$$q = 2\sqrt[3]{\frac{A'BB}{9}},$$

sicque haec combinatio nobis dat hos valores:

$$\int \partial x \sqrt[3]{u} = \sqrt[3]{\frac{A'A'B}{3}} \text{ et } \int \partial x \sqrt[3]{u^2} = 2\sqrt[3]{\frac{A'BB}{9}}.$$

§. 15.

§. 15. Quoniam aequatio quarta cum prima prorsus conuenit, superfluum foret, secundam vel tertiam cum quarta combinare, quoniam eas iam cum prima combinauimus. Sicque pro litteris p et q omnino ternos nacti sumus valores, quos ita coniunctim ob oculos ponamus:

$$\int \partial x \sqrt[3]{u} = \sqrt[3]{\frac{AA}{3B}} = \sqrt[3]{\frac{A'A}{3}} = \sqrt[3]{\frac{A'A'B}{3}} \text{ et}$$

$$\int \partial x \sqrt[3]{u^2} = 2\sqrt[3]{\frac{AB}{9}} = 2\sqrt[3]{\frac{A^2}{9A'}} = 2\sqrt[3]{\frac{A'BB}{9}}.$$

Hinc igitur sumtis cubis sequentes nanciscimur aequationes:

$$\frac{AA}{B} = AA' = A'A'B \text{ et}$$

$$AB = \frac{AA}{A'} = A'BB.$$

§. 16. At relatione inter hos diuersos valores facta omnes hae aequalitates ad vnicam hanc proprietatem reuocantur, qua est $A = A'B$. Substitutis igitur ipsis formulis integralibus consequimur hanc veritatem maxime memorabilem:

$$\int \frac{\partial z}{\sqrt[3]{(1-z^3)}} = \int \frac{\partial z}{\sqrt[3]{(1-z^3)^2}} \cdot \int \frac{z\,\partial z}{\sqrt[3]{(1-z^3)}},$$

et quia A per quadraturam circuli definitur, prodibit valor huius producti:

$$\int \frac{\partial z}{\sqrt[3]{(1-z^3)^2}} \cdot \int \frac{z\,\partial z}{\sqrt[3]{(1-z^3)}} = \frac{2\pi}{3\sqrt{3}},$$

vnde si alterius harum duarum formularum valor innotesceret, simul alterius valor foret cognitus; hoc enim modo ex binis valoribus A et B bini reliqui A' et B' ita determinantur, vt sit $A' = \frac{A}{B}$ et B' et A. Denique etiam operae pretium erit notasse hanc relationem

$$\int \partial x \sqrt[3]{u} \cdot \int \partial x \sqrt[3]{u^2} = \frac{2}{3} A = \frac{4\pi}{9\sqrt{3}}.$$

§. 17.

Euolutio casus quo $\lambda = 4$.

§. 17. Hic iam primo breuitatis gratia ponamus

$$\int \partial x \sqrt[4]{u^2} = q \text{ et } \int \partial x \sqrt[4]{u^3} = r;$$

praeterea vero designemus formulam integralem $\int \frac{z^{\mu-1}\partial z}{\sqrt[4]{(1-z^4)^{4-\nu}}}$, per characterem (μ, ν), quandoquidem iam vidimus litteras μ et ν inter se permutari posse. Deinde aequatio principalis hoc modo repraesentetur:

$$\int \partial x \sqrt[4]{u^\mu} . \int \partial x \sqrt[4]{u^\nu} = \frac{\mu\nu}{\mu+\nu} \int \partial x \sqrt[4]{u^{\mu+\nu}} (\mu, \nu);$$

vbi notetur si $\mu + \nu = \lambda = 4$, fore $\int \partial x \sqrt[4]{u^4} = 1$; sin autem $\mu + \nu = \lambda + \alpha = 4 + \alpha$, erit

$$\int \partial x \sqrt[4]{u^{4+\alpha}} = (1 + \tfrac{\alpha}{4}) \int u^{\frac{\alpha}{4}} \partial x = \frac{\mu+\nu}{4} \int \partial x \sqrt[4]{u^\alpha}.$$

§. 18. Tribuamus nunc litteris μ et ν successiue omnes valores minores quam 4, atque aequatio principalis nobis praebebit sequentes aequationes:

1°. Si $\begin{pmatrix}\mu = 1\\ \nu = 1\end{pmatrix}$, erit $pp = \frac{1}{2} q (1, 1)$, vnde fit $\frac{pp}{q} = \frac{1}{2}(1, 1) = A$.

2°. Si $\begin{pmatrix}\mu = 1\\ \nu = 2\end{pmatrix}$, erit $pq = \frac{2}{3} r (1, 2)$, vnde $\frac{pq}{r} = \frac{2}{3}(1, 2) = B$.

3°. Si $\begin{pmatrix}\mu = 1\\ \nu = 3\end{pmatrix}$, erit $pr = \frac{3}{4} (1, 3) = C$.

4°. Si $\begin{pmatrix}\mu = 2\\ \nu = 2\end{pmatrix}$, erit $qq = \quad (2, 2) = D$.

5°. Si $\begin{pmatrix}\mu = 2\\ \nu = 3\end{pmatrix}$, erit $qr = \frac{6}{5} . \frac{5}{4} p (2, 3)$, vnde fit $\frac{qr}{p} = \frac{3}{2}(2, 3) = E$.

6°. Si $\begin{pmatrix}\mu = 3\\ \nu = 3\end{pmatrix}$, erit $rr = \frac{9}{6} . \frac{6}{4} q (3, 3)$, vnde fit $\frac{rr}{q} = \frac{9}{4}(3, 3) = F$.

§. 19. Hinc igitur nacti ſumus ſex aequationes, ex quibus tres noſtras incognitas p, q et r definiri oportet, quod igitur pluribus modis fieri poteſt, ſiquidem ternae aequationes ſufficiunt. Eligamus igitur eas, quae negotium facillime conficiunt, ac primo quidem quarta nobis ſtatim dat $q = \sqrt{D}$, vnde ex prima elicimus $pp = A\sqrt{D}$, ideoque

$$p = \sqrt{A\sqrt{D}} = \sqrt[4]{AAD},$$

denique ex aequatione ſexta colligimus $rr = F\sqrt{D}$, ideoque $r = \sqrt[4]{FFD}$, ſicque omnes tres formulas tranſcendentes ita determinauimus, vt ſit

$$1^\circ.\ p = \int \partial x \sqrt[4]{u} = \sqrt[4]{AAD} = \sqrt[4]{\tfrac{1}{4}(1,1)^2(2,2)},$$

$$2^\circ.\ q = \int \partial x \sqrt[4]{u^2} = \int \partial x \sqrt[4]{u} = \sqrt{D} = \sqrt{(2,2)},$$

$$3^\circ.\ r = \int \partial x \sqrt[4]{u^3} = \sqrt[4]{DFF} = \tfrac{3}{2}\sqrt[4]{(2,2)(3,3)^2}.$$

20. Hic iam notaſſe iuuabit, valorem formulae (μ, ν) caſu quo $\mu + \nu = \lambda$, in genere per quadraturam circuli exprimi poſſe, cum hoc caſu ſit

$$(\mu, \nu) = \frac{\mu}{\lambda \sin. \frac{\mu\pi}{\lambda}}.$$

Noſtro igitur caſu, quo $\lambda = 4$, erit

$$(2, 2) = \frac{\pi}{4 \sin. \frac{1}{2}\pi} = \frac{\pi}{4},$$

deinde quoque erit

$$(1, 3) = \frac{\pi}{4 \sin. \frac{1}{4}\pi} = \frac{\pi}{2\sqrt{2}}.$$

Hinc igitur patet fore $D = \frac{\pi}{4}$ et $C = \frac{3}{4} \cdot \frac{\pi}{2\sqrt{2}} = \frac{3\pi}{8\sqrt{2}}$, ita vt hae duae littere C et D a ſola quadratura circuli pendeant.

§. 21.

§. 21. Quoniam has tres determinationes, nempe:

$$p = \sqrt[4]{AAD},\ q = \sqrt{D} \text{ et } r = \sqrt[4]{DFF},$$

ex aequationibus 1, 4 et 6ta elicuimus, si eosdem valores in reliquis aequationibus substituamus, reperiemus egregias relationes inter litteras nostras maiusculas. Sic enim secunda aequatio $pq = Br$ dabit $AAD^3 = B^4DFF$, quae reducitur ad hanc: $AD = BBF$; tertia vero aequatio $pr = C$ dabit $ADF = CC$; denique quinta aequatio $qr = Ep$ praebebit $D^3FF = A^2DE$, vnde fit $DF = AEE$. Hoc ergo modo deducti sumus ad tres sequentes relationes:

$$1^\circ.\ AD = BBF,\ 2^\circ.\ ADF = CC \text{ et } 3^\circ.\ DF = AEE,$$

quarum prima ducta in secundam dabit $AD = BC$, at vero secunda ducta in tertiam producit $DF = CE$. Cum igitur sit $AD = BC$, ex prima concluditur quoque fore $C = BF$, ita vt ternae determinationes repertae ad istas ternas reuocentur:

$$1^\circ.\ C = AE,\ 2^\circ.\ C = BF,\ 3^\circ.\ AD = BC,$$

quae reducuntur ad istas tres simplicissimas:

$$1^\circ.\ C = AE,\ 2^\circ.\ C = BF,\ 3^\circ.\ D = BE.$$

§. 22. Quodsi iam in his postremis aequationibus loco litterarum formulas integrales per nostros characteres designatas introducamus, prouenient sequentes relationes:

$$1^\circ.\ (1, 3) = (1, 1)(2, 3),$$
$$2^\circ.\ (1, 3) = 2(1, 2)(3, 3) \text{ et}$$
$$3^\circ.\ (2, 2) = (1, 2)(2, 3).$$

Hinc igitur per ipsas formulas integrales habebimus istas tres relationes maxime memorabiles:

$$1^\circ.\ \frac{\pi}{2\sqrt{2}} = \int \frac{\partial z}{\sqrt[4]{(1-z^4)^3}} \cdot \int \frac{z\,\partial z}{\sqrt[4]{(1-z^4)}} = \int \frac{\partial z}{\sqrt[4]{(1-z^4)^3}} \cdot \int \frac{z z\,\partial z}{\sqrt{(1-z^4)}},$$

2°. $\frac{\pi}{4\sqrt{2}} = \int \frac{\partial z}{\sqrt{(1-z^4)}} \cdot \int \frac{z z \partial z}{\sqrt[4]{(1-z^4)}}$ et

3°. $\frac{\pi}{4} = \int \frac{\partial z}{\sqrt{(1-z^4)}} \cdot \int \frac{z z \partial z}{\sqrt{(1-z^4)}}$,

quarum vltimam iam dudum in medium attuli.

§. 23. Cum igitur ex ſex formulis integralibus quae hic occurrunt, binae, ſcilicet C et D, a quadratura circuli pen deant, ſi modo ex reliquis vnius valor innoteſcat, valore caeterarum inde aſſignari poterunt. Si enim praeter charactere (1, 3) et (2, 2) inſuper hunc (1, 2) tanquam cognitum ſpec temus, reliqui tres per hos ſequenti modo determinabuntur

$$(3, 3) = \frac{(1, 3)}{2(1, 2)}; \quad (2, 3) = \frac{(2, 2)}{(1, 2)}; \quad (1, 1) = \frac{(1, 2)(1, 3)}{(2, 2)}.$$

Euolutio caſus quo $\lambda = 5$.

§. 24. Vocemus hic formulas tranſcendentes quae tas $\int u^{\frac{1}{5}} \partial x = p$, $\int u^{\frac{2}{5}} \partial x = q$, $\int u^{\frac{3}{5}} \partial x = r$, $\int u^{\frac{4}{5}} \partial x = s$. Nu vero character (μ, ν) ſignificet hanc formulam integralem $\int \frac{z^{\mu-1} \partial z}{\sqrt[5]{(1-z^5)^{5-\nu}}}$, quibus poſitis ex aequatione principali de cem ſequentes aequationes nanciſcemur:

1°. Si $\begin{pmatrix}\mu = 1\\ \nu = 1\end{pmatrix}$, erit $p\,p = \frac{1}{2} q\,(1, 1)$, vnde fit $\frac{p\,p}{q} = \frac{1}{2}(1, 1) =$

2°. Si $\begin{pmatrix}\mu = 1\\ \nu = 2\end{pmatrix}$, erit $p\,q = \frac{2}{3} r\,(1, 2)$, ergo $\frac{p\,q}{r} = \frac{2}{3}(1, 2) =$

3°. Si $\begin{pmatrix}\mu = 1\\ \nu = 3\end{pmatrix}$, erit $p\,r = \frac{3}{4} s\,(1, 3)$, ergo $\frac{p\,r}{s} = \frac{3}{4}(1, 3) =$

4°. Si $\begin{pmatrix}\mu = 1\\ \nu = 4\end{pmatrix}$, erit $p\,s = \frac{4}{5}(1, 4) = D$.

5°. Si $\begin{pmatrix}\mu = 2\\ \nu = 2\end{pmatrix}$, erit $q\,q = s\,(2, 2)$, ergo $\frac{q\,q}{s} = (2, 2) =$

6°. Si $\begin{pmatrix}\mu = 2\\ \nu = 3\end{pmatrix}$, erit $q\,r = \frac{6}{5}(2, 3) = F$.

7°. Si $\begin{pmatrix}\mu = 2\\ \nu = 4\end{pmatrix}$, erit $q\,s = \frac{8}{6} \cdot \frac{6}{5} p\,(2, 4)$, ergo $\frac{q\,s}{p} = \frac{8}{5}(2, 4) = G$.

8°. Si $\begin{pmatrix}\mu = 3\\ \nu = 3\end{pmatrix}$, erit $r\,r = \frac{9}{6} \cdot \frac{6}{5} p\,(3, 3)$, ergo $\frac{r\,r}{p} = \frac{9}{5}(3, 3) = H$.

9°. Si $\begin{pmatrix}\mu = 3\\ \nu = 4\end{pmatrix}$, erit $r\,s = \frac{12}{7} \cdot \frac{7}{5} q\,(3, 4)$, ergo $\frac{r\,s}{q} = \frac{12}{5}(3, 4) = I$.

10°. Si $\begin{pmatrix}\mu = 4\\ \nu = 4\end{pmatrix}$, erit $s\,s = \frac{16}{8} \cdot \frac{8}{5} \nu\,(4, 4)$, ergo $\frac{s\,s}{r} = \frac{16}{5}(4, 4) = K$.

§. 25. Quoniam igitur decem adepti ſumus aequationes, ex quibus quatuor quantitates incognitas definiri oportet: eligamus eas, quibus negotium facillime expedietur. Quarta autem aequatio ſtatim dat $s = \frac{D}{p}$; ex ſexta autem fit $r = \frac{F}{q}$, ita vt tantum ſuperſit binas litteras p et q elicere. Deinde vero ex prima deducimus $q = \frac{p\,p}{A}$, ita vt ſit $r = \frac{A\,F}{p\,p}$. Nunc igitur ex ſecunda aequatione fiet $\frac{p^5}{A\,A\,F} = B$, vnde fit $p = \sqrt[5]{(A\,A\,B\,F)}$, quo valore inuento colligitur fore $q = \sqrt[5]{(\frac{B\,B\,F\,F}{A})}$, $r = \sqrt[5]{(\frac{A\,F^3}{B\,B})}$, denique erit $s = \frac{D}{\sqrt[5]{(A\,A\,B\,E)}}$. Sicque omnes quatuor incognitas per quadraturas ordinarias exprimere licebit. Quodſi iam hos valores in reliquis aequationibus ſubſtituamus, orientur ſequentes aequationes: 1°. $C\,D = A\,F$, 2°. $B\,F = E\,D$, 3°. $D = A\,G$, 4°. $F = B\,H$, 5°. $D = B\,I$, 6°. $D\,D = A\,F\,K$, vnde ob $D = A\,G$ eruitur $D\,G = F\,K$.

§. 26. Ecce ergo ſex nouae prodierunt determinationes, quibus decem noſtrae litterae a ſe inuicem pendent, ita vt e x quatuor pro cognitis aſſumtis reliquae ſex definiri queant,

ant; pro cognitis autem imprimis assumi conueniet binas D et F, quippe quae per quadraturam circuli innotescunt, cum sit

$$D = \tfrac{4}{5}(1,4) = \tfrac{4}{5} \cdot \frac{\pi}{5 \text{ sin. } \frac{1}{5}\pi} \text{ et } F = \tfrac{6}{5}(2,3) = \tfrac{6}{5} \cdot \frac{\pi}{5 \text{ sin. } \frac{2}{5}\pi}.$$

Dummodo ergo duae reliquarum etiam vt cognitae spectentur, caeteras omnes per eas definire licebit. At vero sex illae relationes rite inter se comparatae ternas formulas tam ipsi D quam ipsi F aequales suppeditant, quae sunt $D = AG = BI = CK$ et $F = BH = CG = EI$. Hinc igitur, si praeter D et F etiam litterae A et B pro cognitis assumantur, reliquae litterae ex iis determinabuntur vt sequitur: $C = \frac{AF}{D}$, $E = \frac{BF}{D}$, $G = \frac{D}{A}$, $H = \frac{F}{B}$, $I = \frac{D}{B}$ et $K = \frac{DD}{AF}$.

§. 27. Substituamus nunc loco harum litterarum characteres formularum integralium, atque sex sequentes relationes obtinebuntur:

1°. $(1,4) = (1,1)(2,4)$,
2°. $(1,4) = 2(1,2)(3,4)$,
3°. $(1,4) = 3(1,3)(4,4)$,
4°. $(2,3) = (1,2)(3,3)$,
5°. $(2,3) = (1,3)(2,4)$,
6°. $(2,3) = 2(2,2)(3,4)$;

vnde plura egregia theoremata formari possent.

§. 28. Quoniam ambae litterae D et F, seu potius characteres $(1,4)$ et $(2,3)$ ambo peripheriam circuli inuoluunt, eorum ratio, seu fractio $\frac{(1,4)}{(2,3)}$, algebraice exhiberi poterit, quippe cuius valor est $= \frac{\text{sin. } \frac{2}{5}\pi}{\text{sin. } \frac{1}{5}\pi} = 2 \text{ cos. } \frac{1}{5}\pi$. Hinc etiam sequentes rationes

nes inter binas formulas integrales deriuantur:

$$2 \cos. \tfrac{1}{3} = \frac{(1,1)}{(1,3)} = 2\,\frac{(3,4)}{(3.3)} = \frac{(1,2)}{(2,2)} = 3\,\frac{(4,4)}{(2,4)},$$

vnde iterum eximia theoremata formari possent, si praesens nostrum institutum hoc postularet. Pleniorem autem huius argumenti expositionem in aliam occasionem sum dilaturus.

§. 29. Simili modo quo hic casum $\lambda = 5$ euoluimus, etiam tractare liceret sequentes casus, quibus litterae λ maiores valores tribuuntur. Quoniam autem numerus aequationum continuo secundum numeros trigonales increscit, superfluum foret istum laborem hic suscipere, quoniam omnes operationes analyticae, quibus hae solutiones nituntur, iam satis dilucide sunt expositae.

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