PLENIOR EXPOSITIO SERIERVM ILLARVM MEMORABILIVM, QVAE EX VNCIIS POTESTATVM BINOMII FORMANTVR.

Auctore
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§. 1.

Ad fummationem iftarum progreffionum imprimis me pe duxit idonea fignandi ratio, qua vfus fum ad vncias c iuscunque poteftatis Binomii fuccincte repraefentandas. Scilic poteftatem indefinitam Binomii $(1+z)^n$ per fequentem feri exhibui:

$$(1+z)^n = 1 + (\tfrac{n}{1})\, z + (\tfrac{n}{2})\, zz + (\tfrac{n}{3})\, z^3 + (\tfrac{n}{4})\, z^4 + \text{etc}$$

ita vt poteftatis z^p coëfficiens fit $(\frac{n}{p})$, in quo charactere, forma fractionis expreffo, numerus fuperior n denotat ipf exponentem poteftatis, inferior vero p indicat, quotus fit i coëfficiens ab initio numeratus. Conftat autem ex euolutio huius poteftatis femper effe vt fequitur:

$$(\tfrac{n}{0}) = 1,\; (\tfrac{n}{1}) = n,\; (\tfrac{n}{2}) = \tfrac{n}{1}\cdot\tfrac{n-1}{2},\; (\tfrac{n}{3}) = \tfrac{n}{1}\cdot\tfrac{n-1}{2}\cdot\tfrac{n-}{3}$$

et in genere

$$(\tfrac{n}{p}) = \tfrac{n}{1}\cdot\tfrac{n-1}{2}\cdot\tfrac{n-2}{3}\cdot\ldots\ldots\tfrac{n-p+1}{p},$$

deinde cum vltimus terminus Binomii euoluti fit z^n, e $(\frac{n}{n}) = 1$; et quia vnciae, ab vltimo termino regrediendo, de dem ordinem seruant atque ab initio, erit $(\frac{n}{n-1}) = (\frac{n}{1})$; $(\frac{n}{n-2}) = (\frac{n}{2})$, atque in genere $(\frac{n}{n-p}) = (\frac{n}{p})$. Caeterum quoque hinc manifestum est valorem huius characteris $(\frac{n}{p})$ semper in nihilum abire, quoties fuerit p vel numerus negatiuus, vel positiuus maior quam n.

§. 2. His positis contemplatus sum seriem, cuius singuli termini sunt producta ex binis vnciis duarum quarumcunque potestatum Binomii, ordine inuicem iunctis, cuiusmodi in genere est haec progressio:

$$s = (\frac{m}{0})(\frac{n}{p}) + (\frac{m}{1})(\frac{n}{p+1}) + (\frac{m}{2})(\frac{n}{p+2}) + (\frac{m}{3})(\frac{n}{p+3}) + \text{etc.}$$

donec perueniatur ad terminos euanescentes, quemadmodum etiam termini, qui primum praecederent, euanescunt, atque ostendi talis progressionis summam semper esse $s = (\frac{m+n}{m+p})$, vel etiam $s = (\frac{m+n}{n-p})$. Demonstratio quidem huius veritatis ita comparata videtur, vt tantum pro exponentibus integris m et n valeat; veruntamen iam ostendi, eandem summationem etiam pro exponentibus fractis locum habere, si modo valor characteris $(\frac{m+n}{m+p})$ per notas methodos interpolationum rite definiatur.

§. 3. Ista autem interpolatio commodissime instituitur per formulas integrales logarithmicas. Notum enim est, si ponatur breuitatis gratia $l\frac{1}{x} = u$, atque integralia sequentia perpetuo a termino $x = 0$ vsque ad terminum $x = 1$ extendantur, fore vt sequitur: $\int u \partial x = 1$; $\int u u \partial x = 1.2$; $\int u^3 \partial x = 1.2.3$; $\int u^4 \partial x = 1.2.3.4$, atque in genere

$$\int u^p \partial x = 1.2.3.4 \ldots p.$$

Praeterea vero erit $\int u^0 \partial x = 1$. Sin autem exponens p denotet numerum integrum negatiuum quemcunque, valor integralis $\int u^p \partial x$, semper erit infinitus. Cum enim in genere sit

$$\int u^{p+1}\partial x = 1.\,2.\,3.\,4.\,\dots\,(p+1) = (p+1)\int u^{p}\partial x,$$

erit vicissim

$$\int u^{p}\partial x = \frac{1}{p+1}\int u^{p+1}\partial x.$$

Quare si sumamus $p = -1$, prodit

$$\int \frac{\partial x}{u} = \frac{1}{0}\int u^{0}\partial x = \frac{1}{0} = \infty.$$

Deinde sumto $p = -2$, habebitur:

$$\int \frac{\partial x}{u^{2}} = -\frac{1}{1}\int \frac{\partial x}{u} = -\frac{1.\,1}{0.\,1} = \infty.$$

Vnde patet etiam omnia sequentia integralia euadere infinita. Quando autem p denotat numerum fractum, talis euolutio non amplius locum habere potest, sed contentos nos esse oportet ea quantitate transcendente, quae per formulam $\int u^{p}\partial x$ exprimitur. Ita iam dudum innotuit, si fuerit $p = -\frac{1}{2}$, tum esse $\int \frac{\partial x}{\sqrt{u}} = \sqrt{\pi}$, denotante π peripheriam circuli cuius diameter est $= 1$. Hinc ergo per reductionem ante allatam erit $\int \partial x \sqrt{u} = \frac{1}{2}\sqrt{\pi}$, similique modo porro est

$$\int u^{\frac{3}{2}}\partial x = \frac{1.\,3}{2.\,2}\sqrt{\pi}, \text{ atque vlterius}$$

$$\int u^{\frac{5}{2}}\partial x = \frac{1.\,3.\,5}{2.\,2.\,2}\sqrt{\pi} \text{ et}$$

$$\int u^{\frac{7}{2}}\partial x = \frac{1.\,3.\,5.\,7}{2.\,2.\,2.\,2}\sqrt{\pi},$$

etc.

Quando autem p eiusmodi est fractio, cuius denominator est maior quam 2, tum valores huiusmodi formularum integralium reducuntur ad quadraturas magis transcendentes.

§. 4. His expositis summatio progressionis supra allatae per huiusmodi formulas integrales exhiberi poterit: facile enim perspicitur fore

$$s = \left(\frac{m+n}{m+p}\right) = \frac{\int u^{m+n}\partial x}{\int u^{m+p}\partial x \,.\, \int u^{n-p}\partial x}.$$

Si enim m, n et p fuerint numeri integri positiui, erit vtique $\int u^{m+}$

$$\int u^{m+n} \partial x = 1.\ 2.\ 3.\ 4.\ \ldots\ldots (m+n).$$

Simili modo erit

$$\int u^{m+p} \partial x = 1.\ 2.\ 3.\ 4.\ \ldots\ldots (m+p);$$

$$\int u^{n-p} \partial x = 1.\ 2.\ 3.\ 4.\ \ldots\ldots (n-p);$$

vnde ſequitur fore

$$\frac{\int u^{m+n} \partial x}{\int u^{m+p} \partial x} = (m+p+1)(m+p+2).\ \ldots\ldots (m+n);$$

vbi factorum numerus eſt $= n-p$, quippe qui, ordine retrogrado ſcripti, ſunt

$$(m+n)(m+n-1)(m+n-2).\ \ldots\ldots (m+p+1).$$

Verum hoc productum ſi inſuper diuidatur per

$$\int u^{n-p} \partial x = 1.\ 2.\ 3.\ \ldots\ldots (n-p),$$

vbi factorum numerus pariter eſt $n-p$, reperietur fore

$$\frac{\int u^{m+n} \partial x}{\int u^{m+p} \partial x . \int u^{n-p} \partial x} = \frac{m+n}{1} \cdot \frac{m+n-1}{2} \cdot \frac{m+n-2}{3} \cdot \frac{m+n-3}{4} \ldots \frac{m+p+1}{n-p},$$

atque haec forma manifeſto eſt valor huius characteris $(\frac{m+n}{n-p})$, qui pariter ſummam quaeſitam s indicat. Quamquam autem haec demonſtratio ad numeros integros reſtringi videtur, tamen per principium continuitatis iſta expreſſio, per formulas integrales exhibita, etiam veritati conformis manere debet, quicunque numeri fracti pro litteris m, n et p accipiantur.

§. 5. Huc fere redeunt, quae non ita pridem circa ſummationem huiusmodi progreſſionum ſum commentatus. Nunc autem mihi propoſitum eſt in easdem ſummas per methodum maxime diuerſam, cuius iam nonnulla dedi ſpecimina, inquirere; quo pacto non ſolum ſummatio hic tradita maxime confirmabitur et illuſtrabitur, ſed etiam pro caſibus exponentium fractorum eae curuae algebraicae reperientur, a quarum quadratura ſummationes pendent, cum ante iſtae ſummae per

qua-

quadraturas curuarum transcendentium exprimerentur, ita vt ista noua methodus maximam vtilitatem in Analysin sit illatura; ea autem nititur reductione formularum integralium satis quidem cognita, quam autem ad nostrum vsum in sequentibus Lemmatibus sum accommodaturus.

Lemma I.

§. 6. Si ponatur $V = x^a (1 - x^b)^{\frac{c}{b}}$, erit

$$l V = a l x + \tfrac{c}{b} l (1 - x^b),$$

ac differentiando

$$\frac{\partial V}{V} = \frac{a \partial x}{x} - \frac{c x^{b-1} \partial x}{1 - x^b},$$

hinc per V multiplicando, iterumque integrando, perueniemus ad hanc reductionem:

$$V = x^a (1 - x^b)^{\frac{c}{b}} = a \int x^{a-1} \partial x (1 - x^b)^{c:b} - c \int x^{a+b-1} \partial x (1 - x^b)^{\frac{c-b}{b}}$$

hincque binas sequentes reductiones deducimus:

$$\text{I. } \int x^{a-1} \partial x (1 - x^b)^{\frac{c}{b}} = \tfrac{1}{a} x^a (1 - x^b)^{\frac{c}{b}} + \tfrac{c}{a} \int x^{a+b-1} \partial x (1 - x^b)^{\frac{c}{b} - 1}$$

$$\text{II. } \int x^{a+b-1} \partial x (1 - x^b)^{\frac{c}{b} - 1} = -\tfrac{1}{c} x^a (1 - x^b)^{\frac{c}{b}} + \tfrac{a}{c} \int x^{a-1} \partial x (1 - x^b)^{\frac{c}{b}}$$

Corollarium.

§. 7. Quodsi haec integralia a termino $x = 0$ vsque ad terminum $x = 1$ extendi debeant, atque omnes exponentes a, b etc. fuerint positiui, tum in vtraque reductione membrum algebraicum penitus ex comparatione tollitur, quippe quod euanescit, facto tam $x = 0$ quam $x = 1$, atque binae reductiones inuentae ita se habebunt:

$$\text{I. } \int x^{a-1}\,\partial x\,(1-x^b)^{\frac{c}{b}} = \frac{c}{a}\int x^{a+b-1}\,\partial x\,(1-x^b)^{\frac{c}{b}-1} \text{ et}$$

$$\text{II. } \int x^{a+b-1}\,\partial x\,(1-x^b)^{\frac{c}{b}-1} = \frac{a}{c}\int x^{a-1}\,\partial x\,(1-x^b)^{\frac{c}{b}}.$$

Sin autem exponentes a, b et c non fuerint positiui, in istis reductionibus membrum algebraicum seu absolutum praetermitti nequit, quandoquidem id vel casu $x = 0$ vel casu $x = 1$ in infinitum excrescit. Hic autem semper exponens b tanquam positiuus spectari poterit.

Lemma II.

§. 8. Posito vt ante $V = x^a (1 - x^b)^{\frac{c}{b}}$, si ambae fractiones, quas pro $\frac{\partial V}{V}$ inuenimus, ad communem denominatorem redigamus, habebimus $\frac{\partial V}{V} = \frac{a\,\partial x - (a+c)\,x^b\,\partial x}{x\,(1-x^b)}$. Quodsi iam iterum per V multiplicemus et integremus, perveniemus ad hanc aequationem:

$$V = x^a (1-x^b)^{\frac{c}{b}} = a\int x^{a-1}\,\partial x\,(1-x^b)^{\frac{c}{b}-1}$$
$$- (a+c)\int x^{a+b-1}\,\partial x\,(1-x^b)^{\frac{c}{b}-1},$$

vnde sequuntur hae duae aequationes:

$$\text{I. } \int x^{a-1}\,\partial x\,(1-x^b)^{\frac{c}{b}-1} = \frac{1}{a}\,x^a\,(1-x^b)^{\frac{c}{b}}$$
$$+ \left(\frac{a+c}{a}\right)\int x^{a+b-1}\,\partial x\,(1-x^b)^{\frac{c}{b}-1} \text{ et}$$

$$\text{II. } \int x^{a+b-1}\,\partial x\,(1-x^b)^{\frac{c}{b}-1} = \frac{-1}{a+c}\,x^a\,(1-x^b)^{\frac{c}{b}}$$
$$+ \frac{a}{a+c}\cdot\int x^{a-1}\,\partial x\,(1-x^b)^{\frac{c}{b}-1}.$$

Corollarium.

§. 9. Quodſi haec integralia, quemadmodum in ſequentibus perpetuo aſſumemus, a termino $x = 0$ vsque ad terminum $x = 1$ extendi debeant, atque exponentes a etc. fuerint poſitiui, membra abſoluta praetermittere licebit, ita vt tum ſequentes reductiones locum ſint habiturae:

$$\text{I. } \int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \frac{a+c}{a}\int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1} \text{ et}$$

$$\text{II. } \int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \frac{a}{a+c}\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1}.$$

Lemma III.

§. 10. Poſito denuo $V = x^a(1-x^b)^{\frac{c}{b}}$, quoniam ſupra inuenimus $\frac{\partial V}{V} = \frac{a\,\partial x - (a+c)\,x^b\,\partial x}{x(1-x^b)}$, ſi hic loco prioris membri $a\,\partial x$ ſcribamus $(a+c)\,\partial x - c\,\partial x$, fiet

$$\frac{\partial V}{V} = \frac{\partial x\,(a+c)}{x} - \frac{c\,\partial x}{x(1-x^b)},$$

quae aequatio per V multiplicata et integrata praebet:

$$V = x^a(1-x^b)^{\frac{c}{b}} = (a+c)\int x^{a-1}\,\partial x\,(1-x^b)^{\frac{c}{b}}$$
$$- c\int x^{a-1}\,\partial x\,(1-x^b)^{\frac{c}{b}-1},$$

vnde ſequentes duae reductiones obtinentur:

$$\text{I. } \int x^{a-1}\,\partial x\,(1-x^b)^{\frac{c}{b}} = \frac{1}{a+c}\,x^a(1-x^b)^{\frac{c}{b}}$$
$$+ \frac{c}{a+c}\int x^{a-1}\,\partial x\,(1-x^b)^{\frac{c}{b}-1} \text{ et}$$

$$\text{II. } \int x^{a-1}\,\partial x\,(1-x^b)^{\frac{c}{b}-1} = -\frac{1}{c}\,x^a(1-x^b)^{\frac{c}{b}}$$
$$+ \frac{a+c}{c}\int x^{a-1}\,\partial x\,(1-x^b)^{\frac{c}{b}}.$$

Corol-

Corollarium.

§. 11. Quodsi igitur exponentes a etc. fuerint positiui, et integralia extendi debeant ab $x = 0$ ad $x = 1$, omisso membro absoluto hae reductiones nascentur:

$$\text{I. } \int x^{a-1} \partial x (1 - x^b)^{\frac{c}{b}} = \frac{c}{a+c} \int x^{a-1} \partial x (1 - x^b)^{\frac{c}{b} - 1} \text{ et}$$

$$\text{II. } \int x^{a-1} \partial x (1 - x^b)^{\frac{c}{b} - 1} = \frac{a+c}{c} \int x^{a-1} \partial x (1 - x^b)^{\frac{c}{b}}.$$

Problema 1.

Si, integratione ab $x = 0$ vsque ad $x = 1$ extensa, exponentes a et c fuerint positiui, cognitusque fuerit valor formulae integralis $\int x^{a-1} \partial x (1 - x^b)^{\frac{c}{b} - 1} = \Delta$, per eundem exprimere omnes formulas integrales in hac forma generali contentas:

$$\int x^{a+ib-1} \partial x (1 - x^b)^{\frac{c}{b} - 1}.$$

Solutio.

§. 12. Hic vtendum erit reductione posteriore Lemmatis secundi, in Corollario tradita, quae est

$$\int x^{a+b-1} \partial x (1 - x^b)^{\frac{c}{b} - 1} = \frac{a}{a+c} \int x^{a-1} \partial x (1 - x^b)^{\frac{c}{b} - 1},$$

vbi continuo exponentem a numero b augeamus; et cum per hypothesin sit

$$\int x^{a-1} \partial x (1 - x^b)^{\frac{c}{b} - 1} = \Delta,$$

reductiones reperiuntur sequentes:

$$\text{I. } \int x^{a+b-1} \partial x (1 - x^b)^{\frac{c}{b} - 1} = \frac{a}{a+c} \cdot \Delta.$$

$$\text{II. } \int x^{a+2b-1} \partial x (1 - x^b)^{\frac{c}{b} - 1} = \frac{a}{a+c} \cdot \frac{a+b}{a+b+c} \cdot \Delta.$$

III.

$$\text{III.} \int x^{a+3b-1} \partial x (1-x^b)^{\frac{c}{b}-1} = \frac{a}{a+c} \cdot \frac{a+b}{a+b+c} \cdot \frac{a+2b}{a+2b+c} \cdot \Delta.$$

$$\text{IV.} \int x^{a+4b-1} \partial x (1-x^b)^{\frac{c}{b}-1} = \frac{a}{a+c} \cdot \frac{a+b}{a+b+c} \cdot \frac{a+2b}{a+2b+c} \cdot \frac{a+3b}{a+3b+c} \cdot \Delta.$$

etc. etc.

cuius progreſſionis lex manifeſto in oculos incurrit.

Corollarium 1.

§. 13. Quodſi fuerit $a+c=b$, ideoque $c=b-a$, erit $\Delta = \int \frac{x^{a-1}\partial x}{(1-x^b)^{\frac{a}{b}}}$; vbi quidem tenendum eſt, non ſolum exponentem a debere eſſe poſitiuum, ſed etiam minorem quam b, quia etiam c debet eſſe poſitiuum. Haec autem formula commode ad quadraturam circuli reuocari poteſt, ad quod oſtendendum ponatur $\frac{x}{\sqrt[b]{(1-x^b)}} = y$, vt poſito $x=0$ fiat etiam $y=0$, at facto $x=1$ euadet $y=\infty$; tum autem erit $\Delta = \int \frac{y^a \partial x}{x}$, ſumtisque poteſtatibus exponentis b, erit $y^b = \frac{x^b}{1-x^b}$, vnde reperitur $x^b = \frac{y^b}{1+y^b}$, hincque ſumendis logarithmis erit $b\,l\,x = b\,l\,y - l(1+y^b)$, vnde differentiando colligitur:

$$\frac{\partial x}{x} = \frac{\partial y}{y} - \frac{y^{b-1}\partial y}{1+y^b} = \frac{\partial y}{y(1+y^b)},$$

quo valore ſubſtituto erit $\Delta = \int \frac{y^{a-1}\partial y}{1+y^b}$, quod integrale cum ab $y=0$ vsque ad $y=\infty$ extendi debeat, alia occaſione oſtendi eius valorem eſſe $= \frac{\pi}{b \sin. \frac{a\pi}{b}}$.

Corol-

Corollarium 2.

§. 14. Quamobrem si in genere statuamus $c = b - a$, ita vt sit $\Delta = \frac{\pi}{b \sin. \frac{a\pi}{b}}$, singulae reductiones in problemate inventae ita se habebunt:

$$\text{I. } \int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = \frac{a}{b}\Delta.$$

$$\text{II. } \int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = \frac{a}{b}\cdot\frac{a+b}{2b}\cdot\Delta.$$

$$\text{III. } \int \frac{x^{a+3b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = \frac{a}{b}\cdot\frac{a+b}{2b}\cdot\frac{a+2b}{3b}\cdot\Delta.$$

$$\text{IV. } \int \frac{x^{a+4b-1}\partial x}{\sqrt[b]{(x-x^b)^a}} = \frac{a}{b}\cdot\frac{a+b}{2b}\cdot\frac{a+2b}{3b}\cdot\frac{a+3b}{4b}\cdot\Delta.$$

etc. etc.

vbi euidens est coëfficientes ipsius Δ prorsus conuenire cum vnciis potestatis binomialis $(1-x^b)^{-\frac{a}{b}}$, quippe quae per euolutionem praebet

$$1 + \frac{a}{b}x^b + \frac{a}{b}\cdot\frac{a+b}{2b}\cdot x^{2b} + \frac{a}{b}\cdot\frac{a+b}{2b}\cdot\frac{a+2b}{3b}\cdot x^{3b} + \text{etc.}$$

Problema 2.

§. 15. *Quodsi ponamus breuitatis gratia*

$$(1-x^b)^{-\frac{a}{b}} = 1 + A x^b + B x^{2b} + C x^{3b} + \text{etc.}$$

ita vt sit

$$A = \frac{a}{b},\ B = \frac{a}{b}\cdot\frac{a+b}{2b},\ C = \frac{a}{b}\cdot\frac{a+b}{2b}\cdot\frac{a+2b}{3b},\ \text{etc.}$$

 inuesti-

inuestigare summam huius seriei:

$$S = 1 + A^2 + B^2 + C^2 + D^2 + \text{etc.}$$

Solutio.

Cum igitur sit

$$(1 - x^b)^{-\frac{a}{b}} = 1 + A x^b + B x^{2b} + C x^{3b} + \text{etc.}$$

multiplicemus vtrinque per $\frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}}$, atque integrando habebimus:

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{2a}}} = \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}} + A\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^a}}$$

$$+ B\int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} + C\int \frac{x^{a+3b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} + \text{etc.}$$

Omnes autem has formulas integrales per quantitatem Δ exprimere docuimus, qui valores si substituantur, perueniemus ad sequentem seriem:

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{2a}}} = \Delta + A.\tfrac{a}{b}\Delta + B.\tfrac{a}{b}.\tfrac{a+b}{2b}.\Delta$$

$$+ C.\tfrac{a}{b}.\tfrac{a+b}{2b}.\tfrac{a+2b}{3b}.\Delta \text{ etc.}$$

quae series manifesto reducitur ad hanc:

$$\Delta(1 + A^2 + B^2 + C^2 + D^2 + \text{etc.})$$

vnde seriei nostrae propositae summa quaesita erit

$$S = \frac{1}{\Delta}\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{2a}}},$$

existente $\Delta = \frac{\pi}{b \sin. \frac{a\pi}{b}}$.

Corol-

Corollarium 1.

§. 16. Confideremus hic primo cafum quo $b = 2$, et quia capi debet $a < b$, fit $a = 1$, vnde fit $\Delta = \frac{\pi}{2}$; tum vero pro ferie ipfa habebimus $A = \frac{1}{2}$, $B = \frac{1}{2} \cdot \frac{3}{4}$, $C = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6}$, $D = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8}$, etc. atque feriei

$$1 + A^2 + B^2 + C^2 + D^2 + \text{etc.}$$

fumma erit $S = \frac{2}{\pi} \int \frac{\partial x}{(1 - xx)}$. Eft vero $\int \frac{\partial x}{1 - xx} = \frac{1}{2} l \frac{1 + x}{1 - x}$, qui valor, pofito $x = 1$, abit in infinitum. Eft vero vtique fumma huius feriei:

$$1 + (\tfrac{1}{2})^2 + (\tfrac{1 \cdot 3}{2 \cdot 4})^2 + (\tfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6})^2 + (\tfrac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8})^2 + \text{etc.}$$

infinite magna, quemadmodum alia occafione oftendi.

Corollarium 2.

§. 17. Confideremus quoque cafum $b = 3$, fumamusque $a = 1$, vt exponens $\frac{2a}{b} = \frac{2}{3}$ etiamnunc fit vnitate minor. Hoc igitur cafu pro ipfa ferie habebimus: $A = \frac{1}{3}$, $B = \frac{1}{3} \cdot \frac{4}{6}$, $C = \frac{1}{3} \cdot \frac{4}{6} \cdot \frac{7}{9}$, $D = \frac{1}{3} \cdot \frac{4}{6} \cdot \frac{7}{9} \cdot \frac{10}{12}$, $E = \frac{1}{3} \cdot \frac{4}{6} \cdot \frac{7}{9} \cdot \frac{10}{12} \cdot \frac{13}{15}$, etc. atque ob $\Delta = \frac{2\pi}{3\sqrt{3}}$, fumma feriei $1 + A^2 + B^2 + C^2 +$ etc. erit $S = \frac{3\sqrt{3}}{2\pi} \int \frac{\partial x}{\sqrt[3]{(1 - x^3)^2}}$, quam ergo nunc exprimere licet per quadraturam curuae algebraicae, cuius abfciffae x refpondet applicata $y = \frac{1}{\sqrt[3]{(1 - x^3)^2}}$, pro quo cafu methodus primum tradita praebet quadraturam curuae tranfcendentis.

Scholion.

§. 18. Haec expreffio pro fumma feriei

$$1 + A^2 + B^2 + C^2 + \text{etc.}$$

locum habere nequit, nifi exponens a fuerit pofitiuus, quo

ergo

ergo casu Binomii $1 - x^b$ potestas est negatiua, ideoque se-
ries $1 + A^2 + B^2 + C^2 +$ etc. in infinitum excurrit. Hinc ergo pro vnciis Binomii ad dignitatem positiuam eleuati nihil concludi potest, cum tamen hic casus priore methodo spon- se obtulerit. Deinde cum summa huius seriei inuenta sit

$$S = \frac{1}{\Delta} \int \frac{x^{a-1} \partial x}{\sqrt[b]{(1 - x^b)^{2a}}}, \text{ existente}$$

$$\Delta = \int \frac{x^{a-1} \partial x}{\sqrt[b]{(1 - x^b)^a}};$$

euidens est, si esset $a = b$, quo casu foret $A = 1$, $B =$ $C = 1$, etc. tum summam seriei quadratorum manifesto fo infinitam, id quod multo magis eueniret, si esset $a > b$. Qu etiam, si foret $2a = b$, siue $a = \frac{1}{2} b$, in Corollario primo v dimus etiam hanc summam esse infinitam. Quamobrem su matio hic inuenta restringitur ad hos arctos limites 1°. $a >$ et 2°. $a < \frac{1}{2} b$. Quemadmodum autem hinc etiam summae d finiri queant, quando a est numerus negatiuus, deinceps v debimus.

Problema 3.

§. 19. *Si maneat, vt ante*

$$(1 - x^b)^{-\frac{a}{b}} = 1 + A x^b + B x^{2b} + C x^{3b} + \text{etc.}$$

atque insuper ponatur:

$$(1 - x^b)^{-\frac{\alpha}{b}} = 1 + \mathfrak{A} x^b + \mathfrak{B} x^{2b} + \mathfrak{C} x^{3b} + \text{etc.}$$

ita vt sit

$$\mathfrak{A} = \frac{\alpha}{b}, \; \mathfrak{B} = \frac{\alpha}{b} \cdot \frac{\alpha + b}{2b}, \; \mathfrak{C} = \frac{\alpha}{b} \cdot \frac{\alpha + b}{2b} \cdot \frac{\alpha + 2b}{3b}, \text{ etc.}$$

inuenire summam seriei ex his binis vnciarum seriebus composita

$$S = 1 + \mathfrak{A} A + \mathfrak{B} B + \mathfrak{C} C + \mathfrak{D} D + \text{etc.}$$

Sol

Solutio.

Posito vt in antecedente Problemate

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = \Delta,$$

ita vt sit $\Delta = \frac{\pi}{b \sin. \frac{a\pi}{b}}$, si quidem fuerit $a > 0$, reductiones ibi adhibitae dabunt:

$$\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = A\Delta,$$

$$\int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = B\Delta,$$

$$\int \frac{x^{a+3b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = C\Delta,$$

$$\int \frac{x^{a+4b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = D\Delta,$$

etc. etc.

Cum igitur sit

$$(1-x^b)^{-\frac{\alpha}{b}} = 1 + \mathfrak{A}x^b + \mathfrak{B}x^{2b} + \mathfrak{C}x^{3b} + \text{etc.}$$

si vtrinque multiplicemus per $\frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}}$ et integremus a termino $x = 0$ vsque ad $x = 1$, perueniemus ad sequentem seriem:

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} = \Delta + \mathfrak{A}A\Delta + \mathfrak{B}B\Delta + \mathfrak{C}C\Delta + \text{etc.}$$

quae est ipsa series quaesita per Δ multiplicata, ideoque eius summa $= \Delta$ S. Hinc ergo vicissim concludimus fore

$$S = 1 + \mathfrak{A}A + \mathfrak{B}B + \mathfrak{C}C + \text{etc.} = \frac{1}{\Delta}\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}}.$$

Haec autem summatio pariter locum habere nequit, nisi sit $a > 0$. At vero circa exponentem α hic nobis nihil praescribitur; vnde pro eo tam numeros positiuos quam negatiuos accipere licebit. Id tantum hic obseruandum occurrit: nisi fuerit $a + \alpha < b$, summam seriei propositae semper esse infinite magnam.

Corollarium 1.

§. 20. Quoniam a contineri debet intra limites 0 et b, sumamus $b = 2$, capique oportet $a = 1$, vnde fit $A = \frac{1}{2}$, $B = \frac{1}{2} \cdot \frac{3}{4}$, $C = \frac{1.3.5}{2.4.6}$, $D = \frac{1.3.5.7}{2.4.6.8}$, etc. praeterea vero habebimus:

$$\Delta = \int \frac{\partial x}{\sqrt{(1 - xx)}} = \frac{\pi}{2}.$$

Hinc igitur, quicunque valor ipsi α tribuatur, seriei quaesitae

$$S = 1 + \tfrac{1}{2}\mathfrak{A} + \tfrac{1.3}{2.4}\mathfrak{B} + \tfrac{1.3.5}{2.4.6}\mathfrak{C} + \text{etc.}$$

summa erit

$$S = \frac{2}{\pi}\int \frac{\partial x}{\sqrt{(1 - xx)^{1+\alpha}}}.$$

Vnde patet, dummodo fuerit $1 + \alpha < 2$, ideoque $\alpha < 1$, summam semper fore finitam.

Corollarium 2.

§. 21. Manente igitur $a = 1$ et $b = 2$, quia debet esse $\alpha < 1$, quosdam casus euoluamus.

I.

I. Sit $a = 0$.

Hinc fiet $\mathfrak{A} = 0$, $\mathfrak{B} = 0$, etc. ficque feries fummanda erit $S = 1$, noftra autem formula praebet $S = \frac{2}{\pi}\int\frac{\partial x}{\sqrt{(1 - xx)}}$. Eft vero pro terminis fummationis praefcriptis $\int\frac{\partial x}{\sqrt{(1 - xx)}} = \frac{\pi}{2}$, vnde fit $S = 1$, id quod egregie conuenit.

II. Sit $a = -1$.

Hoc cafu fiet $\mathfrak{A} = -\frac{1}{2}$, $\mathfrak{B} = -\frac{1 \cdot 1}{2 \cdot 4}$, $\mathfrak{C} = -\frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}$, etc. vnde feries fummanda erit

$$S = 1 - \frac{1 \cdot 1}{2 \cdot 2} - \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1 \cdot 1}{2 \cdot 4} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} - \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} - \text{etc.}$$

At vero noftra formula praebet

$$S = \frac{2}{\pi}\int\frac{\partial x}{\sqrt{(1 - xx)^0}} = \frac{2}{\pi},$$

quae fumma egregie conuenit cum ea, quam ex formulis integralibus logarithmicis eruimus.

III. Sit $a = -2$.

Fiet hic $\mathfrak{A} = -1$, $\mathfrak{B} = 0$, $\mathfrak{C} = 0$, etc. Series ergo fummanda erit $S = 1 - \frac{1}{2} = \frac{1}{2}$; formula vero noftra integralis praebet $S = \frac{2}{\pi}\int\partial x\sqrt{(1 - xx)}$. Iam vero ex Corollario Lemmatis tertii habemus hanc reductionem:

$$\int x^{a-1}\partial x(1 - x^b)^{\frac{c}{b}} = \frac{c}{a + c}\int x^{a-1}\partial x(1 - x^b)^{\frac{c}{b} - 1},$$

quae, ad noftrum cafum accommodata, ponendo $a = 1$, $b = 2$, $c = 1$, dat

$$\int\partial x\sqrt{(1 - xx)} = \frac{1}{2}\int\frac{\partial x}{\sqrt{(1 - xx)}} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4},$$

hinc ergo fit $S = \frac{1}{2}$.

IV. Sit $a = -3$.

Hoc ergo cafu fiet

$\mathfrak{A} = -\frac{3}{2}$, $\mathfrak{B} = +\frac{3 \cdot 1}{2 \cdot 4}$, $\mathfrak{C} = \frac{3 \cdot 1 \cdot 1}{2 \cdot 4 \cdot 6}$, $\mathfrak{D} = \frac{3 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6 \cdot 8}$, $\mathfrak{E} = \frac{3 \cdot 1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10}$, etc.

vnde

vnde series summanda erit

$$S = 1 - \tfrac{1}{2} \cdot \tfrac{3}{2} + \tfrac{1.3}{2.4} \cdot \tfrac{3.1}{2.4} + \tfrac{1.3.5}{2.4.6} \cdot \tfrac{3.1.1}{2.4.6} + \tfrac{1.3.5.7}{2.4.6.8} \cdot \tfrac{3.1.1.3}{2.4.6.8} + \text{etc.}$$

at vero formula nostra integralis praebet:

$$S = \tfrac{2}{\pi} \int \partial x \, (1 - x x).$$

Est vero pro terminis summationis stabilitis $\int \partial x (1 - x x) = \tfrac{2}{3}$, quamobrem summa quaesita erit $S = \tfrac{4}{3\pi}$.

V. Sit $\alpha = -4$.

Hoc ergo casu fiet

$$\mathfrak{A} = -2, \ \mathfrak{B} = 1, \ \mathfrak{C} = 0, \ \mathfrak{D} = 0, \ \text{etc.}$$

vnde series summanda erit

$$S = 1 - 1 + \tfrac{1.3}{2.4} = \tfrac{1.3}{2.4} = \tfrac{3}{8},$$

at vero formula integralis praebet:

$$S = \tfrac{2}{\pi} \int \partial x \, \sqrt{(1 - x x)^3} = \tfrac{2}{\pi} \int \partial x \, (1 - x x)^{\frac{3}{2}}.$$

Nunc vero per reductionem casu III. adhibitam, sumto $a = 1$, $b = 2$ et $c = 3$, habebimus

$$\int \partial x \, (1 - x x)^{\frac{3}{2}} = \tfrac{3}{4} \int \partial x \, \sqrt{(1 - x x)}.$$

Vidimus autem esse $\int \partial x \sqrt{(1 - x x)} = \tfrac{\pi}{4}$, vnde erit

$$\int \partial x \, (1 - x x)^{\frac{3}{2}} = \tfrac{3\pi}{16},$$

ex quo colligitur summa quaesita $S = \tfrac{3}{8}$, id quod egregie convenit.

Problema 4.

§. 22. *Retentis litteris tam Latinis A, B, C, D, etc. quam Germanicis $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, etc. et iisdem valoribus, quos ipsis in praecedente problemate assignauimus, inuestigare summas sequentium serierum ex illis compositarum:*

$S' =$

$$S' = A + \mathfrak{A} B + \mathfrak{B} C + \mathfrak{C} D + \mathfrak{D} E + \text{etc.}$$
$$S'' = B + \mathfrak{A} C + \mathfrak{B} D + \mathfrak{C} E + \mathfrak{D} F + \text{etc.}$$
$$S''' = C + \mathfrak{A} D + \mathfrak{B} E + \mathfrak{C} F + \mathfrak{D} G + \text{etc.}$$
etc. etc.

Solutio.

Posito iterum $\Delta = \int \frac{x^{a-1} \partial x}{\sqrt[b]{(1-x^b)^a}}$; in praecedente problemate vidimus esse

$$\int \frac{x^{a+b-1} \partial x}{\sqrt[b]{(1-x^b)^a}} = A \Delta,$$
$$\int \frac{x^{a+2b-1} \partial x}{\sqrt[b]{(1-x^b)^a}} = B \Delta,$$
$$\int \frac{x^{a+3b-1} \partial x}{\sqrt[b]{(1-x^b)^a}} = C \Delta.$$
etc. etc.

Cum iam posuerimus

$$(1-x^b)^{\frac{-\alpha}{b}} = 1 + \mathfrak{A} x^b + \mathfrak{B} x^{2b} + \mathfrak{C} x^{3b} + \text{etc.}$$

multiplicemus vtrinque statim per $\frac{x^{a+b-1} \partial x}{\sqrt[b]{(1-x^b)^a}}$, et integrando nanciscemur sequentem formam:

$$\int \frac{x^{a+b-1} \partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} = A\Delta + \mathfrak{A} B \Delta + \mathfrak{B} C \Delta + \mathfrak{C} D \Delta + \mathfrak{D} E \Delta + \text{etc.}$$

quae series manifesto est $= \Delta S'$. Hinc ergo concludimus fore

$$S' = \frac{1}{\Delta} \int \frac{x^{a+b-1} \partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}},$$

vbi, vt hactenus, eſt

$$\Delta = \int \frac{x^{a-1} \partial x}{\sqrt[b]{(1-x^b)^a}} = \frac{\pi}{b \text{ ſin.} \frac{a\pi}{b}}.$$

Pro ſecunda ſerie inuenienda multiplicetur forma illa:

$$(1-x^b)^{-\frac{\alpha}{b}} = 1 + \mathfrak{A} x^b + \mathfrak{B} x^{2b} + \mathfrak{C} x^{3b} + \text{etc.}$$

per formulam $\frac{x^{a+2b-1} \partial x}{\sqrt[b]{(1-x^b)^a}}$, et ſingulis terminis integratis deducemur ad ſequentem formam:

$$\int \frac{x^{a+2b-1} \partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} = B\Delta + \mathfrak{A} C\Delta + \mathfrak{B} D\Delta + \mathfrak{C} E\Delta + \mathfrak{D} F\Delta + \text{etc.}$$

quae eſt ſecunda ſeries propoſita in Δ ducta, ideoque $\Delta S''$, vnde concludimus fore

$$S'' = \frac{1}{\Delta} \int \frac{x^{a+2b-1} \partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}}.$$

Ex his iam ſatis perſpicitur, quomodo omnium ſequentium ſerierum propoſitarum ſummae aſſignari queant: reperitur enim vt ſequitur:

$$S''' = \frac{1}{\Delta} \int \frac{x^{a+3b-1} \partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}};$$

$$S^{IV} = \frac{1}{\Delta} \int \frac{x^{a+4b-1} \partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}};$$

$$S^{V} = \frac{1}{\Delta} \int \frac{x^{a+5b-1} \partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}};$$

etc.

hinc

hincque porro in genere concluditur fore

$$S^{(n)} = \frac{1}{\Delta} \int \frac{x^{a+nb-1} \partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}}.$$

Verum hic adhuc conditio supra praescripta locum habet, qua valor exponentis a inter limites o et b contineri debet. Deinde vero circa exponentem α pariter notari oportet, has summas finitas esse non posse, nisi sit $\alpha + a < b$. Quamobrem nunc dispiciamus, quemadmodum has summationes etiam ad alios valores exponentis a accommodari conveniat.

Problema 5.

§. 23. *Si exponens* a *fuerit numerus negatiuus, minor tamen quam b, ita vt sit $a + b > 0$, inuenire summam seriei*

$$S = 1 + \mathfrak{A}A + \mathfrak{B}B + \mathfrak{C}C + \mathfrak{D}D + \text{etc.}$$

vbi litterae maiusculae eosdem habeant valores vt hactenus, scilicet:

$$(1-x^b)^{-\frac{a}{b}} = 1 + Ax^b + Bx^{2b} + Cx^{3b} + Dx^{4b} + \text{etc.}$$

$$(1-x^b)^{-\frac{\alpha}{b}} = 1 + \mathfrak{A}x^b + \mathfrak{B}x^{2b} + \mathfrak{C}x^{3b} + \mathfrak{D}x^{4b} + \text{etc.}$$

Solutio.

Cum exponens $a + b$ sit positiuus, reductiones supra §. 12. exhibitas a secunda inchoemus, ac iam ponamus: $\int \frac{x^{a+b-1} \partial x}{\sqrt[b]{(1-x^b)^{a}}} = \Delta'$, vnde reductiones supra exhibitae reuocabuntur ad sequentes:

∫

$$\int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = \frac{a+b}{2b}\Delta' = \frac{b}{a}\,\mathrm{B}\,\Delta',$$

$$\int \frac{x^{a+3b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = \frac{a+b}{2b}\cdot\frac{a+2b}{3b}\Delta' = \frac{b}{a}\,\mathrm{C}\,\Delta',$$

$$\int \frac{x^{a+4b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = \frac{a+b}{2b}\cdot\frac{a+2b}{3b}\cdot\frac{a+3b}{4b}\Delta' = \frac{b}{a}\,\mathrm{D}\,\Delta',$$

$$\int \frac{x^{a+5b-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = \frac{a+b}{2b}\cdot\frac{a+2b}{3b}\cdot\frac{a+3b}{4b}\cdot\frac{a+4b}{5b}\Delta' = \frac{b}{a}\,\mathrm{E}\,\Delta'.$$

etc. etc.

His praenotatis confideremus hanc aequationem:

$$(1-x^b)^{-\frac{\alpha}{b}} - 1 = \mathfrak{A}x^b + \mathfrak{B}x^{2b} + \mathfrak{C}x^{3b} + \mathfrak{D}x^{4b} + \text{etc.}$$

quam per $\dfrac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}}$ multiplicemus et integremus, atque obtinebimus fequentem aequationem:

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} - \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}}$$
$$= \mathfrak{A}\Delta' + \frac{b}{a}\mathfrak{B}\,\mathrm{B}\,\Delta' + \frac{b}{a}\mathfrak{C}\,\mathrm{C}\,\Delta' + \frac{b}{a}\mathfrak{D}\,\mathrm{D}\,\Delta' + \text{etc.}$$

vbi loco primi termini $\mathfrak{A}\Delta'$ fcribamus $\frac{b}{a}\mathfrak{A}\,\mathrm{A}\,\Delta'$, vt feries ad hanc formam redigatur:

$$\frac{b}{a}\Delta'(\mathfrak{A}\,\mathrm{A} + \mathfrak{B}\,\mathrm{B} + \mathfrak{C}\,\mathrm{C} + \mathfrak{D}\,\mathrm{D} + \text{etc.}) = \frac{b}{a}\Delta'(\mathrm{S}-1).$$

Quo-

Quoniam autem exponens a ſupponitur negatiuus, vnde ambae formulae integrales euaderent infinitae, vtamur reductione in Lemmate I. tradita:

$$\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}} = \tfrac{1}{a}x^a(1-x^b)^{\frac{c}{b}} + \tfrac{c}{a}\int x^{a+b-1}\partial x(1-x^b)^{\frac{c}{b}-1},$$

et facta applicatione ad priorem formulam integralem, ſumendo $c = -a-\alpha$, erit

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} = \tfrac{1}{a}x^a(1-x^b)^{\frac{-a-\alpha}{b}}$$

$$-\tfrac{(a+\alpha)}{a}\int x^{a+b-1}\partial x(1-x^b)^{\frac{-a-\alpha-b}{b}}.$$

Pro altera autem formula noſtra integrali ſumi debet $c = -a$, fietque

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = \tfrac{1}{a}x^a(1-x^b)^{\frac{-a}{b}}\int x^{a+b-1}\partial x(1-x^b)^{\frac{-a-b}{b}}.$$

Cum iam exponens $a+b$ ſit poſitiuus, per reductionem in Corollario Lemmatis III. quae erat

$$\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}-1} = \tfrac{a+c}{c}\int x^{a-1}\partial x(1-x^b)^{\frac{c}{b}},$$

pro caſu poſterioris formulae habebimus

$$\int x^{a+b-1}\partial x(1-x^b)^{\frac{-a-b}{b}} = -\tfrac{b}{a}\int x^{a+b-1}\partial x(1-x^b)^{\frac{-a}{b}} = -\tfrac{b}{a}\Delta',$$

ſicque formula noſtra integralis poſterior ita exprimetur, vt ſit

$$\int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^a}} = \tfrac{1}{a}x^a(1-x^b)^{\frac{-a}{b}} + \tfrac{b}{a}\Delta',$$

qui valor a priore formula integrali ſubtractus relinquit pro parte ſiniſtra hanc expreſſionem:

$$\tfrac{1}{a} x^a (1 - x^b)^{\frac{-a-\alpha}{b}} - \tfrac{(a+\alpha)}{b} \int \frac{x^{a+b-1}\, \partial x}{\sqrt[b]{(1 - x^b)^{a+\alpha+\beta}}}$$

$$= -\tfrac{1}{a} x^a (1 - x^b)^{\frac{-a}{b}} - \tfrac{b}{a} \Delta'.$$

Hic quidem, quoniam a supponitur negatiuum, vtrum-que membrum absolutum, posito $x = 0$, abit in infinitum; bina coniunctim ita repraesententur:

$$\tfrac{1}{a} x^a (1 - x^b)^{\frac{-a}{b}} [(1 - x^b)^{\frac{-\alpha}{b}} - 1],$$

quae forma, sumto x infinite paruo, ob $(1 - x^b)^{\frac{\alpha}{b}} = 1 + \frac{\alpha}{b} x^b$ etc. transmutatur in hanc: $\frac{\alpha}{ab} x^{a+b} (1 - x^b)^{\frac{-a}{b}}$, quae si $a + b > 0$, facto $x = 0$, vtique euanescit, prouti conditio integrationis postulat. Posito autem $x = 1$ quoque totum membrum absolutum euanescit; quamobrem pro membro dextro nostrae aequationis habebimus:

$$-\tfrac{b}{a} \Delta' - (\tfrac{a+\alpha}{a}) \int \frac{x^{a+b-1}\, \partial x}{\sqrt[b]{(1 - x^b)^{a+\alpha+\beta}}},$$

cui si membrum sinistrum $\frac{b}{a} \Delta' (S - 1)$ aequale ponatur, impetrabimus istum valorem:

$$S = -\frac{(a+\alpha)}{b\,\Delta'} \int \frac{x^{a+b-1}\, \partial x}{\sqrt[b]{(1 - x^b)^{a+\alpha+\beta}}},$$

quae expressio iam pro omnibus casibus valet, quibus $a + b$ est numerus positiuus.

Corollarium 1.

§. 24. Quoniam in paragrapho praecedente inuenimus

$$\int \frac{x^{a+b-1}\, \partial x}{\sqrt[b]{(1 - x^b)^{a+b}}} = -\frac{b}{a} \Delta',$$

notum

notum est istius formulae integralis valorem, ab $x = 0$ vsque ad $x = 1$ extensum, reduci ad hanc formam: $\frac{\pi}{b \sin. \frac{(a+b)}{b} \pi}$, vnde innotescit quantitas charactere Δ' contenta, quippe quae erit $\Delta' = \frac{-\pi a}{b b \sin. \frac{(a+b)}{b} \pi}$, qui valor ita reducitur:

$$\Delta' = \frac{\pi a}{b b \sin. \frac{a \pi}{b}}$$

Corollarium 2.

§. 25. Restituamus autem loco Δ' ipsam formulam integralem, vnde prodiit, atque summa inuenta S hoc modo per binas formulas integrales exprimetur:

$$S = -\frac{(a+\alpha)}{b} \int \frac{x^{a+b-1} \partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+b}}} : \int \frac{x^{a+b-1} \partial x}{\sqrt[b]{(1-x^b)^{a}}},$$

quae etiam hoc modo exprimi potest:

$$S = \frac{(a+\alpha)}{a} \int \frac{x^{a+b-1} \partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+b}}} : \int \frac{x^{a+b-1} \partial x}{\sqrt[b]{(1-x^b)^{a+b}}},$$

quae expressio ergo valet, quando $a + b > 0$, etiamsi forte a sit negatiuum; at vero pro casibus, quibus exponens a ipse est negatiuus, pro eadem serie inuenimus summam

$$S = \int \frac{x^{a-1} \partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} : \int \frac{x^{a-1} \partial x}{\sqrt[b]{(1-x^b)^{a}}}.$$

Corollarium 3.

§. 26. Quodsi has binas formas accuratius perpendamus, mox deprehendemus, formam hic inuentam ex praecedente

dente facile deriuari posse ope reductionis prioris in Corollario Lemmatis primi monstratae, vbi erat

$$\int x^{a-1} \partial x (1 - x^b)^{\frac{c}{b}} = \frac{c}{a} \int x^{a+b-1} \partial x (1 - x^b)^{\frac{c}{b} - 1}.$$

Quodsi enim hanc reductionem applicemus ad formam supra pro S inuentam, pro numeratore erit $c = -a - \alpha$, vnde ipse numerator hoc modo transmutatur:

$$\int \frac{x^{a-1} \partial x}{\sqrt[b]{(1 - x^b)^{a+\alpha}}} = -\frac{(a+\alpha)}{a} \int \frac{x^{a+b-1} \partial x}{\sqrt[b]{(1 - x^b)^{a+\alpha+\beta}}}.$$

Deinde vero pro denominatore erit $c = -a$, ideoque ipse denominator:

$$\int \frac{x^{a-1} \partial x}{\sqrt[b]{(1 - x^b)^a}} = -\int \frac{x^{a+b-1} \partial x}{\sqrt[b]{(1 - x^b)^{a+b}}};$$

vbi euidens est, si numerator per denominatorem diuidatur, eum ipsum valorem resultare, quem hoc problemate sumus nacti.

Scholion.

§. 27. Quamquam igitur expressio supra inuenta pro summatione seriei:

$$S = 1 + \mathfrak{A} A + \mathfrak{B} B + \mathfrak{C} C + \text{etc.}$$

quae ita se habet:

$$S = \int \frac{x^{a-1} \partial x}{\sqrt[b]{(1 - x^b)^{a+\alpha}}} : \int \frac{x^{a-1} \partial x}{\sqrt[b]{(1 - x^b)^a}},$$

tantum valet pro casibus quibus $a > 0$, tamen ex ea facile immediate deduci potuisset expressio pro S hic inuenta, quae etiam valet, dummodo fuerit $a + b > 0$, quam hic non sine longis ambagibus sumus adepti; at vero ratio hic difficulter perspicitur, ob quam tali reductione vti liceat, quandoquidem reduc-

reductio §. 7. tradita subsistere nequit, nisi exponens a fuerit positiuus, propterea quod pars absoluta est neglecta, vnde vtique reductio tam numeratoris quam denominatoris seorsim spectata foret erronea; verum ambo errores, tam in numeratore quam in denominatore commissi, feliciter se mutuo compensant. Quamobrem hac noua methodo tuto vti poterimus, quando exponenti a adhuc maiores valores negatiui tribuuntur.

Problema 6.

§. 28. *Retineant litterae maiusculae, tam Latinae quam Germanicae, eosdem valores, quos ipsis hactenus assignauimus, definire summam seriei:*

$$S = 1 + \mathfrak{A} A + \mathfrak{B} B + \mathfrak{C} C + \mathfrak{D} D + \text{etc.}$$

quando exponens a *valores negatiuos quantumuis magnos accipit.*

Solutio.

Pro casibus, quibus exponens a est positiuus, summa istius seriei ita exprimitur, vt sit

$$S = \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha}}} : \int \frac{x^{a-1}\partial x}{\sqrt[b]{(1-x^b)^{a}}}.$$

Deinde vero pro valoribus negatiuis ipsius a, si modo fuerit $a+b>0$, per reductionem §. 7. modo inuenimus:

$$S = \left(\frac{a+\alpha}{a}\right) \int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+\beta}}} : \int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+b}}}.$$

Sin autem demum formula $a+2b$ fiat positiua, reductionem expositam ad formam proxime praecedentem applicemus, sumique debebit $a = a+b$ et $c = -a-\alpha-b$, pro numeratore; at vero pro denominatore $c = -a-b$; vnde reperitur

$$\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+b}}} = -\frac{(a+\alpha+b)}{a+b}\int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+2b}}}$$ et

$$\int \frac{x^{a+b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+b}}} = -\int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+2b}}},$$

quibus valoribus in postrema expressione pro S substitutis, pro casu $a+2b>0$, reperiemus

$$S = \frac{a+\alpha}{a}\cdot\frac{a+\alpha+b}{a+b}\int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+2b}}} : \int \frac{x^{a+2b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+2b}}}.$$

Sin autem $a+3b$ positiuum obtineat valorem, similis reductio perducet ad sequentem expressionem:

$$S = \frac{a+\alpha}{a}\cdot\frac{a+\alpha+b}{a+b}\cdot\frac{a+\alpha+2b}{a+2b}\int \frac{x^{a+3b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+3b}}} : \int \frac{x^{a+3b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+3b}}}.$$

Similique modo si demum formula $a+4b$ ad positiuum valorem assurgat, summa quaesita reperietur

$$S = \frac{a+\alpha}{a}\cdot\frac{a+\alpha+b}{a+b}\cdot\frac{a+\alpha+2b}{a+2b}\cdot\frac{a+\alpha+3b}{a+3b}\int \frac{x^{a+4b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+\alpha+4b}}} : \int \frac{x^{a+4b-1}\partial x}{\sqrt[b]{(1-x^b)^{a+4b}}}.$$

In omnibus his formulis denominatores ad arcum circularem se reduci patiuntur. Cum enim forma generalis denominatorum sit $\int \frac{x^{a+nb-1}\partial x}{\sqrt[b]{(1-x)^{a+nb}}}$, per ea quae supra sunt ostensa patet eius valorem esse $\frac{\pi}{b\,\text{sin.}\,(\frac{a+nb}{b})\pi}$; vbi cum sit $\text{sin.}\,(\frac{a+nb}{b})\pi = \text{sin.}\,(n\pi + \frac{a}{b}\pi)$, euidens est casibus, quibus n est numerus par, fore denominatorem $= \frac{\pi}{b\,\text{sin.}\,\frac{a\pi}{b}}$; casibus autem, quibus n est numerus impar, denominator erit: $= \frac{-\pi}{b\,\text{sin.}\,\frac{a\pi}{b}}$.

Caete-

Caeterum omnes iſtae formulae prorſus inter ſe conuenire ſunt cenſendae, quoniam omnes ex prima per reductiones ſupra traditas ſunt deductae, ſi modo partes abſolutae negligantur. Tanta enim eſt vis harum formularum, vt, etiamſi reductiones iſtae tam pro numeratore quam denominatore ſeorſim ſumtae falſae eſſent futurae, tamen hi duo errores ſe mutuo iterum deſtruant.

Exemplum.

§. 29. Sumamus $b = 2$, ſitque $a = -5$ et $\alpha = -4$, vnde hae ſeries naſcuntur:

$$(1-xx)^{\frac{5}{2}} = 1 - \tfrac{5}{2}xx + \tfrac{5.3}{2.4}x^4 - \tfrac{5.3.1}{2.4.6}x^6 - \tfrac{5.3.1.1}{2.4.6.8}x^8 - \tfrac{5.3.1.1.3}{2.4.6.8.10}x^{10} - \text{etc. et}$$

$$(1-xx)^2 = 1 - 2xx + x^4,$$

vnde ſeries ſummanda erit

$$S = 1 + 5 + \tfrac{15}{8} = \tfrac{63}{8}.$$

Iam quoniam hic demum $a + 3b$ fit quantitas poſitiua, nempe $a + 3b = 1$, vtendum erit formula ſecunda, vnde colligitur:

$$S = \frac{9.7.5}{5.3.1}\int \frac{\partial x}{\sqrt{(1-xx)^{-3}}} : \int \frac{\partial x}{\sqrt{(1-xx)}},$$

vbi denominator eſt $\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{2}$; at vero numerator eſt $\int \partial x \sqrt{(1-xx)^3}$, quae formula per reductionem priorem §. 10. poſito $a = 1$ et $b = 2$ praebet:

$$\int \partial x (1-xx)^{\frac{c}{2}} = \tfrac{c}{1+c}\int \partial x (1-xx)^{\frac{c}{2}-1},$$

vnde ob $c = 3$ fiet

$$\int \partial x (1-xx)^{\frac{3}{2}} = \tfrac{3}{4}\int \partial x (1+xx)^{\frac{1}{2}}.$$

Porro autem quia hic $c = 1$, erit

$$\int \partial x (1-xx)^{\frac{1}{2}} = \tfrac{1}{2}\int \frac{\partial x}{\sqrt{(1-xx)}} = \tfrac{\pi}{4},$$

vnde

vnde pro numeratore habebimus:

$$\int \partial x\,(1 - x\,x)^{\frac{3}{2}} = \tfrac{3\pi}{16},$$

quibus valoribus fubftitutis fumma quaefita prodit:

$$S = \tfrac{9.\,7.\,5}{5.\,3.\,1}\,.\,\tfrac{3\pi}{16} : \tfrac{\pi}{2} = \tfrac{63}{8},$$

id quod egregie conuenit cum vera fumma.

Alia methodus earundum ferierum fummas inueniendi.

§. 30. Praecedentes fummas elicuimus ex ipfa indole ferierum, qua finguli termini funt producta ex binis vnciis duarum poteftatum Binomii. Quoniam autem non ita pridem (*) demonftraui, fi feries ita formetur, vt fit

$$S = 1 + (\tfrac{m}{1})(\tfrac{n}{1}) + (\tfrac{m}{2})(\tfrac{n}{2}) + (\tfrac{m}{3})(\tfrac{n}{3}) + (\tfrac{m}{4})(\tfrac{n}{4}) + \text{etc.}$$

tum fore $S = (\frac{m+n}{m})$, vel etiam $S = (\frac{m+n}{n})$, cuius valor more folito euolutus praebet:

$$S = \tfrac{m+n}{1}\,.\,\tfrac{m+n-1}{2}\,.\,\tfrac{m+n-2}{3}\,.\,\tfrac{m+n-3}{4}\,\ldots\ldots\,\tfrac{m+1}{n},$$

qui ergo valor, quoties m et n funt numeri integri, femper facile affignari poteft. Quando autem pro his numeris fractiones accipiuntur, huius expreffionis valorem fequenti modo per formulas integrales exhibui, vt fit

$$S = \frac{\int u^{m+1}\,\partial x}{\int u^m\,\partial x\,.\int u^n\,\partial x},$$

exiftente $u = l\,\frac{1}{x}$, tum vero integralibus ab $x = 0$ vsque ad $x = 1$ extenfis.

§. 31. Ifta autem expreffio ad noftrum inftitutum non fatis idonea videtur, propterea quod quadraturas curuarum tranfcendentium inuoluit; at vero, re penitius confiderata, inueni valo-

(*) V. Acta Acad. Imp. Sc. pro anno 1781. P. I, pag. 94 et 95.

valorem eiusdem formulae $(\frac{m+n}{n})$ etiam posse ad quadraturas curuarum algebraicarum reuocari, quae adeo simpliciores prodierunt quam illae, quas methodo praecedenti sumus adepti, neque etiam eo incommodo laborant, vt pro diuersis exponentibus alias atque alias reductiones postulent. Hanc igitur nouam methodum hic clarius sum expositurus.

§. 32. Haec autem methodus deducta est ex reductionibus supra §. 12. allatis, vbi posuimus

$$\Delta = \int x^{a-1} \partial x \, (1 - x^b)^{\frac{c}{b} - 1}.$$

Hic autem sumamus statim $b = 1$ et $a = 1$, ita vt sit:

$$\Delta = \int \partial x \, (1 - x)^{c-1} = \tfrac{1}{c},$$

tum autem reductiones §. 12. allatae sequenti modo se habebunt:

$$\int x \, \partial x \, (1 - x)^{c-1} = \frac{1}{c+1} \cdot \Delta,$$

$$\int x x \, \partial x \, (1 - x)^{c-1} = \frac{1}{c+1} \cdot \frac{2}{c+2} \cdot \Delta,$$

$$\int x^3 \, \partial x \, (1 - x)^{c-1} = \frac{1}{c+1} \cdot \frac{2}{c+2} \cdot \frac{3}{c+3} \cdot \Delta,$$

$$\int x^4 \, \partial x \, (1 - x)^{c-1} = \frac{1}{c+1} \cdot \frac{2}{c+2} \cdot \frac{3}{c+3} \cdot \frac{4}{c+4} \cdot \Delta,$$

etc. etc.

vnde concluditur in genere fore:

$$\int x^\lambda \, \partial x \, (1 - x)^{c-1} = \frac{1}{c+1} \cdot \frac{2}{c+2} \cdot \frac{3}{c+3} \cdots \frac{\lambda}{c+\lambda} \cdot \Delta.$$

§. 33. Quodsi iam postrema formula inuertatur, reperietur

$$\frac{\Delta}{\int x^\lambda \, \partial x \, (1 - x)^{c-1}} = \frac{c+1}{1} \cdot \frac{c+2}{2} \cdot \frac{c+3}{3} \cdot \frac{c+4}{4} \cdots \frac{c+\lambda}{\lambda},$$

hoc autem productum, si numeratores ordine inuerso scribantur, hanc induet formam:

$$\frac{c+\lambda}{1} \cdot \frac{c+\lambda-1}{2} \cdot \frac{c+\lambda-2}{3} \cdot \frac{c+\lambda-3}{4} \cdots \frac{c+1}{\lambda},$$

 quam

quamobrem cum summa quaesita $S = (\frac{m+n}{n})$ euoluta dederit

$$S = \frac{m+n}{1} \cdot \frac{m+n-1}{2} \cdot \frac{m+n-2}{3} \cdot \ldots \cdot \frac{m+1}{n},$$

illa forma manifesto in hanc transformatur, sumendo $c = m$ $\lambda = n$, ex quibus fit $\Delta = \frac{1}{m}$, et ipsa summa quaesita exprimetur hoc modo: $S = \frac{1}{m \int x^n \partial x (1-x)^{m-1}}$, atque cum ambos numeros m et n inter se permutare liceat, erit etiam

$$S = \frac{1}{m \int x^m \partial x (1-x)^{n-1}}.$$

§. 34. Haec expressio, quoties m et n sunt numeri integri, manifesto veram summam suppeditat. Sit exempli gratia $m = 4$ et $n = 3$, et quia

$$(1+z)^4 = 1 + 4z + 6zz + 4z^3 + z^4 \text{ et}$$
$$(1+z)^3 = 1 + 3z + 3zz + z^3,$$

erit series summanda $S = 1 + 3 \cdot 4 + 3 \cdot 6 + 1 \cdot 4 = 35$. At vero per formulam integralem priorem habemus $S = \frac{1}{4 \int x^3 \partial x (1-x)^3}$, per formulam autem posteriorem erit $S = \frac{1}{3 \int x^4 \partial x (1-x)^2}$. Est vero pro priore $\int x^3 \partial x (1-x)^3 = \frac{1}{140}$, pro altera vero est

$$\int x^4 \partial x (1-x)^2 = \frac{1}{5} - \frac{2}{6} + \frac{1}{7} = \frac{1}{105},$$

ita vt ex vtraque formula prodeat $S = 35$.

§. 35. In reductionibus quidem, vnde has expressiones hausimus, integralia ita accipi assumsimus, vt a termino $x = 0$ vsque ad $x = 1$ extendantur. Verum hic eadem circumstantia commode vsu venit, quam in praecedente solutione obseruauimus, quod formula hic inuenta etiam locum obtineat, etiamsi exponentes fuerint negatiui, quibus quippe casibus eam regulam obseruare non licet; namque hic etiam gemini

mini errores se mutuo destruunt. Ita si fuerit $m = -4$ et $n = 3$, fiet

$(1+z)^{-4} = 1 - 4z + 10zz - 20z^3 + 35z^4 - 56z^5 + \text{etc. et}$

$(1+z)^3 = 1 + 3z + 3zz + z^3$,

sicque series summanda erit

$$S = 1 - 3 \cdot 4 + 3 \cdot 10 - 1 \cdot 20 = -1.$$

At vero posterior formula integralis dat $S = \frac{1}{3\int \frac{\partial x}{x^4}(1-x)^2}$.

Est vero

$$\int \frac{\partial x}{x^4}(1-x)^2 = \frac{-1}{3x^3} + \frac{2}{2xx} - \frac{1}{x},$$

quae expressio euanescit facto $x = \infty$, ea vero posito $x = 1$ dat $= \frac{1}{3}$. Sin autem ambo numeri m et n sumerentur negatiui, summa seriei manifesto fieret infinita.

§. 36. Cum autem casus, quibus m et n sunt numeri integri, nullam difficultatem pariant, vsus praecipuus nostrae formulae tum locum habebit, quando numeri m et n sunt fracti. Ita si fuerit $m = \frac{1}{2}$ et $n = \frac{1}{2}$, ob

$$(1+z)^{\frac{1}{2}} = 1 + \tfrac{1}{2}z - \tfrac{1 \cdot 1}{2 \cdot 4}zz + \tfrac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}z^3 - \tfrac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}z^4 + \text{etc. et}$$

$$(1+z)^{-\frac{1}{2}} = 1 - \tfrac{1}{2}z + \tfrac{1 \cdot 3}{2 \cdot 4}zz - \tfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}z^3 + \tfrac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}z^4 - \text{etc.}$$

erit series summanda:

$$S = 1 - \tfrac{1}{2} \cdot \tfrac{1}{2} - \tfrac{1 \cdot 1}{2 \cdot 4} \cdot \tfrac{1 \cdot 3}{2 \cdot 4} - \tfrac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} \cdot \tfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} - \tfrac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \tfrac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} - \text{etc.}$$

At vero formula integralis prior nobis dat $S = \frac{2}{\int \frac{\partial x}{\sqrt{x(1-x)}}}$. Iam vero posito $x = yy$, fit

$$\int \frac{\partial x}{\sqrt{x(1-x)}} = 2\int \frac{\partial y}{\sqrt{(1-yy)}} = \pi,$$

ita

ita vt hinc fiat $S = \frac{2}{\pi}$, quam fummam iam fupra pro eodem cafu inuenimus.

§. 37. Pro aliis cafibus plurimum iuuabit reductiones fupra expofitas in vfum vocare, quod quo facilius fieri poffit confideremus hanc formam generalem: $\int x^q \partial x (1-x)^r$, atque fex illae reductiones fupra in Corollariis Lemmatum allatae praebebunt fequentes:

$$\text{I. } \int x^q \partial x (1-x)^r = \frac{r}{q+1} \int x^{q+1} \partial x (1-x)^{r-1},$$

$$\text{II. } \int x^q \partial x (1-x)^r = \frac{q}{r+1} \int x^{q-1} \partial x (1-x)^{r+1},$$

$$\text{III. } \int x^q \partial x (1-x)^r = \frac{q+r+2}{q+1} \int x^{q+1} \partial x (1-x)^r,$$

$$\text{IV. } \int x^q \partial x (1-x)^r = \frac{q}{q+r+1} \int x^{q-1} \partial x (1-x)^r,$$

$$\text{V. } \int x^q \partial x (1-x)^r = \frac{r}{q+r+1} \int x^q \partial x (1-x)^{r-1},$$

$$\text{VI. } \int x^q \partial x (1-x)^r = \frac{q+r+2}{r+1} \int x^q \partial x (1-x)^{r+1},$$

§. 38. Ope harum reductionum prior expreffio pro fumma inuenta $S = \frac{1}{m \int x^n \partial x (1-x)^{m-1}}$, vbi $q = n$ et $r = m-1$ in fequentes fex formas transfundi poterit:

$$\text{I. } S = \frac{n+1}{m(m-1) \int x^{n+1} \partial x (1-x)^{m-2}},$$

$$\text{II. } S = \frac{1}{n \int x^{n-1} \partial x (1-x)^m},$$

$$\text{III. } S = \frac{n+1}{m(m+n+1) \int x^{n+1} \partial x (1-x)^{m-1}},$$

$$\text{IV. } S = \frac{m+n}{m n \int x^{n-1} \partial x (1-x)^{m-1}},$$

V.

$$\text{V. } S = \frac{m+n}{m(m-1)\int x^n \partial x (1-x)^{m-2}},$$

$$\text{VI. } S = \frac{1}{(m+n+1)\int x^n \partial x (1-x)^m}.$$

quae eaedem formae etiam ex posteriore sequuntur.

§. 39. Reductiones autem istae semper ita in vsum trahi possunt, vt in formula integrali ambo exponentes ipsius x et $1-x$ intra terminos 0 et -1 redigantur, quippe quae formae praecipue considerari solent. Ita si fuerit $m = \frac{7}{2}$ et $n = 4$, hinc fiet:

$$(1+z)^{\frac{7}{2}} = 1 + Az + Bzz + Cz^3 + \text{etc. et}$$

$$(1+z)^4 = 1 + 4z + 6zz + 4z^3 + z^4$$

ita vt series summanda sit $S = 1 + 4A + 6B + 4C + D$; at vero erit $S = \frac{2}{7\int x^4 \partial x (1-x)^{\frac{5}{2}}}$, vbi $q = 4$ et $r = \frac{5}{2}$. Primo ergo exponentem q vsque ad nihilum deprimere poterimus, id quod ope reductionis IV fit, hinc enim erit

$$\int x^4 \partial x (1-x)^{\frac{5}{2}} = \tfrac{8}{15}\int x^3 \partial x (1-x)^{\frac{5}{2}};$$

porro vero

$$\int x^3 \partial x (1-x)^{\frac{5}{2}} = \tfrac{6}{13}\int xx \partial x (1-x)^{\frac{5}{2}};$$

deinde

$$\int xx \partial x (1-x)^{\frac{5}{2}} = \tfrac{4}{11}\int x \partial x (1-x)^{\frac{5}{2}};$$

denique

$$\int x \partial x (1-x)^{\frac{5}{2}} = \tfrac{2}{9}\int x \partial x (1-x)^{\frac{5}{2}};$$

 ficſicque iam habemus:

$$S = \frac{2.15.13.11.9}{7.8.6.4.2\int \partial x (1-x)^{\frac{5}{2}}}.$$

§. 40. Cum porro r ſit $= \frac{5}{2}$, deprimetur hic exponens per reductionem V, vnde ob $q = 0$ et $r = \frac{5}{2}$, fit

$$\int \partial x (1-x)^{\frac{5}{2}} = \tfrac{5}{7}\int \partial x (1-x)^{\frac{3}{2}};$$

eodem modo erit

$$\int \partial x (1-x)^{\frac{3}{2}} = \tfrac{3}{5}\int \partial x (1-x)^{\frac{1}{2}};$$

denique

$$\int \partial x (1-x)^{\frac{1}{2}} = \tfrac{1}{3}\int \frac{\partial x}{\sqrt{(1-x)}};$$

ex quibus conficitur:

$$S = \frac{2.15.13.11.9.7.5.3}{7.8.6.4.2.5.3.1\int \frac{\partial x}{\sqrt{(1-x)}}}.$$

Eſt vero

$$\int \frac{\partial x}{\sqrt{(1-x)}} = 2 - 2\sqrt{(1-x)},$$

ſicque eius valor erit $= 2$, et euoluto calculo reperietur $S = \frac{6435}{128}$.

§. 41. Cum iam fuerit $m = \frac{7}{2}$, erit

$$A = \tfrac{7}{2},\ B = \tfrac{7.5}{2.4},\ C = \tfrac{7.5.3}{2.4.6},\ D = \tfrac{7.5.3.1}{2.4.6.8},$$

ita vt noſtra ſeries ſummanda ſit:

$$S = 1 + 14 + \tfrac{105}{4} + \tfrac{35}{4} + \tfrac{35}{128} = \tfrac{6435}{128},$$

id quod egregie conuenit cum ſumma ante inuenta.

§. 42. Quod porro ad eas ſeries attinet, quas per terminum generalem $\left(\frac{m}{x}\right)\left(\frac{n}{p+x}\right)$ indicauimus, vbi loco x ordine

dine scribendi sunt numeri 0, 1, 2, 3, 4, etc. ita vt sit $S = \int(\frac{m}{x})(\frac{n}{p+x})$, ostendi fore $S = (\frac{m+n}{m+x})$ vel etiam $S = (\frac{m+n}{n-p})$; vnde patet hanc summam eandem fore, ac si series proposita esset $(\frac{m+p}{x})(\frac{n-p}{x})$. Quamobrem ad hanc summam formulae nostrae supra datae accommodabuntur, si in iis loco litterarum m et n scribantur hi valores, $m+p$ et $n-p$, sicque ex priore formula haec summa erit:

$$S = \frac{1}{(m+p)\int x^{n-p}\partial x(1-x)^{m+p-1}},$$

ex posteriore autem erit:

$$S = \frac{1}{(n-p)\int x^{m+p}\partial x(1-x)^{n-p-1}},$$

sicque totum hoc argumentum ad finem perductum est censendum.

§. 42. Vnicum tantum casum hic adiecisse operae erit pretium, quo $m+n=1$, ideoque $m=1-n$, eritque summa seriei ex formula IV. §. 38.

$$S = \frac{1}{n(1-n)\int\frac{x^{n-1}\partial x}{(1-x)^n}},$$

quod integrale commode per arcum circuli exprimi poterit: erit enim $\int\frac{x^{n-1}\partial x}{(1-x)^n} = \frac{\pi}{\text{sin.}\, n\pi}$, ita vt summa seriei propositae sit $S = \frac{\text{sin.}\, n\pi}{mn\pi}$. Vnde si fuerit $m=\frac{1}{2}$ et $n=\frac{1}{2}$, erit $S=\frac{4}{\pi}$, quae est summa seriei:

$$1 + (\tfrac{1}{2})^2 + (\tfrac{1.1}{2.4})^2 + (\tfrac{1.1.3}{2.4.6})^2 + (\tfrac{1.1.3.5}{2.4.6.8})^2 + \text{etc.}$$

vti iam supra notauimus. Deinde si sumamus $m=\frac{1}{3}$ et $n=\frac{2}{3}$ ob

 $(1+z)$

$$(1+z)^{\frac{1}{3}} = 1 + \tfrac{1}{3}z - \tfrac{1.2}{3.6}zz + \tfrac{1.2.5}{3.6.9}z^3 - \text{etc. et}$$

$$(1+z)^{\frac{2}{3}} = 1 + \tfrac{2}{3}z - \tfrac{2.1}{3.6}zz + \tfrac{2.1.4}{3.6.9}z^3 - \tfrac{2.1.4.7}{3.6.9.12}z^4 + \text{etc.}$$

feries fummada erit

$$S = 1 + \tfrac{1.2}{3.3} + \tfrac{1.2}{3.6}\cdot\tfrac{2.1}{3.6} + \tfrac{1.2.5}{3.6.9}\cdot\tfrac{2.1.4}{3.6.9} + \tfrac{1.2.5.8}{3.6.9.12}\cdot\tfrac{2.1.4.7}{3.6.9.12} + \text{etc.}$$

cuius ergo fumma, ob fin. $\frac{1}{3}\pi = \frac{\sqrt{3}}{2}$, per quadraturam circuli exprimi poterit, eritque $S = \frac{9\sqrt{3}}{4\pi}$. Eodem modo fi fumamus $m = \frac{1}{4}$ et $n = \frac{3}{4}$, erit feries fummanda

$$S = 1 + \tfrac{1}{4}\cdot\tfrac{3}{4} + \tfrac{1.3}{4.8}\cdot\tfrac{3.1}{4.8} + \tfrac{1.3.7}{4.8.12}\cdot\tfrac{3.1.5}{4.8.12} + \tfrac{1.3.7.11}{4.8.12.16}\cdot\tfrac{3.1.5.9}{4.8.12.16} + \text{etc.}$$

cuius autem fumma, ob fin. $\frac{1}{4}\pi = \frac{1}{\sqrt{2}}$, erit $S = \frac{8\sqrt{2}}{3\pi}$, feries autem ifta ita fuccincte exhiberi poteft:

$$S = 1 + \tfrac{3}{4^2}\left(1 + \tfrac{1.3}{8^2} + \tfrac{1.3.5.7}{8^2.12^2} + \tfrac{1.3.5.7.9.11}{8^2.12^2.16^2} + \text{etc.}\right).$$

EX-