consentanea deprehenditur, ita ut ista expressio latius pateat, quam omnes casus speciales junctim sumti, unde eam per conjecturam conclusimus; namque in omnibus casibus specialibus littera i necessario denotabat numeros integros tantum positivos.

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4) Demonstratio Theorematis insignis per conjecturam eruti, circa integrationem formulae

$$\int \frac{\partial \phi \cos. i \phi}{(1 + aa - 2a \cos. \phi)^{n+1}}$$

M. S. Academiae exhib. die 10 *Septembris* 1778.

§. 83. Cum nuper hanc formulam integralem tractassem, ac potissimum in ejus valorem inquisivissem, quem accipit, si integrale a termino $\phi = 0$ ad terminum $\phi = 180^\circ$ usque extendatur; ex pluribus casibus, quos evolvere licuit, conclusi ejus integrale in genere ita expressum iri

$$\frac{\pi a^i}{(1 - aa)^{2n+1}} V,$$

ubi V denotat summam hujus seriei

$$V = \left(\frac{n-i}{0}\right)\left(\frac{n+i}{i}\right) + \left(\frac{n-i}{1}\right)\left(\frac{n+i}{i+1}\right) a^2 + \left(\frac{n-i}{2}\right)\left(\frac{n+i}{i+2}\right) a^4 + \text{etc.}$$

Hic scilicet isti characteres clausulis inclusi designant coefficientes potestatis binomialis, dum statuimus

$$(1 + x)^m = 1 + \left(\frac{m}{1}\right) x + \left(\frac{m}{2}\right) x^2 + \left(\frac{m}{3}\right) x^3 + \left(\frac{m}{4}\right) x^4 + \text{etc.}$$

§. 84. Circa hanc autem formulam integralem ante omnia tenendum est, litteram i perpetuo significare numeros integros, quandoquidem in analysi constanter assumitur, casu $\phi = 180^\circ$ semper esse $\sin. i\phi = 0$; tum vero etiam ejus valores perpetuo ut positivi spectari possunt, propterea quod $\cos. (-i\phi) = \cos. (+i\phi)$. Interim tamen mox ostendemus nostram formam integralem etiam veritati esse consentaneam, quamvis litterae i valores negativi tribuantur. Ad hoc ostendendum circa characteres in subsidium vocatos sequentia sunt observanda.

1°). Si p et q designent numeros integros, ac primo quidem positivos, quoniam in evolutione potestatis binomialis omnes termini primum antecedentes sunt nulli, quoties fuerit q numerus negativus, semper erit $(\frac{p}{q}) = 0$.

2°). Quia coefficiens tam primi termini quam ultimi semper est unitas, erit tam $(\frac{p}{0}) = 1$ quam $(\frac{p}{p}) = 1$.

3°). Quia termini ultimum sequentes pariter sunt nulli, quoties fuerit $q > p$, valor characteris $(\frac{p}{q})$ semper pro nihilo haberi poterit.

4°). Quia in evolutione potestatis binomialis coefficientes ordinem tenent retrogradum, hinc sequitur semper fore $(\frac{p}{q}) = (\frac{p}{p-q})$. Sin autem superior numerus p fuerit negativus, ob rationem praecedentem semper etiam erit $(\frac{-p}{-q}) = 0$.

5°). At si q denotet numeros positivos, character $(\frac{-p}{q})$, perpetuo dabit valores alternatim positivos et negativos; cum sit

$$(\frac{-p}{0}) = 1;\ (\frac{-p}{1}) = -p;\ (\frac{-p}{2}) = +\frac{p(p+1)}{1.2};\ (\frac{-p}{3}) = -\frac{p(p+1)(p+2)}{1.2.3}\ \text{etc.}$$

Atque hinc

6°). In genere tales characteres, ubi superior numerus est negativus, ad positivos reduci poterunt, cum sit $(\frac{-p}{q}) = \pm(\frac{p+q-1}{q})$, ubi signum $+$ valet si q fuerit numerus par, inferius $-$ vero, si impar.

§. 85 His proprietatibus circa characteres hic adhibitos notatis, in forma nostra integrali loco i scribamus $-i$, eritque

$$\int \frac{\partial \phi \cos. - i\phi}{(1 + aa - 2a\cos.\phi)^{n+1}} = \frac{\pi a^{-i}}{(1 - aa)^{2n+1}} V,$$

existente

$$V = \left(\tfrac{n+i}{0}\right)\left(\tfrac{n-i}{-i}\right) + \left(\tfrac{n+i}{1}\right)\left(\tfrac{n-i}{-i+1}\right) a^2 + \left(\tfrac{n+i}{2}\right)\left(\tfrac{n-i}{-i+2}\right) a^4$$
$$+ \left(\tfrac{n+i}{3}\right)\left(\tfrac{n-i}{-i+3}\right) a^6 + \text{etc.}$$

ubi posteriores factores evanescunt, quamdiu denominatores sunt negativi: primum igitur membrum significatum habens erit $\left(\frac{n+i}{i}\right) \left(\frac{n-i}{-i+i}\right) a^{2i}$, cujus valor erit $\left(\frac{n+i}{i}\right) a^{2i}$; sequentia autem membra erunt

$$\left(\tfrac{n+i}{i+1}\right)\left(\tfrac{n-i}{-i+i+1}\right) a^{2i+2} = \left(\tfrac{n+i}{i+1}\right)\left(\tfrac{n-i}{1}\right) a^{2i+2},$$

tum vero $\left(\frac{n+i}{i+2}\right)\left(\frac{n-i}{2}\right) a^{2i+4}$, etc. Hoc igitur modo erit

$$V = a^{2i}\left[\left(\tfrac{n+i}{i}\right)\left(\tfrac{n-i}{0}\right) + \left(\tfrac{n+i}{i+1}\right)\left(\tfrac{n-i}{1}\right) a^2 + \left(\tfrac{n+i}{i+2}\right)\left(\tfrac{n-i}{2}\right) a^4 + \text{etc.}\right]$$

qui valor ductus in $\frac{\pi a^{-i}}{(1 - aa)^{2n+1}}$ praebet hanc formam

$$\frac{\pi a^i}{(1-aa)^{2n+1}}\left[\left(\tfrac{n+i}{i}\right)\left(\tfrac{n-i}{0}\right) + \left(\tfrac{n+i}{i+1}\right)\left(\tfrac{n-i}{1}\right) a^2 + \left(\tfrac{n+i}{i+2}\right)\left(\tfrac{n-i}{2}\right) a^4 + \text{etc.}\right]$$

quae prorsus congruit cum nostra formula valori positivo ipsius i respondente, qui egregius consensus haud contemnendum firmamentum pro veritate nostrae formae integralis continet.

§. 86. Praeterea vero circa formam nostram integralem imprimi notari debet, seriem pro V supra datam semper alicubi abrumpi quoties n fuerit numerus integer positivus, quippe quod eveniet, quando vel in priore factore, cujus forma est $\left(\frac{n-i}{\lambda}\right)$, pervenitur ad terminum quo $\lambda > n - i$, vel in posteriore factore, cujus forma est $\left(\frac{n+i}{i+\lambda}\right)$, evadet $\lambda > n$; quae proprietas eo magis est observanda, quod, si series V in infinitum porrigeretur, parum lucrati essemus censendi, id quod praecipue de iis casibus est notandum, quibus n foret numerus fractus, quos ergo casus penitus ab instituto nostro removemus, ita ut pro n tantum numeros integros simus assumturi.

§. 87. Consideremus ergo etiam casus, quibus n est numerus negativus, ac primo quidem jam per se clarum est, quamdiu is minor fuerit quam i, ideoque $n+i$ etiamnum numerus positivus, tum seriem pro V datam adeo citius abruptum iri; tum igitur demum in infinitum excurret, quando etiam $n+i$ fuerit numerus positivus. His autem casibus forma integralis supra data ita transformari potest, ut abruptio pariter locum inveniat.

§. 88. Ad hoc ostendendum statuamus $n = -m-1$, ut formula nostra integralis evadat

$$\int \partial \phi \cos. i\phi\, (1 + aa - 2a\cos.\phi)^m,$$

ejusque igitur valor $= \pi a^i\, (1-aa)^{2m+1}\, \text{V}$, existente jam

$$\text{V} = \left(\tfrac{-m-1-i}{0}\right)\left(\tfrac{-m-1+i}{i}\right) + \left(\tfrac{-m-1-i}{i}\right)\left(\tfrac{-m-1+i}{i+1}\right) a^2$$
$$+ \left(\tfrac{-m-1-i}{2}\right)\left(\tfrac{-m-1+i}{i+2}\right) a^4 + \left(\tfrac{-m-1-i}{3}\right)\left(\tfrac{-m-1+i}{i+3}\right) a^6 + \text{etc.}$$

quae series manifesto in infinitum excurrit, quam autem ope sequentes lemmatis transformare poterimus.

Lemma.

§. 89. Ista series per characteres hic introductos procedens

$$\mathfrak{s} = \left(\tfrac{f}{0}\right)\left(\tfrac{h}{e}\right) + \left(\tfrac{f}{1}\right)\left(\tfrac{h}{e+1}\right)x + \left(\tfrac{f}{2}\right)\left(\tfrac{h}{e+2}\right)x^2 + \left(\tfrac{f}{3}\right)\left(\tfrac{h}{e+3}\right)x^3 + \text{etc.}$$

in hanc sui similem transmutari potest

$$\text{♂} = \left(\frac{-h-1}{0}\right)\left(\frac{-f-1}{e}\right) + \left(\frac{-h-1}{1}\right)\left(\frac{-f-1}{e+1}\right)x + \left(\frac{-h-1}{2}\right)\left(\frac{-f-1}{e+2}\right)x^2 + \text{etc.}$$

quandoquidem inter earum valores ♄ et ♂ ista relatio semper locum habere, non ita pridem a me est demonstrata

$$\left(\frac{e+f}{1}\right)\text{♄} = \left(\frac{e-h-1}{e}\right)(1-x)^{f+h+1}\,\text{♂},$$

cujus demonstratio profundissimae est indaginis, dum adeo per aequationes differentiales secundi gradus procedit.

§. 90. Applicemus jam istud lemma ad casum nostrum propositum, atque ut series ♄ cum nostro V consentiens reddatur, ut fiat ♄ $= V$, sumi debet $f = -m-1-i$, $h = -m-1+i$, $e = i$ et $x = a\,a$, unde altera series ♂ hanc accipiet formam

$$\text{♂} = \left(\frac{m-i}{0}\right)\left(\frac{m+i}{i}\right) + \left(\frac{m-i}{1}\right)\left(\frac{m+i}{i+1}\right)aa + \left(\frac{m-i}{2}\right)\left(\frac{m+i}{i+2}\right)a^4 + \text{etc.}$$

quae series jam certe abrumpitur alicubi, propterea quod hic m denotat numerum integrum positivum: at vero relatio inter superiorem $V =$ ♄ et novam hanc seriem ♂ ita se habebit

$$\left(\frac{-m-1}{i}\right)V = \left(\frac{m}{i}\right)(1-a\,a)^{-2m-1}\,\text{♂}.$$

§. 91. Hinc igitur formulae nostrae integralis hujus

$$\int \partial\Phi \cos.i\Phi(1+a\,a-2\,a\cos.\Phi)^m = \frac{\left(\frac{m}{i}\right)\pi a^i\,\text{♂}}{\left(\frac{-m-1}{i}\right)},$$

ubi ♂ denotat seriem modo ante §. 89. expositam, qui valor cum factorem habeat $\left(\frac{m}{i}\right)$ semper evanescet, quamdiu fuerit $i > m$, ita ut his casibus valor integralis semper nihilo sit aequalis. Ceterum hic notasse juvabit, facta evolutione esse

$$\left(\frac{m}{i}\right):\left(\frac{-m-1}{i}\right) = \pm\frac{m(m-1)\ \ldots\ldots\ (m-i+1)}{(m+1)(m+2)\ \ldots\ldots\ (m+i)},$$

ubi signum superius $+$ valet si i fuerit numerus par, inferius $-$

vero si impar. His circa indolem nostri theorematis notatis, ipsam ejus demonstrationem aggrediamur, quam quo clarior evadat in varias partes distribuamus.

Demonstrationis pars prima.

§. 92. Quoniam valorem nostrum integralem ad duas formulas accommodavimus, eas distinctionis gratia signis ⊙ et ℭ designemus, sitque

$$\odot = \int \frac{\partial \Phi \cos. i \Phi}{(1 + aa - 2a \cos. \Phi)^{n+1}} \left[\begin{matrix} \text{a } \Phi = 0 \\ \text{ad } \Phi = 180^\circ \end{matrix}\right],$$

$$\mathfrak{C} = \int \partial \Phi \cos. i \Phi (1 + aa - 2a \cos. \Phi)^m \left[\begin{matrix} a \Phi = 0 \\ ad \Phi = 180^\circ \end{matrix}\right],$$

quarum posterior ℭ in priorem ⊙ convertitur si loco m scribamus $-n-1$; modo autem vidimus, has duas formulas a se invicem pendere, unde a posteriori tanquam simpliciori, siquidem denominatore $(1-aa)^{2n+1}$ caret, incipiamus, quam quo simpliciorem reddamus statuamus $\frac{a}{1+aa} = b$; sic enim habebimus

$$\mathfrak{C} = (1 + aa)^m \int \partial \Phi \cos. i \Phi (1 - 2b \cos. \Phi)^m;$$

cujus ergo integrale nobis erit investigandum.

§. 93. Ante omnia igitur conveniet potestatem $(1 - 2b \cos. \Phi)^m$ evolvi, unde fiet

$$(1 - 2b \cos. \Phi)^m = 1 - (\tfrac{m}{1}) 2b \cos. \Phi + (\tfrac{m}{2}) 4 b^2 \cos. \Phi^2 - (\tfrac{m}{3}) 8 b^3 \cos. \Phi^3 + \text{etc.}$$

cujus ergo terminus quicunque erit $\pm (\frac{m}{\lambda}) 2^\lambda b^\lambda \cos. \Phi^\lambda$; ubi signum + valet si λ fuerit numerus par, alterum vero − si impar. Jam quia hic potestates ipsius cos. Φ occurrunt, eas per praecepta satis cognita in cosinus simplices converti oportet, quibus fit

$$2^2\cos.\phi^2 = 2\cos.2\phi + 1\left(\tfrac{2}{1}\right),$$
$$2^3\cos.\phi^3 = 2\cos.3\phi + 2\left(\tfrac{3}{1}\right)\cos.\phi,$$
$$2^4\cos.\phi^4 = 2\cos.4\phi + 2\left(\tfrac{4}{1}\right)\cos.2\phi + 1\left(\tfrac{4}{2}\right).$$
$$2^5\cos.\phi^5 = 2\cos.5\phi + 2\left(\tfrac{5}{1}\right)\cos.3\phi + 2\left(\tfrac{5}{2}\right)\cos.\phi,$$
$$2^6\cos.\phi^6 = 2\cos.6\phi + 2\left(\tfrac{6}{1}\right)\cos.4\phi + 2\left(\tfrac{6}{2}\right)\cos.2\phi + 1\left(\tfrac{6}{3}\right),$$
etc. etc.

Ubi notandum, in potestatibus paribus postremum membrum cos. $0\phi = 1$ dimidio tantum coefficiente esse affectum. Hinc igitur in genere erit

$$2^\lambda\cos.\phi^\lambda = 2\cos.\lambda\phi + 2\left(\tfrac{\lambda}{1}\right)\cos.(\lambda+2)\phi + 2\left(\tfrac{\lambda}{2}\right)\cos.(\lambda-4)\phi + 2\left(\tfrac{\lambda}{3}\right)\cos.(\lambda-6)\phi + \text{etc.}$$

ubi notetur, quoties fuerit λ numerus par, puta $\lambda = 2i$, ultimum membrum fore tantum $1 . \left(\frac{2i}{i}\right)$ cos. 0ϕ.

§. 94 Postquam igitur omnes cosinuum potestates ad cosinus simplices fuerint reductae, integrationes nostrae semper ad talem formam redigentur $\int \partial\phi \cos. i\phi \cos. \lambda\phi$, de qua forma hic imprimis est notandum, ejus integrale a $\phi = 0$ ad $\phi = 280°$ extensum semper esse nullum, solo casu $\lambda = i$ excepto, Cum enim sit

$$\cos. i\phi \cos. \lambda\phi = \tfrac{1}{2}\cos.(i+\lambda)\phi + \tfrac{1}{2}\cos.(i-\lambda)\phi,$$

erit illud integrale indefinitum

$$= \frac{\sin.(i+\lambda)\phi}{2(i+\lambda)} + \frac{\sin.(i-\lambda)\phi}{2(i-\lambda)},$$

quod pro termino $\phi = 0$ manifesto evanescit; pro altero vero termino $\phi = 180° = \pi$, ob i et λ numeros integros, manifestum est, hoc integrale denuo evanescere, solo casu excepto quo $\lambda = i$. Si enim $i - \lambda$ ut infinite parvum spectetur, puta $= \omega$, pars posterior hujus integralis erit $\frac{\sin.\omega\phi}{2\omega} = \frac{\pi}{2}$, id quod etiam inde patet, quod sit

$$\cos. i\phi^2 = \tfrac{1}{2} + \tfrac{1}{2}\cos. 2i\phi,$$

ideoque

$$\int \partial\phi \cos. i\phi^2 = \tfrac{1}{2}\phi + \tfrac{1}{4}\sin. 2i\phi = \tfrac{1}{2}\pi.$$

§. 95. Ad integrale igitur quaesitum obtinendum, ex potestate $(1 - 2b\cos.\phi)^m$ evoluta, eos tantum terminos, qui $\cos. i\phi$ continent, excerpsisse sufficiet, cum reliqui omnes nihil plane producant, qui si junctim sumti praebeant $N\cos. i\phi$, totum nostrum integrale pro ☾ erit

$$☾ = (1 + aa)^m . \tfrac{1}{2} N\pi;$$

quocirca nobis incumbet, in omnes superioris formae partes inquirere, quae formula $\cos. i\phi$ erunt affectae; unde evidens est, quamdiu in illo termino generali $\pm \left(\frac{m}{\lambda}\right) 2^\lambda b^\lambda \cos.\phi^\lambda$ exponens λ minor fuerit quam i, inde nihil plane in integrale inferri.

§. 96. Primus igitur terminus, qui hic in computum venit, erit $\pm \left(\frac{m}{i}\right) 2^i b^i \cos.\phi^i$, pro quo signum superius $+$ valebit si i fuerit numerus par, inferius $-$ vero si impar. Hinc autem par superiorem reductionem proveniet

$$2^i \cos.\phi^i = 2\cos. i\phi,$$

ita ut hinc pro N oriatur pars prima $\pm \left(\frac{m}{i}\right) 2 b^i$. Tum vero ex termino immediate sequente, qui erit

$$\mp \left(\frac{m}{i+1}\right) 2^{i+1} b^{i+1} \cos.\phi^{i+1},$$

nullus angulus $i\phi$ oritur, cum sit

$$2^{i+1}\cos.\phi^{i+1} = 2\cos.(i+1)\phi + 2\left(\frac{i+1}{1}\right)\cos.(i-1)\phi + \text{etc.}$$

At vero terminus sequens

$$\pm \left(\frac{m}{i+2}\right) 2^{i+2} b^{i+2} \cos.\phi^{i+2}, \text{ ob}$$

$$2^{i+2}\cos.\phi^{i+2} = 2\cos.(i+2)\phi + 2\left(\frac{i+2}{1}\right)\cos. i\phi + \text{etc.}$$

partem hinc in litteram N resultantem dat

$$2\left(\frac{i+2}{1}\right)\left(\frac{m}{i+2}\right) b^{i+2}.$$

Simili modo ex casu $\lambda = i + 3$ nihil nascitur. At ex sequente

$$\pm\left(\frac{m}{i+4}\right) 2^{i+4} b^{i+4} \cos. \Phi^{i+4}, \text{ ob}$$

$$2^{i+4} \cos. \Phi^{i+4} = 2 \cos. (i+4)\Phi + 2\left(\frac{i+4}{1}\right) \cos. (i+2)\Phi + 2\left(\frac{i+4}{2}\right) \cos. i\Phi + \text{etc.}$$

pars ad litteram N accedens erit

$$2\left(\frac{i+4}{2}\right)\left(\frac{m}{i+4}\right) b^{i+4}.$$

Eodem modo ex casu $\lambda = i + 6$ pars ad litteram N accedens erit

$$2\left(\frac{i+6}{3}\right)\left(\frac{m}{i+6}\right) b^{i+6}, \text{ et ita porro.}$$

§. 97. His igitur omnibus partibus colligendis, nanciscemur valorem completum litterae N, qui erit

$$N = \pm 2 b^i\left[\left(\frac{m}{i}\right)+\left(\frac{i+2}{1}\right)\left(\frac{m}{i+2}\right)b^2+\left(\frac{i+4}{2}\right)\left(\frac{m}{i+4}\right)b^4+\left(\frac{i+6}{3}\right)\left(\frac{m}{i+6}\right)b^6+\text{etc.}\right]$$

ubi notasse juvabit esse, ut sequitur

$$\left(\frac{i+2}{1}\right)\left(\frac{m}{i+2}\right) = \left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right),$$

$$\left(\frac{i+4}{2}\right)\left(\frac{m}{i+4}\right) = \left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right),$$

$$\left(\frac{i+6}{3}\right)\left(\frac{m}{i+6}\right) = \left(\frac{m}{3}\right)\left(\frac{m-3}{i+3}\right),$$

etc.

Per hos igitur valores erit

$$N = \pm 2 b^i\left[\left(\frac{m}{0}\right)\left(\frac{m}{i}\right)+\left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right)b^2+\left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right)b^4+\left(\frac{m}{3}\right)\left(\frac{m-3}{i+3}\right)b^6 \text{ etc.}\right]$$

quo valore invento, erit integrale nostrum quaesitum

$$C = \pm\pi(1+aa)^m b^i\left[\left(\frac{m}{0}\right)\left(\frac{m}{i}\right)+\left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right)b^2 + \text{etc.}\right]$$

quae series manifesto abrumpitur, quoties fuerit m numerus integer positivus. Statim enim atque in hoc charactere $(\frac{m-\lambda}{i+\lambda})$ denominator $i+\lambda$ superare incipit numeratorem $m-\lambda$, valor ejus in nihilum abit.

Demonstrationis pars secunda.

§. 98. Ut autem hanc integralis expressionem ad solam litteram a revocamus, prouti in nostro theoremate supra est repraesentata, hic loco b restituamus valorem assumtum $\frac{a}{1+aa}$, fietque

$$☾ = \pm \pi a^i (1+aa)^{m-i} [(\frac{m}{0})(\frac{m}{i}) + (\frac{m}{1})(\frac{m-1}{i+1})\frac{a^2}{(i+aa)^2} + (\frac{m}{2})(\frac{m-2}{i+2})\frac{a^4}{(1+aa)^4} + \text{etc.}]$$

ubi, ut formam supra datam eliciamus, potestates ipsius $1+aa$ evolvi oportet. Hunc in finem statuamus $☾ = \pm \pi a^i A$, ita ut jam sit

$$A = (\frac{m}{0})(\frac{m}{i})(1+aa)^{m-i} + (\frac{m}{1})(\frac{m-1}{i+1})a^2(1+aa)^{m-i-2} + (\frac{m}{2})(\frac{m-2}{i+2})a^4(1+aa)^{m-i-4} + (\frac{m}{3})(\frac{m-3}{i+3})a^6(1+aa)^{m-i-6} + \text{etc.}$$

Facta autem harum potestatum evolutione, fiat

$$A = \alpha + \beta a^2 + \gamma a^4 + \delta a^6 + \varepsilon a^8 + \zeta a^{10} + \eta a^{12} + \text{etc.}$$

quarum litterarum α, β, γ, δ, etc. valores investigemus.

§. 99. Primo igitur statim patet esse $\alpha = (\frac{m}{0})(\frac{m}{i})$; deinde vero reperietur

$$\beta = (\frac{m}{0})(\frac{m}{i})(\frac{m-i}{1}) + (\frac{m}{1})(\frac{m-1}{i+1}).$$

At vero pars posterior per priorem divisa, facta evolutione, praebet $\frac{m-i-1}{i+1}$, quo observato erit

$$\beta = \frac{m}{i+1}(\frac{m}{0})(\frac{m}{i})(\frac{m-i}{1}),$$

quod reducitur ad $\beta = (\frac{m}{1})(\frac{m}{i+1})$. Simili modo littera γ constabit ex tribus partibus: erit enim

$$\gamma = \left(\frac{m}{0}\right)\left(\frac{m}{i}\right)\left(\frac{m-i}{2}\right) + \left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right)\left(\frac{m-i-2}{1}\right) + \left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right),$$

ubi pars secunda per primam divisa dat $\frac{2(m-i-2)}{i+1}$. At tertius terminus per primum divisus praebet $\frac{(m-i-2)(m-i-3)}{(i+1)(i+2)}$, unde fit

$$\gamma = 1 + \frac{2(m-i-2)}{i+1} + \frac{(m-i-2)(m-i-3)}{(i+1)(i+2)}.$$

At vero est

$$1 + \frac{m-i-2}{i+1} = \frac{m-1}{i+1}, \text{ et}$$

$$\left(\frac{m-i-2}{i+1}\right)\left(1 + \frac{m-i-3}{i+2}\right) = \frac{m-1}{i+2} \cdot \frac{m-i-2}{i+1}.$$

unde colligitur

$$\gamma = \frac{m-1}{i+1} \cdot \frac{m}{i+2} \left(\frac{m}{0}\right)\left(\frac{m}{i}\right)\left(\frac{m-i}{2}\right),$$

quae expressio contrahitur in hanc $\left(\frac{m}{2}\right)\left(\frac{m}{i+2}\right)$.

§. 100. Cum igitur sit

$$\alpha = \left(\frac{m}{0}\right)\left(\frac{m}{i}\right), \quad \beta = \left(\frac{m}{1}\right)\left(\frac{m}{i+1}\right), \quad \gamma = \left(\frac{m}{2}\right)\left(\frac{m}{i+2}\right),$$

hinc jam satis tuto concludere liceret, fore

$$\delta = \left(\frac{m}{3}\right)\left(\frac{m}{i+3}\right), \quad \varepsilon = \left(\frac{m}{4}\right)\left(\frac{m}{i+4}\right), \text{ etc.}$$

Verum ne hic quicquam conjecturae vel inductioni tribuamus, in genere pro valore litterae A investigemus coëfficientem potestatis indefinitae $a^{2\lambda}$, quem vocemus $= \lambda$, eritque

$$A = \left(\frac{m-i}{\lambda}\right)\left(\frac{m}{0}\right)\left(\frac{m}{i}\right) + \left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right)\left(\frac{m-i-2}{\lambda-1}\right) + \left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right)\left(\frac{m-i-4}{\lambda-2}\right)$$
$$+ \left(\frac{m}{3}\right)\left(\frac{m-3}{i+3}\right)\left(\frac{m-i-6}{\lambda-3}\right) + \text{etc.}$$

§. 101. Hujus seriei pro A inventae singulos terminos sub hac forma generali complecti licet $\left(\frac{m}{\theta}\right)\left(\frac{m-\theta}{i+\theta}\right)\left(\frac{m-i-2\theta}{\lambda-\theta}\right)$, quae secundum factores evoluta transmutatur in hanc formam

$$\frac{m(m-1) \;.\;.\;.\;.\;.\;.\;.\;.\; (m-i-\lambda-\theta+1)}{1 \;.\;.\;.\; \theta \times 1 \;.\;.\;.\;.\;.\; (i+\theta) \times 1 \;.\;.\; (\lambda-\theta)},$$

ubi numeratoris factores ab m incipientes continuo unitate decrescent usque ad ultimum $(m-i-\lambda-\theta+1)$. Jam ista fractio supra et infra multiplicetur per hoc productum

$$\lambda(\lambda-1) \;.\;.\;.\;.\;.\;.\; (\lambda-\theta+1),$$

ac prodibit ista fractio

$$\frac{\lambda(\lambda-1) \;.\;.\;.\; (\lambda-\theta+1) \times m(m-1) \;.\;.\;.\; (m-i-\lambda-\theta+1)}{1.2.3 \;.\;.\;.\; \theta + 1.2.3. \;.\;.\;.\; (i+\theta) \times 1.2.3 \;.\;.\;.\; \lambda},$$

in qua primo continetur character $(\frac{\lambda}{\theta})$, deinde etiam ibi continetur character $(\frac{m}{\lambda})$; quod restat dabit characterem $(\frac{m-\lambda}{i+\theta})$, sicque habebitur forma A generalis $= (\frac{\lambda}{\theta})(\frac{m}{\lambda})(\frac{m-\lambda}{i+\theta})$. Unde si loco θ successive scribamus 0, 1, 2, 3, etc., quia in singulis terminis communis inest factor $(\frac{m}{\lambda})$, erit valor litterae

$$A = (\frac{m}{\lambda})[(\frac{\lambda}{0})(\frac{m-\lambda}{i}) + (\frac{\lambda}{1})(\frac{m-\lambda}{i+1}) + (\frac{\lambda}{2})(\frac{m-\lambda}{i+2}) + \text{etc.}]$$

Verum ante aliquod tempus demonstravi, hujus similis seriei

$$(\frac{p}{0})(\frac{q}{r}) + (\frac{p}{1})(\frac{q}{r+1}) + (\frac{p}{2})(\frac{q}{r+2}) + (\frac{p}{3})(\frac{q}{r+3}) + \text{etc.}$$

summam semper esse $= (\frac{p+q}{p+r}) = (\frac{p+q}{q-r})$. Facta ergo applicatione, erit $p = \lambda$, $q = m - \lambda$, $r = i$: sicque finito modo habebimus

$$A = (\frac{m}{\lambda})(\frac{m}{\lambda+i}) = (\frac{m}{\lambda})(\frac{m}{m-\lambda-i}),$$

quae est demonstratio conjecturae supra allatae et ex valoribus α, β, γ, conclusae.

§. 102. Quod si jam hic loco λ successive scribamus numeros 0, 1, 2, 3, etc., nanciscemur verum valorem seriei, quam sub littera A complexi; erit scilicet

$$\mathrm{A} = (\frac{m}{0})(\frac{m}{i}) + (\frac{m}{1})(\frac{m}{i+1})\, a^2 + (\frac{m}{2})(\frac{m}{i+2})\, a^4$$
$$+ (\frac{m}{3})(\frac{m}{i+3})\, a^6 + \text{etc.}$$

atque hinc valor integralis sub signo $\complement$ indicatae formulae erit

$$\complement = \pm \pi a^i [(\tfrac{m}{0})(\tfrac{m}{i}) + (\tfrac{m}{1})(\tfrac{m}{i+1})a^2 + (\tfrac{m}{2})(\tfrac{m}{i+2})a^4 + \text{etc.}]$$

quae expressio manifesto semper abrumpitur, quoties m est numerus integer positivus. Hic autem meminisse oportet, signi ambigui $\pm$ superius locum habere quando i fuerit numerus par, inferius vero si impar.

Demonstrationis pars tertia.

§. 103. Ista forma, quam pro valore integrali $\complement$ hic sumus adepti multo adeo est simplicior ea, quam theorema nostrum nobis suppeditaverat, quippe quae, si loco $\mathfrak{S}$ seriem quam designat scribamus, erit

$$\complement = \frac{\pi a^i (\tfrac{m}{i})}{(\tfrac{-m-1}{i})} [(\tfrac{m-i}{0})(\tfrac{m+i}{i}) + (\tfrac{m-i}{1})(\tfrac{m+i}{i+1})a^2 + (\tfrac{m-i}{2})(\tfrac{m+i}{i+2})a^4 + \text{etc.}]$$

Superest igitur, ut perfectum consensum inter has duas expressiones specie multum a se invicem discrepantes ostendamus. Hic autem plurimum notasse juvabit, esse $(\tfrac{-m-1}{i}) = \pm (\tfrac{m+i}{i})$, propterea quod supra §. 88. jam observavimus, esse in genere $(\tfrac{-p}{q}) = \pm (\tfrac{p+q-1}{q})$, ubi signum superius valet si fuerit q numerus par, inferius vero si impar; quo notato posterior forma pro $\complement$ inventa erit

$$\complement = \pm \frac{\pi a^i (\tfrac{m}{i})}{(\tfrac{m+i}{i})} [(\tfrac{m-i}{0})(\tfrac{m+i}{i}) + (\tfrac{m-i}{1})(\tfrac{m+i}{i+1}) a^2 + \text{etc.}].$$

§. 104. Quoniam nunc ambae formae affectae sunt signo ambiguo $\pm$, demonstrandum nobis incumbit, si utramque expressionem per $(\tfrac{m+i}{i})$ multiplicemus, duas sequentes series inter se prorsus esse aequales

$$\text{I. } \left(\frac{m}{0}\right)\left(\frac{m}{i}\right)\left(\frac{m+i}{i}\right) + \left(\frac{m}{1}\right)\left(\frac{m}{i+1}\right)\left(\frac{m+i}{i}\right) a^2$$
$$+ \left(\frac{m}{2}\right)\left(\frac{m}{1+2}\right)\left(\frac{m+i}{i}\right) a^4 + \text{etc.}$$
$$\text{II. } \left(\frac{m-i}{0}\right)\left(\frac{m+i}{i}\right)\left(\frac{m}{i}\right) + \left(\frac{m-i}{1}\right)\left(\frac{m+i}{i+1}\right)\left(\frac{m}{i}\right) a^2$$
$$+ \left(\frac{m-i}{2}\right)\left(\frac{m+i}{i+2}\right)\left(\frac{m}{i}\right) a^4 + \text{etc.}$$

ubi aequalitas primorum terminorum ob $\left(\frac{m}{0}\right)$ et $\left(\frac{m-i}{0}\right) = 1$ sponte se prodit; deinde vero non difficulter aequalitas inter terminos secundos ipso $a\ a$ affectos ostendi poterit, similique modo etiam de sequentibus hoc idem est tenendum.

§. 105. Verum ne etiam hic inductione uti cogamur, convenientiam binorum terminorum eadem potestate $a^{2\lambda}$ demonstremus. In priore vero serie ista potestas $a^{2\lambda}$ hunc habet coëfficientem $\left(\frac{m}{\lambda}\right)\left(\frac{m}{i+\lambda}\right)\left(\frac{m+i}{i}\right)$; in altera vero ejusdem coëfficiens est $\left(\frac{m-i}{\lambda}\right)$ $\left(\frac{m+i}{i+\lambda}\right)\left(\frac{m}{i}\right)$. Evolvatur igitur uterque in factores simplices, ac prior deducit ad hanc fractionem

$$\frac{m \ \ldots \ (m-\lambda+1) \times m \ \ldots \ (m-i-\lambda+1) \times (m+i) \ \ldots \ (m+1)}{1 \ \ldots \ \lambda \times 1) \ \ldots \ (i+\lambda) \times 1 \ \ldots \ i};$$

posterior vero praebet istam

$$\frac{(m-i) \ \ldots \ (m-i-\lambda+1) \times (m+i) \ \ldots \ (m-\lambda+1) \times m \ \ldots \ (m-i+1)}{1 \ \ldots \ \lambda \times 1 \ \ldots \ (i+\lambda) \times 1 \ \ldots \ i};$$

ubi denominatores utrinque manifesto sunt iidem, ita ut tantum aequalitas inter numeratores sit demonstranda.

§. 106. Primo autem in priore numeratore tertius factor generalis cum primo conjunctus praebet hoc productum

$$(m+i) \ \ldots \ (m-\lambda+1),$$

quod etiam in forma posteriori occurrit: his igitur sublatis aequalitatem monstrari oportet inter partes residuas quae sunt, in priori forma $m \quad . \quad . \quad . \quad . \quad . \quad (m-i-\lambda+1)$

in altera $m \ldots (m-i+1) \times (m-i) \ldots (m-i-l+1)$

quae nunc iterum est manifesta. Sic igitur veritas nostri theorematis, quod demonstrandum suscepimus, jam rigide est ob oculos posita pro formula integrali

$$☾ = \int \partial \phi \cos. i \phi (1 + aa - 2 a \cos. \phi)^n \left[\begin{matrix} a\phi = 0 \\ ad\phi = \pi \end{matrix} \right].$$

Demonstrationis pars quarta.

§. 107. Invento valore formulae ☾, tota demonstratio jam confecta est censenda, quandoquidem jam initio ex valore formulae ⊙ ille rite est derivatus. Interim tamen hic quoque vicissim ex valore ☾ alterum valorem ⊙ derivari conveniet. Utamur autem forma simpliciori ipsius ☾, ad quem nos ipsa demonstratio immediate perduxit, qui erat

$$☾ = \pm \pi a^i [(\tfrac{m}{0})(\tfrac{m}{i}) + (\tfrac{m}{1})(\tfrac{m}{i+1}) a^2 + (\tfrac{m}{2})(\tfrac{m}{i+2}) a^4 + \text{etc.}]$$

ubi signum superius valet si i fuerit numerus par, inferius si impar.

§. 108. Ex hoc jam valore formulae ☾ alterius formulae ⊙ valor deducitur, si modo loco m scribamus $-n-1$, qui ergo valor hinc erit

$$⊙ = \pm \pi a^i [(\tfrac{-n-1}{0})(\tfrac{-n-1}{i}) + (\tfrac{-n-1}{1})(\tfrac{-n-1}{i+1}) a^2 + (\tfrac{-n-1}{2})(\tfrac{-n-1}{i+2}) a^4 + \text{etc.}]$$

quae autem series nunc in infinitum progreditur, siquidem n fuerit numerus integer positivus; quamobrem hanc seriem in aliam converti oportet, quae abrumpatur, quoties n fuerit numerus integer positivus, id quod ope lemmatis supra initio allati praestari poterit.

§. 109. Seriem igitur hic inventam cum serie ♄ in lemmate comparemus, id quod fit statuendo

$$f = -n-1,\ h = -n-1 \text{ et } e = i,$$

ita ut jam sit $\odot = \pm \pi a^i$ ♄. Ex his autem valoribus altera series signo ♂ notata fiet, ob

$$-h-1 = n, -f-1 = n, \text{ et } x = a^2,$$

$$♂ = (\tfrac{n}{0})(\tfrac{n}{i}) + (\tfrac{n}{1})(\tfrac{n}{i+1})a^2 + (\tfrac{n}{2})(\tfrac{n}{i+2})a^4 + \text{etc.}$$

At vero relatio inter has duas series erit

$$\left(\frac{i-n-1}{i}\right) ♄ = \frac{(\frac{n+i}{i})\, ♂}{(1-aa)^{2n+1}};$$

ubi notetur, cum supra jam observaverimus esse

$$(\tfrac{-p}{q}) = \pm(\tfrac{p+q-1}{q}), \text{ hic fore } (\tfrac{-n-1+i}{i}) = \pm(\tfrac{n}{i});$$

ubi iterum signum superius valet, si i fuerit numerus par. Hinc igitur erit

$$♄ = \pm \frac{(\frac{n+i}{i})\, ♂}{(\frac{n}{i})(1-aa)^{2n+1}}.$$

§. 110. Substituatur igitur iste valor loco ♄, quo ipso duplex signorum ambiguitas e medio tolletur, loco ♂ autem series modo data scribatur, atque pro $\odot$ sequentem nanciscemur expressionem

$$\odot = \frac{\pi a^i (\frac{n+i}{i})}{(\frac{n}{i})(1-aa)^{2n+1}}\left[(\tfrac{n}{0})(\tfrac{n}{i}) + (\tfrac{n}{1})(\tfrac{n}{i+1})a^2 + (\tfrac{n}{2})(\tfrac{n}{i+2})a^4 + \text{etc.}\right]$$

quae series manifesto semper abrumpitur, quoties n fuerit numerus integer positivus. Verumtamen hoc laborat defectu, quod casibus quibus $n < i$, ob $(\frac{n}{i}) = 0$, infinita evadere videtur. Verum notandum est, his casibus etiam omnes terminos seriei δ in nihilum abire; ex quo necesse est, ut in ejus verum valorem totiusque expressionis inquiramus. At vero reliquis casibus, quibus $n > i$ haec expressio adeo illi quam in theoremate dedimus praeferenda videtur.

§. 111. Ostendi ergo hic debet, omnes terminos nostrae seriei ita transformari posse, ut per denominatorem $(\frac{n}{i})$ divisionem admittant. At vero quilibet nostrae seriei terminus sub hac forma continetur $(\frac{n}{\lambda})(\frac{n}{i+\lambda})$, quae per factorem comunem $(\frac{n+i}{i})$ multiplicata fit $(\frac{n+i}{i})(\frac{n}{\lambda})(\frac{n}{i+\lambda})$, quae in factores evoluta ad hanc fractionem reducitur

$$\frac{(n+i)\ldots\ldots(n+1)\times n\ldots\ldots(n-\lambda+1)\times n\ldots\ldots(n-i-\lambda+1)}{1\ldots\ldots i\times 1\ldots\ldots\lambda\times 1\ldots\ldots(i+\lambda)};$$

ubi tam numerator quam denominator tres habet factores principales; factores autem singulares in numeratore continuo unitate decrescunt, in denominatore unitate increscunt. Cum igitur sit $(\frac{n}{i}) = \frac{n\ldots\ldots(n-i+1)}{1\ldots\ldots i}$, superior fractio per hanc divisa, ob

$$\frac{n\ldots\ldots(n-i-\lambda+1)}{n\ldots\ldots(n-i+1} = (n-i)\ldots\ldots(n-i-\lambda+1),$$

proveniet

$$\frac{(n+i)\ldots\ldots(n+i)\times n\ldots\ldots(n-\lambda+1)\times(n-i)\ldots\ldots(n-i-\lambda+1)}{1.2.3\ldots\ldots\lambda\times 1.2.3\ldots\ldots(i+\lambda)},$$

quae manifesto in hanc transit (ob duo priores factores cohaerentes)

$$\frac{n+i)\ldots\ldots\ldots(n-\lambda+1)\times(n-i)\ldots(n-i-\lambda+1)}{1.2.\ldots\ldots\lambda\times 1.2\ldots\ldots(i+\lambda)},$$

ita ut omnibus ad characteres reductis, sit forma generalis cujusque termini $= (\frac{n+i}{i+\lambda})(\frac{n-i}{\lambda})$.

§. 112. Nunc igitur loco λ successive scribantur valores 0, 1, 2, 3, etc. atque valor integralis formulae ϖ prodibit, prorsus uti in theoremate est enunciatus, scilicet

$$\odot = \frac{\pi a^i}{(1-aa)^{2n+1}} \left[\left(\frac{n-i}{0}\right)\left(\frac{n+i}{i}\right) + \left(\frac{n-i}{1}\right)\left(\frac{n+i}{i+1}\right)a^2 + \left(\frac{n-i}{2}\right)\left(\frac{n+i}{i+2}\right)a^4 + \text{etc.}\right]$$

quae expressio jam non solum semper abrumpitur, quoties n fuerit numerus integer positivus, nec ullo amplius laborat defectu, cum omnibus casibus valorem ipsius $\odot$ determinatum exhibeat, sicque adeo nostrum theorema, quod antea sola conjectura innitebatur, solidissima demonstratione est confirmatum.