2.) Methodus succinctior comparationes quantitatum transcendentium in forma $\int \frac{P \partial z}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}}$ contentarum inveniendi.

M. S. Academiae exhib. die 3 Nov. 1777.

In Capite VI. Sect. II. Institutionum mearum Calculi Integralis Tom. I. insignes tradidi comparationes inter quantitates maxime transcendentes, ad quam deductus eram methodo penitus indirecta. Postquam igitur non ita pridem illustris *de la Grange* methodum maxime ingeniosam excogitasset easdem comparationes inveniendi, totum hoc argumentum multo succinctius et elegantius tractari poterit, quam mihi quidem tum temporis licebat, unde sequentia Supplementa Geometris haud displicebunt.

Hypothesis 1.

§. 80. Denotet hic perpetuo character $\Pi : z$ valorem formulae integralis $\int \frac{\partial z}{\sqrt{(\alpha + \beta z + \gamma zz + \delta z^3 + \varepsilon z^4)}}$, ita sumtae ut evanescat posito $z = 0$. Ponatur autem brevitatis gratia $\alpha +$ Fig. 13. $\beta z + \gamma zz + \delta z^3 + \varepsilon z^4 = Z$, ita ut sit $\Pi : z = \int \frac{\partial z}{\sqrt{Z}}$. Tum vero concipiatur super axe oz exstructa ejusmodi curva O Z, cujus singuli arcus O Z abscissis $oz = z$ respondentes exprimantur per formulam $\Pi : z = \int \frac{\partial z}{\sqrt{Z}}$; atque haec curva ista insigni proprietate erit praedita, ut sumto in ea pro lubitu arcu quocunque F G, a quovis alio puncto X semper arcus X Y illi arcui F G aequalis geometrice abscindi possit, cujus demonstrationem solutio sequentis problematis suppeditabit.

Problema 1.

Si in curva modo descripta proponatur arcus quicunque F G, innumerabiles alios arcus X Y in eadem curva geometrice assignare, qui singuli eidem arcui F G sint aequales.

Solutio.

§. 81. Ductis ex punctis F et G ad axem $o\,z$ applicatis $F\,f$ et $G\,g$, vocentur abscissae $o\,f = f$ et $o\,g = g$, eruntque arcus $O\,F = \Pi : f$ et $O\,G = \Pi : g$, unde longitudo arcus propositi F G erit $= \Pi : g - \Pi : f$. Simili modo pro quovis arcu quaesito X Y vocentur abscissae $o\,x = x$ et $o\,y = y$, eruntque arcus $O\,X = \Pi : x$ et $O\,Y = \Pi : y$, ideoque arcus $X\,Y = \Pi : y - \Pi : x$, qui cum aequalis esse debeat arcui F G, habebitur ista aequatio $\Pi : y - \Pi : x = \Pi : g - \Pi : f$, cui satisfieri oportet.

§. 82. Quoniam puncta F et G considerantur ut fixa, dum puncta X et Y per totam curvam variari possunt, differentiatio nobis praebebit hanc aequationem $\partial \,.\, \Pi : y - \partial \,.\, \Pi : x = 0$. Quare cum sit per hypothesin

$$\Pi : x = \int \frac{\partial x}{\sqrt{X}} \text{ et } \Pi : y = \int \frac{\partial y}{\sqrt{Y}},$$

existente

$$X = \alpha + \beta x + \gamma x x + \delta x^3 + \varepsilon x^4 \text{ et}$$
$$Y = \alpha + \beta y + \gamma y y + \delta y^3 + \varepsilon y^4,$$

solutio problematis perducta est ad hanc aequationem differentialem $\frac{\partial y}{\sqrt{Y}} - \frac{\partial x}{\sqrt{X}} = 0$.

§. 83. Hic jam methodum ill. *de la Grange* in subsidium vocantes statuamus $\frac{\partial x}{\sqrt{X}} = \partial t$, eritque $\frac{\partial y}{\sqrt{Y}} = \partial t$. Hic scilicet novum elementum ∂t in calculum introducimus, quod in sequentibus differentiationibus ut constans tractetur; tum igitur habebimus

$$\frac{\partial x}{\partial t} = \sqrt{X} \text{ et } \frac{\partial y}{\partial t} = \sqrt{Y}.$$

Quod si ergo porro statuamus $y + x = p$ et $y - x = q$, habebimus hinc

$$\frac{\partial p}{\partial t} = \sqrt{Y} + \sqrt{X} \text{ et } \frac{\partial q}{\partial t} = \sqrt{Y} - \sqrt{X},$$

quarum formularum productum praebet

$$\frac{\partial p \partial q}{\partial t^2} = Y - X.$$

Valoribus ergo loco Y et X substitutis erit

$$\frac{2 p \partial q}{\partial t^2} = \beta (y - x) + \gamma (y^2 - x^2) + \delta (y^3 - x^3) + \varepsilon (y^4 - x^4).$$

Quare cum sit

$$y = \frac{p + q}{2} \text{ et } x = \frac{p - q}{2} \text{ erit}$$

$$y - x = q,\ y^2 - x^2 = p q,\ y^3 - x^3 = \tfrac{1}{4} q (3 p p + q q) \text{ et } y^4 - x^4 = \tfrac{1}{2} p q (p p + q q),$$

quibus substitutis factaque divisione per q habebitur

$$\frac{\partial p \partial q}{q \partial t^2} = \beta + \gamma p + \tfrac{1}{4} \delta (3 p p + q q) + \tfrac{1}{2} \varepsilon p (p p + q q),$$

cujus aequationis plurimus erit usus in sequenti calculo.

§. 84. Jam sumtis quadratis primae aequationes dabunt

$$\frac{\partial x^2}{\partial t^2} = X \text{ et } \frac{\partial y^2}{\partial t^2} = Y,$$

quae denuo differentientur, quem in finem ponamus brevitatis gratia

$$\partial X = X' \partial x \text{ et } \partial Y = Y' \partial y,$$

atque hinc nanciscemur

$$\frac{2\partial\partial x}{\partial t^2} = X' \text{ et } \frac{2\partial\partial y}{\partial t^2} = Y',$$

quibus additis erit

$$\frac{2\partial\partial p}{\partial t^2} = X' + Y'.$$

Cum igitur sit

$$X' = \beta + 2\gamma x + 3\delta x x + 4\varepsilon x^3 \text{ et}$$
$$Y' = \beta + 2\gamma y + 3\delta y y + 4\varepsilon y^3, \text{ erit}$$
$$\frac{2\partial\partial p}{\partial t^2} = 2\beta + 2\gamma(x+y) + 3\delta(x^2+y^2) + 4\varepsilon(x^3+y^3).$$

Introducendo igitur litteras p et q ut ante, fiet

$$x + y = p, \quad x^2 + y^2 = \tfrac{1}{2}(pp + qq),$$
$$x^3 + y^3 = \tfrac{1}{4}p(pp + 3qq),$$

sicque ista aequatio hanc induet formam

$$\frac{2\partial\partial p}{\partial t^2} = 2\beta + 2\gamma p + \tfrac{3}{2}\delta(pp+qq) + \varepsilon p(pp+3qq).$$

§. 85. Ab hac jam postrema aequatione subtrahatur praecedens bis sumta, ac remanebit

$$\frac{2\partial\partial p}{\partial t^2} - \frac{2\partial p\partial q}{q\partial t^2} = \delta qq + 2\varepsilon pqq.$$

Hinc per qq dividendo habebimus

$$\frac{1}{\partial t^2}\cdot\left(\frac{2\partial\partial p}{qq} - \frac{2\partial p\partial q}{q^3}\right) = \delta + 2\varepsilon p,$$

cujus utrumque membrum manifesto integrationem admittit, si ducatur in elementum ∂p. Hoc enim facto aequatio integralis erit

$$\frac{\partial p^2}{qq\partial t^2} = C + \delta p + \varepsilon pp.$$

§. 86. Initio autem vidimus esse $\frac{\partial p}{\partial t} = \sqrt{X} + \sqrt{Y}$, hincque statim pervenimus ad aequationem integralem algebraicam hanc

$$\frac{(\sqrt{X}+\sqrt{Y})^2}{qq} = C + \delta p + \varepsilon pp.$$

Quare cum sit $p = x + y$ et $q = y - x$, haec aequatio evoluta fiet

$$\frac{X + Y + 2\sqrt{XY}}{(y-x)^2} = C + \delta(x+y) + \varepsilon(x+y)^2,$$

ubi constantem per integrationem ingressam secundum indolem problematis ita definiri oportet, ut dum punctum X incidit in punctum F, punctum Y in ipsum punctum G cadat, sive ut facto $x = f$ fiat $y = g$.

§. 87. Cum jam sit

$$X + Y = 2\alpha + \beta(x+y) + \gamma(x^2+y^2) + \delta(x^3+y^3) + \varepsilon(x^4+y^4),$$

si terminos $\delta(x+y) + \varepsilon(x+y)^2$ in alteram partem transferimus, perveniemus ad hanc aequationem

$$\frac{2\alpha + \beta(x+y) + \gamma(x^2+y^2) + \delta xy(x+y) + 2\varepsilon xxyy + 2\sqrt{XY}}{(y-x)^2} = C.$$

Subtrahamus autem insuper utrinque γ, et loco $C - \gamma$ scribamus Δ, hocque modo nostra aequatio reducetur ad hanc formam satis concinnam

$$\frac{2\alpha + \beta(x+y) + 2\gamma xy + \delta xy(x+y) + 2\varepsilon xxyy + 2\sqrt{XY}}{(y-x)^2} = \Delta.$$

§. 88. Quia nunc Δ ita determinari debet, ut sumto $x = f$ fiat $y = g$, si secundum analogiam statuamus

$$\alpha + \beta f + \gamma ff + \delta f^3 + \varepsilon f^4 = F \text{ et}$$
$$\alpha + \beta g + \gamma gg + \delta g^3 + \varepsilon g^4 = G,$$

erit ista constans Δ ita expressa

$$\Delta = \frac{2\alpha + \beta(f+g) + 2\gamma fg + \delta fg(f+g) + 2\varepsilon ffgg + 2\sqrt{FG}}{(g-f)^2}.$$

Hac igitur aequatione inventa, si ipsi x pro lubitu tribuatur valor quicunque, inde elici poterit valor ipsius y, ita ut alter terminus X arcus quaesiti X Y pro arbitrio assumi possit. Verum facile patet, istam determinationem in calculos perquam molestos praecipitare, quandoquidem aequatio inventa quadratis sumendis ab irrationalitate $\sqrt{XY}$ liberari deberet. Sequenti autem modo ista investigatio sublevari poterit.

§. 89. Quoniam ista formula

$$2\alpha + \beta(x+y) + 2\gamma xy + \delta xy(x+y) + 2\varepsilon xxyy$$

essentialiter in calculum ingreditur, ejus loco brevitatis gratia scribamus hunc characterem $[x, y]$, cujus ergo valor erit cognitus, etiam si loco x et y aliae litterae accipiantur. Hoc igitur modo aequatio inventa ita referri poterit

$$\frac{[x, y] + 2\sqrt{XY}}{(y-x)^2} = \frac{[f, g] + 2\sqrt{FG}}{(g-f)^2},$$

quae ergo aequatio exprimit relationem inter bina ordinata x et y, ut problemati satisfiat, hoc est, ut fiat

$$\Pi : y - \Pi : x = \Pi : g - \Pi : f.$$

Quare cum hic etiam sequatur

$$\Pi : y - \Pi : g = \Pi : x - \Pi : f,$$

aequatio hinc ista exsurget

$$\frac{[g, y] + 2\sqrt{GY}}{(y-g)^2} = \frac{[f, x] + 2\sqrt{FX}}{(x-f)^2}.$$

§. 90. Ex hac jam aequatione cum priore conjuncta facile eliminari poterit formula radicalis $\sqrt{Y}$, sicque aequatio habebitur tantum litteram y tanquam incognitam involvens, unde ejus valor haud difficulter definiri potest. Calculum autem hunc instituenti patebit, tantum ad aequationem quadraticam perveniri, ita ut bini valores pro puncto Y reperiantur, quemadmodum rei natura postulat, dum sumto puncto X alterum punctum Y tam dextrorsum quam sinistrorsum cadere poterit. Hinc autem calculo fusius non immoramur, quandoquidem hic potissimum est propositum, totam hujus problematis solutionem per methodum directam a priori repetere.

Hypothesis 2.

Fig. 14. §. 91. Constituta super axe $o\,z$ curva O Z in priori hypothesi descripta, concipiatur super eodem axe alia curva insuper descripta $\mathfrak{O}\,\mathfrak{Z}$, ita comparata, ut abscissae $o\,z = z$ respondeat arcus $\mathfrak{O}\,\mathfrak{Z} = \phi : z$, ita ut sit

$$\phi : z = \int \frac{\partial z\,(\mathfrak{A} + \mathfrak{B}\,z + \mathfrak{C}\,z\,z + \mathfrak{D}\,z^3 + etc.)}{\sqrt{Z}},$$

integrali hoc pariter ita sumto ut evanescat posito $z = 0$, existente ut ante

$$Z = \alpha + \beta\,z + \gamma\,z\,z + \delta\,z^3 + \varepsilon\,z^4.$$

Pro numeratore autem ponamus brevitatis gratia

$$\mathfrak{A} + \mathfrak{B}\,z + \mathfrak{C}\,z\,z + \mathfrak{D}\,z^3 + \text{etc.} = \mathfrak{Z},$$

ita ut sit $\phi : z = \int \frac{\mathfrak{Z}\,\partial z}{\sqrt{Z}}$.

§. 92. Ista jam curva hac ratione descripta hac insigni proprietate erit praedita, ut, si in priore curva rescissi fuerint arcus F G et X Y inter se aequales, productis iisdem applicatis in nova curva, arcuum hoc modo rescissorum $\mathfrak{F}\,\mathfrak{G}$ et $\mathfrak{X}\,\mathfrak{Y}$ differentia vel algebraice vel saltem per logarithmos et arcus circulares assignari possit, cujus rei veritatem solutio sequentis problematis demonstrabit.

Problema 2.

Si in curva secundum primam hypothesin descripta abscissi fuerint duo arcus aequales F G et X Y, iisque in curva modo descripta respondeant arcus $\mathfrak{F}\,\mathfrak{G}$ et $\mathfrak{X}\,\mathfrak{Y}$, quibus scilicet eaedem abscissae in axe conveniant, differentiam inter hos binos arcus investigare.

Solutio.

§. 93. Quia igitur hic quaeritur differentia inter arcus $\mathfrak{F}\mathfrak{G}$ et $\mathfrak{X}\mathfrak{Y}$, ponatur ea $= V$, quae-ergo spectari poterit tanquam certa functio ipsarum x et y, si quidem puncta $\mathfrak{F}$ et $\mathfrak{G}$ tanquam fixa consideramus. Cum igitur sit arcus

$$\mathfrak{F}\mathfrak{G} = \Phi : g - \Phi : f \text{ et arcus}$$

$$\mathfrak{X}\mathfrak{Y} = \Phi : y - \Phi : x,$$

habebimus

$$\Phi : y - \Phi : x = \Phi : g - \Phi : f + V,$$

unde differentiando habebimus

$$\frac{\mathfrak{Y}\,\partial y}{\sqrt{Y}} - \frac{\mathfrak{X}\,\partial x}{\sqrt{X}} = \partial V,$$

quia litteras f et g pro constantibus habemus.

§. 94. Ponamus nunc ut supra factum est

$$\frac{\partial x}{\sqrt{X}} = \frac{\partial y}{\sqrt{Y}} = \partial t,$$

et haec aequatio induet istam formam

$$(\mathfrak{Y} - \mathfrak{X})\,\partial t = \partial V.$$

Verum in solutione primi problematis deducti fuimus ad hanc aequationem finalem

$$\frac{\partial p^2}{qq\,\partial t^2} = C + \delta p + \varepsilon pp,$$

unde fit

$$\frac{\partial p}{\partial t} = \sqrt{(C + \delta p + \varepsilon pp)} = \sqrt{(\Delta + \gamma + \delta p + \varepsilon pp)},$$

atque hinc colligimus

$$\partial t = \frac{\partial p}{q\sqrt{(\Delta + \gamma + \delta p + \varepsilon pp)}},$$

ubi est $p = x + y$ et $q = y - x$. Hoc ergo valore inducto aequatio differentialis resolvenda est

$$\partial V = \frac{(\mathfrak{Y} - \mathfrak{X})\,\partial p}{q\sqrt{(\Delta + \gamma + \delta p + \varepsilon pp)}},$$

ubi est

$$\mathfrak{X} = \mathfrak{A} + \mathfrak{B}\,x + \mathfrak{C}\,x\,x + \mathfrak{D}\,x^3 + \text{etc.}$$

similique modo

$$\mathfrak{Y} = \mathfrak{A} + \mathfrak{B}\,y + \mathfrak{C}\,y\,y + \mathfrak{D}\,y^3 + \text{etc.},$$

quousque libuerit continuando.

§. 95. Quod si jam hos valores substituamus, habebimus

$$\mathfrak{Y} - \mathfrak{X} = \mathfrak{B}\,(y - x) + \mathfrak{C}\,(y^2 - x^2) + \mathfrak{D}\,(y^3 - x^3)$$
$$+ \mathfrak{E}\,(y^4 - x^4) + \text{etc.}$$

unde si loco x et y introducamus quantitates p et q, ob $x = \frac{p-q}{2}$ et $y = \frac{p+q}{2}$, orientur sequentes valores

$$y - x = q,\ y^2 - x^2 = p\,q,\ y^3 - x^3 = \tfrac{1}{4}q\,(3\,p\,p + q\,q),$$
$$y^4 - x^4 = \tfrac{1}{2}p\,q\,(p\,p + q\,q),\ y^5 - x^5 = \tfrac{1}{16}q\,(5\,p^4 + 10\,p\,p\,q\,q + q^4).$$

§. 96. Quantitas ergo V per sequentes formulas integrales secundum numerum litterarum $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. determinatur

$$\mathrm{V} = \mathfrak{B}\int\frac{\partial p}{\sqrt{(\Delta + \gamma + \delta p + \varepsilon p p)}} + \mathfrak{C}\int\frac{p\,\partial p}{\sqrt{(\Delta + \gamma + \delta p + \varepsilon p p)}}$$
$$+ \tfrac{1}{4}\mathfrak{D}\int\frac{(3\,p\,p + q\,q)\,\partial p}{\sqrt{(\Delta + \gamma + \delta p + \varepsilon p p)}} + \tfrac{1}{2}\mathfrak{E}\int\frac{p\,(p\,p + q\,q)\,\partial p}{\sqrt{(\Delta + \gamma + \delta p + \varepsilon p p)}}$$
$$+ \tfrac{1}{16}\mathfrak{F}\int\frac{(5\,p^4 + 10\,p\,p\,q\,q + q^4)\,\partial p}{\sqrt{(\Delta + \gamma + \delta p + \varepsilon p p)}} + \text{etc.}$$

Quarum formularum duae priores jam absolute exhiberi possunt, sive algebraice, quod evenit si $\varepsilon = 0$, sive per logarithmos, si valor ipsius ε fuerit positivus, sive per arcus circulares, si valor ipsius ε fuerint negativus. Reliquae vero formulae exigunt relationem inter p et q, quam deinceps investigabimus. Hic tantum notetur, potestates solas pares ipsius q in has formulas ingredi.

§. 97. Hic autem littera Δ eundem valorem constantem designat, quem supra jam definivimus, qui erat

$$\Delta = \frac{2\alpha + \beta(f+g) + 2\gamma fg + \delta fg(f+g) + 2\varepsilon ffgg + 2\sqrt{FG}}{(g-f)^2}.$$

Praeterea vero cum esse debeat

$$\phi : y - \phi : x = \phi : g - \phi : f + V,$$

evidens est, casu quo $x = f$ et $y = g$ fieri debere $V = 0$; quamobrem formulae illae integrales pro V inventae ita capi debebunt, ut posito $p = f + g$ et $q = g - f$ valor ipsius V evanescat.

Analysis
pro investiganda relatione inter p et q.

§. 98. Quia jam invenimus aequationem finitam inter x et y, ex ea quoque ponendo $y = \frac{p+q}{2}$ et $x = \frac{p-q}{2}$ relatio inter litteras p et q derivari posset; verum hoc calculos nimis taediosos postularet, quamobrem aliam viam ineamus istam relationem ex formulis differentialibus deducendi. Cum enim sit $\frac{\partial p}{\partial q} = \frac{\partial y + \partial x}{\partial y - \partial x}$, ob proportionem

$$\partial x : \partial y = \sqrt{X} : \sqrt{Y} \text{ erit } \frac{\partial p}{\partial q} = \frac{\sqrt{Y} + \sqrt{X}}{\sqrt{Y} - \sqrt{X}};$$

supra autem invenimus esse

$$\frac{\sqrt{Y} + \sqrt{X}}{q} = \sqrt{(\Delta + \gamma + \delta p + \varepsilon pp)},$$

ubi Δ eandem denotat constantem, quam modo ante definivimus.

§. 99. Nunc igitur fractio pro $\frac{\partial p}{\partial q}$ inventa supra et infra multiplicetur per $\sqrt{Y} + \sqrt{X}$, et cum sit

$$(\sqrt{Y} + \sqrt{X})^2 = qq(\Delta + \gamma + \delta p + \varepsilon pp),$$

habebimus hanc aequationem

$$\frac{\partial p}{\partial q} = \frac{qq(\Delta + \gamma + \delta p + \varepsilon pp)}{Y - X},$$

cujus denominatorem jam supra §. 83. evolvimus, ubi invenimus esse

$$Y - X = \beta q + \gamma p q + \tfrac{1}{4}\delta q(3pp + qq) + \tfrac{1}{2}\varepsilon p q(pp + qq),$$

quo valore substituto erit

$$\frac{\partial p}{\partial q} = \frac{q(\Delta + \gamma + \delta p + \varepsilon p p)}{\beta + \gamma p + \frac{1}{4}\delta(3pp + qq) + \frac{1}{2}\varepsilon p(pp + qq)},$$

quae reducitur ad hanc formam

$$2q\partial q = \frac{[2\beta + 2\gamma p + \frac{1}{2}\delta(3pp + qq) + \varepsilon p(pp + qq)]\partial p}{\Delta + \gamma + \delta p + \varepsilon pp}.$$

100. Transferamus terminos qui continent qq a dextra in sinistram partem ut obtineamus hanc aequationem

$$2q\partial q - \frac{qq\partial p(\frac{1}{2}\delta + \varepsilon p)}{\Delta + \gamma + \delta p + \varepsilon pp} = \frac{(2\beta + 2\gamma p + \frac{3}{2}\delta pp + \varepsilon p^3)\partial p}{\Delta + \gamma + \delta p + \varepsilon pp}.$$

Membrum hujus aequationis sinistrum integrabile reddi potest, si per certam functionem ipsius p, quae sit $= \Pi$, multiplicetur, quando fuerit

$$\frac{\partial \Pi}{\Pi} = -\frac{\partial p(\frac{1}{2}\delta + \varepsilon p)}{\Delta + \gamma + \delta p + \varepsilon pp},$$

quae aequatio integrata dat

$$l\Pi = -\tfrac{1}{2}l(\Delta + \gamma + \delta p + \varepsilon pp).$$

Sicque erit multiplicator iste

$$\Pi = \frac{1}{\sqrt{(\Delta + \gamma + \delta p + \varepsilon pp)}};$$

tum autem integrale quaesitum erit

$$\frac{qq}{\sqrt{(\Delta + \gamma + \delta p + \varepsilon pp)}} = \int \frac{(2\beta + 2\gamma p + \frac{3}{2}\delta pp + \varepsilon p^3)\partial p}{(\Delta + \gamma + \delta p + \varepsilon pp)^{\frac{3}{2}}}.$$

§. 101. Hoc postremum integrale manifesto continet formam $\frac{pp}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}$, quippe cujus differentiale est

$$\frac{(2\Delta p+2\gamma p+\frac{3}{2}\delta pp+\varepsilon p^3)\partial p}{(\Delta+\gamma+\delta p+\varepsilon pp)^{\frac{3}{2}}};$$

quare integrale ita potest repraesentari

$$\frac{qq}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}=\frac{pp}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}+\int\frac{(2\beta-2\Delta p)\partial p}{(\Delta+\gamma+\delta p+\varepsilon pp)^{\frac{3}{2}}},$$

quod postremum integrale statuatur $=\frac{m+np}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}$, hujus enim differentiale est

$$\frac{[(\Delta+\gamma)n-\frac{1}{2}\delta m+(\frac{1}{2}\delta n-\varepsilon m)p]\partial p}{(\Delta+\gamma+\delta p+\varepsilon pp)^{\frac{3}{2}}},$$

ideoque fieri debet

$$(\Delta+\gamma)n-\tfrac{1}{2}\delta m=2\beta \text{ et}$$
$$\tfrac{1}{2}\delta n-\varepsilon m=-2\Delta,$$

unde deducuntur valores

$$m=\frac{4\beta\delta+8\Delta\Delta+8\Delta\gamma}{4\Delta\varepsilon+4\gamma\varepsilon-\delta\delta} \text{ et } n=\frac{8\beta\varepsilon+4\Delta\delta}{4\Delta\varepsilon+4\gamma\varepsilon-\delta\delta},$$

quarum fractionum loco in calculo retineamus litteras m et n, consequenter adjecta constante aequatio integralis ita se habebit

$$qq=pp+np+m+C\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}.$$

§. 102. Ista autem constans ita definiri debet, ut posito $p=f+g$ fiat $q=g-f$, ex quo quantitas illa constans ita determinabitur

$$C=-\frac{4fg-n(f+g)-m}{\sqrt{[\Delta+\gamma+\delta(f+g)+\varepsilon(f+g)^2]}}.$$

Hoc ergo valore invento, facile assignari poterunt valores non solum ipsius $q\,q$ sed etiam ejus potestatum parium q^4, q^6, q^8, etc., quibus indigemus. Atque hinc intelligitur pro inveniendo valore ipsius V alias formulas integrales non occurrere nisi quae involvant quantitatem radicalem $\sqrt{(\Delta + \gamma + \delta p + \varepsilon p p)}$, quarum ergo integratio, nisi algebraice institui queat, semper per logarithmos et arcus circulares expediri poterit. Evidens autem est, casu quo $\varepsilon = 0$ omnia integralia algebraica exprimi posse.

§. 103. Quod si ergo pro priori curva O Z fuerit

$$\Pi : z = \int \frac{\partial z}{\sqrt{(\alpha + \beta z + \gamma z^2 + \delta z^3)}},$$

pro altera vero curva

$$\Phi : z = \int \frac{\partial z (\mathfrak{A} + \mathfrak{B} z + \mathfrak{C} z z + \mathfrak{D} z^3 + etc.)}{\sqrt{(\alpha + \beta z + \gamma z z + \delta z^3)}},$$

tum sumtis in priori curva arcubus aequalibus F G et X Y, iis in altera curva respondebunt arcus $\mathfrak{F}\mathfrak{G}$ et $\mathfrak{X}\mathfrak{Y}$, quorum differentia semper geometrice assignari poterit. Interdum etiam fieri potest, ut differentia V in nihilum abeat, id quod quidem semper evenit, sumto $x = f$.

§. 104. Praeterea vero etiam datur alius casus maxime memorabilis, quod differentia illa V algebraice exprimi poterit, qui scilicet semper locum habebit, quando tam in denominatore quam in numeratore tantum potestates pares ipsius z occurrunt, hoc est si fuerit pro curva priore

$$\Pi : z = \int \frac{\partial z}{\sqrt{(\alpha + \gamma z z + \varepsilon z^4)}},$$

pro altera vero curva

$$\Phi : z = \int \frac{\partial z (\mathfrak{A} + \mathfrak{C} z z + \mathfrak{E} z^4 + \mathfrak{G} z^6 + etc.)}{\sqrt{(\alpha + \gamma z z + \varepsilon z^4)}}.$$

His enim casibus, si in priore curva arcus aequales F G et X Y abscindantur, tum arcuum in altera curva respondentium

$\mathfrak{F}\,\mathfrak{G}$ et $\mathfrak{X}\,\mathfrak{Y}$ differentia semper algebraice seu geometrice exhiberi poterit, ad quotcunque terminos etiam numerator $\mathfrak{A} + \mathfrak{C}\,zz + \mathfrak{E}\,z^4 +$ etc. continuetur, atque hic est casus, quem olim tam in calculo integrali quam alibi fusius pertractavi.

§. 105. Ad hoc ostendendum, quia habemus tam $\delta = 0$ quam $\beta = 0$, primo erit

$$qq = pp + m + C\surd(\Delta + \gamma + \varepsilon pp),$$

ita ut hic tantum potestates pares ipsius p occurrant, tum autem pro litteris germanicis $\mathfrak{C}$, $\mathfrak{E}$, $\mathfrak{G}$, etc. formulae integrandae sequenti modo se habebunt:

Pro littera $\mathfrak{C}$ . . . . . $\int \frac{p\,\partial q}{\surd(\Delta + \gamma + \varepsilon pp)}$,

quae per se est absolute integrabilis.

Pro littera $\mathfrak{E}$ . . . . . $\int \frac{p(pp + qq)\,\partial p}{\surd(\Delta + \gamma + \varepsilon pp)}$,

quae loco qq substituto valore induet hanc formam

$$\int \frac{p(2pp + m)\,\partial p}{\surd(\Delta + \gamma + \varepsilon pp)} + C\int p\,\partial p,$$

ubi integratio est manifesta, quod etiam usu venit pro sequentibus formulis litteris $\mathfrak{G}$, $\mathfrak{F}$, affectis. Evidens enim est, si ponatur $\surd(\Delta + \gamma + \varepsilon pp) = s$ fieri

$$pp = \frac{ss - \Delta - \gamma}{\varepsilon}, \text{ et } p\,\partial p = \frac{s\,\partial s}{\varepsilon}, \text{ ideoque}$$

$$\frac{p\,\partial p}{\surd(\Delta + \gamma + \varepsilon pp)} = \frac{\partial s}{\varepsilon},$$

qua substitutione omnes formulae integrandae fiunt rationales et integrae.

§. 106. Cum autem iste posterior casus jam satis prolixe sit tractatus, ac pluribus exemplis a rectificatione Ellipsis et Hyperbolae desumtis illustratus, casus prior quo tantum erat $\varepsilon = 0$ eo majore attentione est dignus, quod quantum equidem scio, a nemine adhuc est observatus, cujus ergo evolutio novae huic methodo unice accepta est referenda. Quemadmodum autem haec deducta sunt ex relatione inter p et q, ita etiam relatio elegantissima erui potest inter has quantitates $p = x + y$ et $u = x\,y$, quam hic subjungamus.

Analysis
pro investiganda relatione inter p et u.

§. 107. Hic pariter primo in relationem inter ∂p et ∂u inquiramus, et cum sit

$$\frac{\partial p}{\partial u} = \frac{\partial x + \partial y}{y \partial x + x \partial y}, \text{ ob}$$

$$\partial x : \partial y = \sqrt{X} : \sqrt{Y} \text{ erit}$$

$$\frac{\partial p}{\partial u} = \frac{\sqrt{X} + \sqrt{Y}}{y\sqrt{X} + x\sqrt{Y}},$$

et sumtis quadratis

$$\frac{\partial p^2}{\partial u^2} = \frac{X + Y + 2\sqrt{XY}}{yyX + xxY + 2xy\sqrt{XY}}.$$

Supra autem vidimus esse

$$(\sqrt{X} + \sqrt{Y})^2 = qq(\Delta + \gamma + \delta p + \varepsilon pp), \text{ existente } q = y - x.$$

Pro denominatore autem utamur relatione §. 87. inventa

$$\Delta = \frac{2\alpha + \beta(x + y) + 2\gamma xy + \delta xy(x + y) + 2\varepsilon xxyy + 2\sqrt{XY}}{(y - x)^2},$$

unde fit

$$2\sqrt{XY} = \Delta qq - 2\alpha - \beta p - 2\gamma u - \delta pu - 2\varepsilon uu,$$

quo valore substituto aequatio nostra erit

$$\frac{\partial p^2}{\partial u^2} = \frac{qq(\Delta + \gamma + \delta p + \varepsilon pp)}{yyX + xxY + \Delta qqu - 2\alpha u - \beta pu - 2\gamma uu - \delta puu - 2\varepsilon u^3}.$$

§. 108. Hic autem substitutis loco X et Y valoribus, habebimus primo

$$yyX + xxY = \alpha(xx + yy) + \beta xy(x + y) + 2\gamma xxyy + \delta xxyy(x + y) + \varepsilon xxyy(xx + yy),$$

quae ob $x+y=p$, $xy=u$ et $xx+yy=pp-2u$, erit

$$yyX+xxY=\alpha(pp-2u)+\beta pu+2\gamma uu+\delta puu$$
$$+\varepsilon uu(pp-2u),$$

unde totus denominator reperietur fore

$$\alpha(pp+4u)+\varepsilon uu(pp-4u)+\Delta qqu,$$

quare cum sit $pp-4u=qq$, nostra fractio erit

$$\frac{\partial p^2}{\partial u^2}=\frac{\Delta+\gamma+\delta p+\varepsilon pp}{\Delta u+\alpha+\varepsilon uu},$$

unde sequitur haec aequatio separata

$$\frac{\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}=\frac{\partial u}{\sqrt{(\alpha+\Delta u+\varepsilon uu)}}:$$

unde deducitur hoc

Theorema memorabile.

§. 109. Si inter binas variabiles x et y habeatur haec aequatio differentialis

$$\frac{\partial x}{\sqrt{(\alpha+\beta x+\gamma xx+\delta x^3+\varepsilon x^4)}}=\frac{\partial y}{\sqrt{(\alpha+\beta y+\gamma yy+\delta y^3+\varepsilon y^4)}},$$

tum posito $x+y=p$ et $xy=u$, inter has variabiles p et u semper locum habebit haec aequatio differentialis

$$\frac{\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}=\frac{\partial u}{\sqrt{(\alpha+\Delta u+\varepsilon uu)}},$$

ubi Δ quidem est constans arbitraria in aequationem posteriorem ingressa, contra vero etiam prior aequatio continet constantem arbitrariam β in altera non occurrentem.

§. 110. Aequationis autem posterioris integratio in promptu est. Si enim utrinque multiplicemus per $\sqrt{\varepsilon}$, integrale per logarithmos ita exprimitur

$$l[p\sqrt{\varepsilon}+\tfrac{\delta}{2\sqrt{\varepsilon}}+\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}]=$$
$$l[u\sqrt{\varepsilon}+\tfrac{\Delta}{2\sqrt{\varepsilon}}+\sqrt{(\alpha+\Delta u+\varepsilon uu)}]+l\Gamma,$$

ideoque integrale ita algebraice exprimetur

$$\varepsilon p + \tfrac{1}{2}\delta + \sqrt{\varepsilon(\Delta + \gamma + \delta p + \varepsilon pp)} = \Gamma[\varepsilon u + \tfrac{1}{2}\Delta + \sqrt{\varepsilon(\alpha + \Delta u + \varepsilon uu)}].$$

Ubi constans ista Γ facile definitur ex conditione, quod posito $x = f$ fieri debet $y = g$, hoc est ut posito $p = f + g$ fiat $u = fg$, quippe ex qua conditione constans prior Δ jam est definita.

§. 111. Quo hinc jam facilius sive p per u sive u per p definiri possit, notatur esse

$$\frac{1}{\varepsilon p + \frac{1}{2}\delta + \sqrt{[\varepsilon(\Delta + \gamma + \delta p + \varepsilon pp)]}} = \frac{\varepsilon p + \frac{1}{2}\delta - \sqrt{[\varepsilon(\Delta + \gamma + \delta p + \varepsilon pp)]}}{\frac{1}{4}\delta\delta - \varepsilon(\Delta + \gamma)} \text{ et}$$

$$\frac{1}{\varepsilon u + \frac{1}{2}\Delta + \sqrt{[\varepsilon(\alpha + \Delta u + \varepsilon uu)]}} = \frac{\varepsilon u + \frac{1}{2}\Delta - \sqrt{[\varepsilon(\alpha + \Delta u + \varepsilon uu)]}}{\frac{1}{4}\Delta\Delta - \alpha\varepsilon}.$$

Hinc igitur per inversionem sequens aequatio resultabit

$$\frac{\varepsilon p + \frac{1}{2}\delta - \sqrt{\varepsilon(\Delta + \gamma + \delta p + \varepsilon pp)}}{\frac{1}{4}\delta\delta - \varepsilon(\Delta + \gamma)} = \frac{1}{\Gamma} \cdot \frac{\varepsilon u + \frac{1}{2}\Delta - \sqrt{\varepsilon(\alpha + \Delta u + \varepsilon uu)}}{\frac{1}{4}\Delta\Delta - \alpha\varepsilon}, \text{ sive}$$

$$\varepsilon p + \tfrac{1}{2}\delta - \sqrt{\varepsilon(\Delta + \gamma + \delta p + \varepsilon pp)} = \frac{\frac{1}{4}\delta\delta - \varepsilon(\Delta + \gamma)}{\Gamma(\frac{1}{4}\Delta\Delta - \alpha\varepsilon)} \times [\varepsilon u + \tfrac{1}{2}\Delta - \sqrt{\varepsilon(\alpha + \Delta u + \varepsilon uu)}],$$

ex quibus duabus aequationibus sine alio negotio sive p per u sive u per p exprimi poterit.

§. 112. Hoc igitur modo loco variabilis p pro invenienda quantitate V facile introduci posset variabilis u, si quidem loco formulae $\frac{\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon p p)}}$ substituatur formula ipsi aequalis $\frac{\partial u}{\sqrt{(\alpha+\Delta u+\varepsilon u u)}}$. Verum hoc modo casus illi, quibus quantitas V fieri potest algebraica, non tam facile patescent; interim tamen etiam hoc modo certi erimus, tam casibus quibus $\varepsilon=0$, quam quo $\beta=0$, $\delta=0$ etc. in serie $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc. tantum potestates pares occurrunt, omnes integrationes algebraice succedere debere. Coronidis loco adhuc aliam relationem inter quantitates p et u investigemus, cujus contemplatio insigne incrementum in integratione aequationum polliceri videtur.

Alia Analysis
pro investigatione relationis inter p et u.

§. 113. Cum sit ut ante vidimus $\frac{\partial p}{\partial u}=\frac{\sqrt{X}+\sqrt{Y}}{y\sqrt{X}+x\sqrt{Y}}$, multiplicemus supra et infra per $\sqrt{X}+\sqrt{Y}$, ut numerator evadat

$$(\sqrt{X}+\sqrt{Y})^2=qq(\Delta+\gamma+\delta p+\varepsilon pp);$$

tum autem denominatur prodibit

$$yX+xY+(x+y)\sqrt{XY},$$

ubi denominatoris pars rationalis dat

$$\alpha p+2\beta xy+\gamma xy(x+y)+\delta xy(xx+yy)+\varepsilon xy(x^3+y^3),$$

quae expressio, ob $x+y=p$, $y-x=q$, et $xy=u$, abit in

$$\alpha p+2\beta u+\gamma pu+\delta u(pp-2u)+\varepsilon pu(pp-3u).$$

Deinde ante vidimus esse

$$2\sqrt{XY}=\Delta qq-2\alpha-\beta p-2\gamma u-\delta pu-2\varepsilon uu,$$

quod ductum in $\frac{1}{2}p$ et superiori additum praebet

$$\tfrac{1}{2}\Delta pqq - \tfrac{1}{2}\beta(pp-4u) + \tfrac{1}{2}\delta u(pp-4u) + \varepsilon pu(pp-4u),$$

quae denominator ob $pp-4u=qq$ induet hanc formam

$$\tfrac{1}{2}\Delta pqq - \tfrac{1}{2}\beta qq + \tfrac{1}{2}\delta uqq + \varepsilon puqq:$$

hinc aequatio erit

$$\frac{\partial p}{\partial u} = \frac{\Delta + \gamma + \delta p + \varepsilon pp}{\frac{1}{2}\Delta p - \frac{1}{2}\beta + \frac{1}{2}\delta u + \varepsilon pu},$$

unde deducitur

$$\partial p(\tfrac{1}{2}\Delta p - \tfrac{1}{2}\beta + \tfrac{1}{2}\delta u + \varepsilon pu) = \partial u(\Delta + \gamma + \delta p + \varepsilon pp),$$

quae ergo certe est integrabilis; id quod adeo inde patet, quod altera variabilis u nusquam ultra primam dimensionem exsurgit.

§. 114. Verum adhuc alio modo relatio inter p et u investigari potest; scilicet aequatio primo inventa

$$\frac{\partial p}{\partial u} = \frac{\sqrt{X} + \sqrt{Y}}{y\sqrt{X} + x\sqrt{Y}},$$

si supra et infra multiplicetur per $\sqrt{Y} - \sqrt{X}$ dabit

$$\frac{\partial p}{\partial u} = \frac{Y - X}{-yX + xY + \sqrt{XY}(y-x)}.$$

Nunc igitur pro numeratore habebimus

$$\beta q + \gamma pq + \delta q(pp-u) + \varepsilon pq(pp-2u).$$

Pro denominatore vero pars rationalis erit

$$-\alpha q + \gamma qu + \delta pqu + \varepsilon qu(pp-u),$$

pars vero irrationalis

$$\tfrac{1}{2}\Delta q^3 - \alpha q - \tfrac{1}{2}\beta pq - \gamma qu - \tfrac{1}{2}\delta pqu - \varepsilon quu,$$

unde totus denominator conficitur

$$\tfrac{1}{2}\Delta q^3 - \alpha q - \tfrac{1}{2}\beta pq + \tfrac{1}{2}\delta pqu + \varepsilon qu(pp - 2u),$$

unde sequitur haec aequatio differentialis

$$\frac{\partial p}{\partial u} = \frac{\beta + \gamma p + \delta(pp - u) + \varepsilon p(pp - 2u)}{\tfrac{1}{2}\Delta(pp - 4u) - 2\alpha - \tfrac{1}{2}\beta p + \tfrac{1}{2}\delta pu + \varepsilon u(pp - 2u)},$$

quae in ordinem redacta ita se habebit

$$\partial p[\Delta(pp - 4u) - 4\alpha - \beta p + \delta pu + 2\varepsilon u(pp - 2u)] =$$
$$2\partial[\beta + \gamma p + \delta(pp - u) + \varepsilon p(pp - 2u)],$$

quae jam ita est comparata, ut nulla via ejus integrationem instituenda perspici queat, etiamsi ejus integrale revera exhibere queamus.

§. 115. Alio insuper modo relationem inter p et u definire licet, si aequationis

$$\frac{\partial p}{\partial u} = \frac{\sqrt{X} + \sqrt{Y}}{y\sqrt{X} + x\sqrt{Y}}$$

posterius membrum supra et infra multiplicemus per $y\sqrt{X} - x\sqrt{Y}$ ut prodeat

$$\frac{\partial p}{\partial u} = \frac{yX - xY + (y - x)\sqrt{XY}}{yyX - xxY}.$$

Nunc enim denominator evadet

$$\alpha pq + \beta qu + \delta quu - \varepsilon pquu.$$

Pro numeratore autem pars rationalis praebet

$$\alpha q - \gamma qu - \delta pqu - \varepsilon qu(pp - u),$$

et pars irrationalis

$$\tfrac{1}{2}\Delta q^3 - \alpha q - \tfrac{1}{2}\beta pq - \gamma qu - \tfrac{1}{2}\delta pqu - \varepsilon quu,$$

totus igitur numerator erit

$$\tfrac{1}{2}\Delta q^3 - \tfrac{1}{2}\beta pq - 2\gamma qu - \tfrac{3}{2}\delta pqu - \varepsilon qupp,$$

ideoque

$$\frac{\partial p}{\partial u} = \frac{\frac{1}{2}\Delta(pp-4u) - \frac{1}{2}\beta p - 2\gamma u - \frac{3}{2}\delta pu - \varepsilon ppu}{\alpha p + \beta u - \delta uu - \varepsilon puu},$$

sive

$$2\,\partial p\,(\alpha p + \beta u - \delta uu - \varepsilon puu) =$$
$$\partial u\,[\Delta(pp - 4u) - \beta p - 4\gamma u - 3\delta pu - 2\varepsilon ppu].$$

Hic autem penitus non patet, quomodo multiplicator hanc aequationem integrabilem reddens investigari debeat, unde nullum est dubium, quin ista contemplatio haud parum ad limites analyseos prolatandos conferre possit.