3.) De formulis integralibus implicatis, earumque evolutione et transformatione. *M. S. Academiae exhib. die* 20 *Aprilis* 1778.

§. 36. Talium formularum implicatarum forma generalis ita exhiberi potest

$$\int p\,\partial x \int q\,\partial x \int r\,\partial x \int s\,\partial x \text{ etc.}$$

ubi quodvis signum integrale omnia sequentia in se complectitur. Ita ad valorem hujus expressionis inveniendum a fine est incipiendum, positoque integrali $\int s\,\partial x = S$ erit

$$\int r\,\partial x \int s\,\partial x = \int S\,r\,\partial x,$$

cujus valor si ponatur $= R$, erit

$$\int q\,\partial x \int r\,\partial x \int s\,\partial x = \int R\,q\,\partial x,$$

quod integrale si ponatur $= Q$, valor ipsius formulae propositae erit $= \int Q\,p\,\partial x$, ubi per se intelligitur, in qualibet integratione more solito constantem arbitrariam in calculum introduci posse.

§. 37. Hic scilicet probe tenendum est, istam expressionem $\int p\,\partial x \int q\,\partial x$ non significare productum ex formula $\int p\,\partial x$ in formulam $\int q\,\partial x$, sed integrale quod oritur, si tota formula differentialis $p\,\partial x \int q\,\partial x$ integretur: at vero si velimus productum talium duarum formularum integralium designare, id interpositione puncti fieri solet hoc modo $\int p\,\partial x \,.\, \int q\,\partial x$, ubi scilicet punctum declarat praecedentia signa integralia non ultra hunc terminum extendi debere, ita haec forma

$$\int p\,\partial x \int q\,\partial x \,.\, \int r\,\partial x \int s\,\partial x$$

exprimit productum, quod oritur si formula $\int p\,\partial x \int q\,\partial x$ multiplicetur per $\int r\,\partial x \int s\,\partial x$.

§. 38. Hic igitur signandi mos prorsus contrarius usu est receptus, atque in formulis differentialibus observari solet, ubi talis expressio $\partial x \partial y \partial z$ denotat productum trium differentialium ∂x, ∂y et ∂z, ita ut singula signa differentiationis tantum litteras immediate sequentes afficiant: at si velimus verbi gratia differentiale hujus expressionis $x \partial y \partial z$ exprimere, hoc interpositione puncti fieri solet $\partial . x \partial y \partial z$, ubi punctum significat, praefixum ∂ complecti totam expressionem sequentem.

§. 39. Tales autem formulae integrales implicatae potissimum nascuntur ex continua integratione aequationum integralium linearium, quarum forma in genere est

$$pz + \frac{q \partial z}{\partial x} + \frac{r \partial \partial z}{\partial x^2} + \frac{s \partial^3 z}{\partial x^3} + \text{etc.} = X,$$

ubi litterae p, q, r, s, etc. sunt functiones datae variabilis x, cujus etiam functio quaecunque sit littera X, altera vero variabilis z ubique unam tantum tenet dimensionem, prouti haec forma generalis hic exhibetur, ad ordinem tertium differentialium refertur, ideoque ternas integrationes postulat, totidemque constantes arbitrarias involvere est censenda, hic scilicet ad methodum integrandi maxime naturalem respicio, quae per ternas integrationes successivas integrale desideratum producat.

§. 40. Tali scilicet aequatione proposita ante omnia nosse oportet multiplicatorem, quo ea reddatur integrabilis, quem ergo supponamus esse $= \partial P$, atque integratione peracta prodeat ista aequatio

$$p' z + \frac{q' \partial z}{\partial x} + \frac{r' \partial \partial z}{\partial x^2} = \int X \partial P,$$

quae aequatio jam est ordinis secundi; quodsi jam ponamus hujus multiplicatorem idoneum esse $= \partial P'$, facta integratione oriatur haec aequatio primi ordinis, quae sit

$$p'' z + \frac{q'' \partial z}{\partial x} = \int \partial P' \int X \partial P,$$

pro qua si $\partial P''$ fuerit multiplicator idoneus, completum integrale induet hanc formam

$$p''' z = \int \partial P'' \int \partial P' \int X \partial P.$$

Sicque quantitas z exprimetur per formulam integralem implicatam.

§. 41. Tali autem forma pro integrali inventa praecipuum negotium huc redit, ut ea ita evolvatur, ut formula continens functionem indefinitam X, quae hic terna signa integralia habet praefixa, plus unico ante se non habeat, quamobrem quemadmodum talis reductio commodissime institui queat, hic ostendere constitui, siquidem nisi certa artificia adhibeantur, hujusmodi operatio calculos maxime molestos postularet.

§. 42. In genere autem hujusmodi formulas implicatas ita repraesentemus

$$\int \partial p \int \partial q \int \partial r \int \partial s \int \partial t \text{ etc.}$$

pro cujus evolutione a casu duorum signorum integralium inchoemus, et quia erit $\int \partial p \int \partial q = \int q \partial p$, reductio vulgaris dat $pq - \int p \partial q$. Jam loco p et q iterum scribamus $\int \partial p$ et $\int \partial q$, atque evolutio ita se habebit

$$\int \partial p \int \partial q = \int \partial p \cdot \int \partial q - \int \partial q \int \partial p,$$

ubi in genere hanc aequalitatem notasse juvabit

$$\int \partial p \int \partial q - \int \partial p \cdot \int \partial q + \int \partial q \int \partial p = 0.$$

§. 43. Consideremus nunc formulam tria signa integralia involventem $= \int \partial p \int \partial q \int \partial r$, et quia ut modo vidimus est $\int \partial q \int \partial r = q r - \int q \partial r$, nostra formula in has partes discerpitur $\int q r \partial p - \int \partial p \int q \partial r$, quae posterior pars reducitur ad hanc formam $p \int q \partial r - \int p q \partial r$, sicque formula nostra erit

$\int q r \partial p - p \int q \partial r + \int p q \partial r$. Quoniam nunc requiritur, ut elementum ∂r in singulis partibus unicum tantum signum integrale habeat praefixum; ponamus $q \partial p = \partial v$ ut sit

$$v = \int q \partial p = \int \partial p \int \partial q, \text{ eritque}$$

$$\int q r \partial p = \int r \partial v = r v - \int v \partial r,$$

hincque colligitur

$$\int p q \partial r - \int v \partial r = \int \partial r (p q - v) = \int \partial r \int p \partial q.$$

Jam loco litterarum finitarum differentialia rursus introducentur, atque valor quaesitus formulae $\int \partial p \int \partial q \int \partial r$ sequenti modo exprimetur

$$\int \partial p \int \partial q . \int \partial r - \int \partial p . \int \partial r \int \partial q + \int \partial r \int \partial q \int \partial p,$$

ubi in singulis membris elemento ∂r unicum signum integrale est praefixum.

§. 44. Inter terna igitur elementa ∂p, ∂q et ∂r sequentem relationem notari operae erit pretium

$$\int \partial p \int \partial q \int \partial r - \int \partial p \int \partial q . \int \partial r + \int \partial p . \int \partial r \int \partial q - \int \partial r \int \partial q \int \partial p = 0,$$

quodsi autem similem reductionem pro casibus plurium signorum integralium exsequi vellemus, in calculos molestissimos ac taediosissimos delaberemur; interim tamen totum hoc negotium per sequentia theoremata facillime et planissime expedietur, et quoniam singula membra ope puncti in duos factores resolvi convenit, ubi talis factor deest, ejus locum unitate supplebimus.

Theorema 1.

§. 45. Pro unico elemento ∂p haec relatio habetur $\int \partial p . 1 - 1 . \int \partial p = 0$, maxime obvia.

Theorema 2.

§. 46. Inter bina elementa ∂p et ∂q semper locum habebit haec relatio

$$\int \partial p \int \partial q \,.\, 1 - \int \partial p \,.\, \int \partial q + 1 \,.\, \int \partial q \int \partial p = 0.$$

Demonstratio.

Ad hoc demonstrandum sufficiet ostendisse, differentiale hujus aequationis esse $= 0$, quoniam vero singula membra binis constant factoribus, seorsim considerentur differentialia ex factoribus prioribus et posterioribus oriunda, hic igitur ex factoribus prioribus oritur differentiale $\partial p (\int \partial q \,.\, 1 - 1 \,.\, \int \partial q) = 0$ per theorema 1. At ex factoribus posterioribus oritur differentiale $- \partial q (\int \partial p \,.\, 1 - 1 \,.\, \int \partial p) = 0$.

Theorema 3.

§. 47. Inter terna elementa ∂p, ∂q et ∂r semper haec relatio locum habet

$$\int \partial p \int \partial q \int \partial r \,.\, 1 - \int \partial p \int \partial q \,.\, \int \partial r + \int \partial p \,.\, \int \partial r \int \partial q - 1 \,.\, \int \partial r \int \partial q \int \partial p = 0.$$

Demonstratio.

Hic iterum seorsim perpendantur differentialia tam ex prioribus quam ex posterioribus factoribus oriunda; ex prioribus autem oritur

$$\partial p (\int \partial q \int \partial r \int \,.\, 1 - \int \partial q \,.\, \int \partial r + 1 \,.\, \int \partial r \int \partial q),$$

cujus valor manifesto ad nihilum redigitur per theorema 2. si scilicet litterae p et q uno gradu promoveantur; tum vero differentiale ex factoribus posterioribus ortum est

$$- \partial r (\int \partial p \int \partial q \,.\, 1 - \int \partial p \,.\, \int \partial q + 1 \,.\, \int \partial q \int \partial p),$$

cujus valor pariter per theorema praecedens evanescit, quoniam igitur ambo differentialia sunt $= 0$, etiam ipsa forma nihilo vel etiam constanti aequalis esse debet, evidens autem est constantem sponte involvi in signis integralibus.

Theorema 4.

§. 48. Inter quaterna elementa ∂p, ∂q, ∂r et ∂s semper ista relatio locum habet

$$\left.\begin{array}{l} \int\partial p\int\partial q\int\partial r\int\partial s.1-\int\partial p\int\partial q\int\partial r.\int\partial s \\ +\int\partial p\int\partial q.\int\partial s\int\partial r-\int\partial p.\int\partial s\int\partial r\int\partial q \\ +1.\int\partial s\int\partial r\int\partial q\int\partial p \end{array}\right\}=0.$$

Demonstratio.

Differentiatio factorum priorum suppeditat sequentem expressionem

$$\partial p(\int\partial q\int\partial r\int\partial s.1-\int\partial q\int\partial r.\int\partial s+\int\partial q.\int\partial r\int\partial s-1\int\partial q\int\partial r\int\partial s),$$

quae ob theorema praecedens ad nihilum reducitur. Simili modo differentiatio factorum posteriorum praebet hanc expressionem

$$-\partial s(\int\partial p\int\partial q\int\partial r.1-\int\partial p\int\partial q.\int\partial r+\int\partial p.\int\partial r\int\partial q-1.\int\partial r\int\partial q\int\partial p),$$

quae ob theorema 3. iterum est $= 0$.

Theorema 5.

§. 49. Inter quina elementa ∂p, ∂q, ∂r, ∂s et ∂t semper haec relatio locum habet

$$\left.\begin{array}{l} \int\partial p\int\partial q\int\partial r\int\partial s\int\partial t.1-\int\partial p\int\partial q\int\partial r\int\partial s.\int\partial t \\ +\int\partial p\int\partial q\int\partial r.\int\partial t\int\partial s-\int\partial p\int\partial q.\int\partial t\int\partial s\int\partial r \\ +\int\partial p.\int\partial t\int\partial s\int\partial r\int\partial q-1.\int\partial t\int\partial s\int\partial r\int\partial q\int\partial p \end{array}\right\}=0.$$

Demonstratio.

Hujus theorematis demonstratio prorsus eodem modo se habet ac theorematum praecedentium; sicque clarissime jam est evictum tales relationes perpetuo veritati esse consentaneas, quotcunque etiam elementis fuerint composita.

§. 50. Quo vis horum theorematum clarius perspiciatur, operae pretium erit, ea per exempla determinata illustrasse; ponamus igitur esse

$$\partial p = x^{\alpha-1}\partial x,\ \partial q = x^{\beta-1}\partial x,\ \partial r = x^{\gamma-1}\partial x,$$
$$\partial s = x^{\delta-1}\partial x,\ \partial t = x^{\varepsilon-1}\partial x,$$

atque ex theoremate primo statim aequatio identica nascitur $\frac{x^\alpha}{\alpha} - \frac{x^\alpha}{\alpha} = 0$. Verum theorema secundum nobis praebet hanc aequationem

$$\frac{x^{\alpha+\beta}}{\beta(\alpha+\beta)} - \frac{x^{\alpha+\beta}}{\alpha\beta} + \frac{x^{\alpha+\beta}}{\alpha(\alpha+\beta)} = 0,$$

unde per $x^{\alpha+\beta}$ dividendo prodit haec aequalitas

$$\frac{1}{\beta(\alpha+\beta)} - \frac{1}{\alpha\beta} + \frac{1}{\alpha(\alpha+\beta)} = 0,$$

cujus veritas satis facile in oculos incurrit.

§. 51. Hae porro positiones in theoremate tertio introductae producent hanc aequationem

$$\frac{x^{\alpha+\beta+\gamma}}{\gamma(\alpha+\beta+\gamma)(\beta+\gamma)} - \frac{x^{\alpha+\beta+\gamma}}{\beta\gamma(\alpha+\beta)} + \frac{x^{\alpha+\beta+\gamma}}{\alpha\beta(\beta+\gamma)} - \frac{x^{\alpha+\beta+\gamma}}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)},$$

unde per $x^{\alpha+\beta+\gamma}$ dividendo prodit haec egregia aequalitas

$$\frac{1}{\gamma(\beta+\gamma)(\alpha+\beta+\gamma)} - \frac{1}{\beta\gamma(\alpha+\beta)} + \frac{1}{\alpha\beta(\beta+\gamma)} - \frac{1}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)} = 0,$$

§. 52. Hae positiones iterum in theoremate quarto substitutae dant hanc aequationem

$$\left.\begin{aligned}&\frac{x^{\alpha+\beta+\gamma+\delta}}{\delta(\delta+\gamma)(\delta+\gamma+\beta)(\delta+\gamma+\beta+\alpha)}-\frac{x^{\alpha+\beta+\gamma+\delta}}{\gamma\delta(\gamma+\beta)(\gamma+\beta+\alpha)}\\&+\frac{x^{\alpha+\beta+\gamma+\delta}}{\beta\gamma(\beta+\alpha)(\gamma+\delta)}-\frac{x^{\alpha+\beta+\gamma+\delta}}{\alpha\beta(\beta+\gamma)(\beta+\gamma+\delta)}\\&+\frac{x^{\alpha+\beta+\gamma+\delta}}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+\delta)}\end{aligned}\right\}=0,$$

quae per $x^{\alpha+\beta+\gamma+\delta}$ divisa producit hanc aequationem

$$\left.\begin{aligned}&\frac{1}{\delta(\delta+\gamma)(\delta+\gamma+\beta)(\delta+\gamma+\beta+\alpha)}-\frac{1}{\delta\gamma(\gamma+\beta)(\gamma+\beta+\alpha)}\\&+\frac{1}{\gamma\beta(\beta+\alpha)(\gamma+\delta)}-\frac{1}{\alpha\beta(\beta+\gamma)(\beta+\gamma+\delta)}\\&+\frac{1}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+\delta)}\end{aligned}\right\}=0.$$

§. 53. Denique eaedem positiones in theoremate quinto substitutae producunt hanc aequationem

$$\left.\begin{aligned}&\frac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\varepsilon(\varepsilon+\delta)(\varepsilon+\delta+\gamma)(\varepsilon+\delta+\gamma+\beta)(\varepsilon+\delta+\gamma+\beta+\alpha)}\\&-\frac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\varepsilon\delta(\delta+\gamma)(\delta+\gamma+\beta)(\delta+\gamma+\beta+\alpha)}\\&+\frac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\delta\gamma(\gamma+\beta)(\gamma+\beta+\alpha)(\delta+\varepsilon)}\\&-\frac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\beta\gamma(\beta+\alpha)(\gamma+\delta)(\gamma+\delta+\varepsilon)}\\&+\frac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\alpha\beta(\beta+\gamma)(\beta+\gamma+\delta)(\beta+\gamma+\delta+\varepsilon)}\\&-\frac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+\delta)(\alpha+\beta+\gamma+\delta+\varepsilon)}\end{aligned}\right\}=0,$$

quae per $x^{\alpha+\beta+\gamma+\delta+\varepsilon}$ divisa dat hanc aequationem maxime notatu dignam

$$\left.\begin{array}{l}\dfrac{1}{\varepsilon(\varepsilon+\delta)(\varepsilon+\delta+\gamma)(\varepsilon+\delta+\gamma+\beta)(\varepsilon+\delta+\gamma+\beta+\alpha)} \\ -\dfrac{1}{\delta\varepsilon(\delta+\gamma)(\delta+\gamma+\beta)(\delta+\gamma+\beta+\alpha)} \\ +\dfrac{1}{\gamma\delta(\gamma+\beta)(\gamma+\beta+\alpha)(\delta+\varepsilon)} \\ -\dfrac{1}{\beta\gamma(\beta+\alpha)(\gamma+\delta)(\gamma+\delta+\varepsilon)} \\ +\dfrac{1}{\alpha\beta(\beta+\gamma)(\beta+\gamma+\delta)(\beta+\gamma+\delta+\varepsilon)} \\ -\dfrac{1}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+\delta)(\alpha+\beta+\gamma+\delta+\varepsilon)}\end{array}\right\} = 0.$$

§. 54. Haec theorematis eo magis sunt memorabilia, quod eorum veritas non nisi per plures ambages in numeris explorari potest, ideoque multo majorem attentionem merentur, quam aliud simile theorema, ad quod nuper sum perductus, quippe cujus demonstratio haud difficulter exhiberi potest, quod ita se habet.

Theorema numericum.

Sumtis pro lubitu quotcunque numeris veluti quatuor $\alpha, \beta, \gamma, \delta$, si hinc totidem alii sequenti modo formentur

$$a = \alpha,\ b = \alpha + \beta,$$
$$c = \alpha + \beta + \gamma \text{ et } d = \alpha + \beta + \gamma + \delta,$$

similique modo etiam isti

$$\mathrm{D} = \delta,\ \mathrm{C} = \delta + \gamma$$
$$\mathrm{B} = \delta + \gamma + \beta \text{ et } \mathrm{A} = \delta + \gamma + \beta + \alpha,$$

tam semper erit

$$\frac{1}{abcd} - \frac{1}{abc\mathrm{D}} + \frac{1}{ab\mathrm{CD}} - \frac{1}{a\mathrm{BCD}} + \frac{1}{\mathrm{ABCD}} = 0.$$

Demonstratio.

§. 55. Binae fractiones priores inventae, ob $D - d = - c$, dant fractionem $- \frac{1}{a b d D}$, quae cum tertia conjuncta producit $\frac{1}{a d C D}$, cui quarta fractio juncta dat $- \frac{1}{d B C D}$, quae [ob $d = A$] a termino ultimo penitus destruitur.

§. 56. Ope superiorum theorematum omnes formulae integrales implicatae, ad quas integratio aequationum linearum perducere solet, facile resolvi poterunt. Pervenitur autem plerumque ad tales formas:

$$Z = \int \partial q \int X \partial p, \quad Z = \int \partial r \int \partial q \int X \partial p,$$

$$Z = \int \partial s \int \partial r \int \partial q \int X \partial p, \quad Z = \int \partial t \int \partial s \int \partial r \int \partial q \int X \partial p \text{ etc.}$$

ubi litterae p, q, r, s, t, etc. sunt functiones datae ipsius X, at vero X functio quaecunque ipsius x; atque hic tota resolutio ita institui debet, ut in singulis membris functio haec indefinita X unicum tantum signum integrale habeat praefixum: hoc igitur, ope superiorum theorematum, facile praestari poterit, si modo ibi loco elementi ∂p scribamus $X \partial p$, quo observato singulae reductiones sequenti modo se habebunt.

I. Resolutio formulae integralis $\int \partial q \int X \partial p$.

§. 57. Si loco ∂p scribamus $X \partial p$ theorema secundum §. 46. nobis suppeditat hanc aequationem:

$$\int X \partial p \int \partial q - \int X \partial p . \int \partial q + \int \partial q \int X \partial p = 0,$$

cujus postremum membrum est ipsa nostra forma reducenda Z, consequenter resolutio statim dat

$$Z = \int \partial q . \int X \partial p - \int X \partial p \int \partial q.$$

ideoque ob $\int \partial q = q$ habebimus

$$Z = q \int X \partial p - \int X q \partial p.$$

Corollarium.

§. 58. Si fuerit $q = p$, erit

$$Z = p \int X \partial p - \int X p \partial p.$$

II. Resolutio formulae implicatae

§. 59. Pro hoc casu sumamus theorema 3. §. 47. unde, si loco ∂p scribatur $X \partial p$, deducimus hanc aequationem

$$\int X \partial p \int \partial q \int \partial r - \int X \partial p \int \partial q . \int \partial r + \int X \partial p . \int \partial r \int \partial q - \int \partial r \int \partial q \int X \partial p = 0,$$

cujus postremum membrum est ipsa forma reducenda Z, hincque adeoque colligitur

$$Z = \int \partial r \int \partial q . \int X \partial p - \int \partial r . \int X \partial p \int \partial q + \int X \partial p \int \partial q \int \partial r,$$

quae ergo reducta dat

$$Z = \int q \partial r . \int X \partial p - r \int X q \partial p + \int X \partial p \int r \partial q.$$

Corollarium.

§. 60. Si ergo hic fuerit $q = r = p$, prodibit ista resolutio:

$$Z = \int \partial p \int \partial p \int X \partial p = \tfrac{1}{2} pp \int X \partial p - p \int X p \partial p + \tfrac{1}{2} \int X p p \partial p.$$

III. Resolutio hujus formulae implicatae

$$Z = \int \partial s \int \partial r \int \partial q \int X \partial p.$$

§. 61. Pro hoc casu sumamus theorema 4. §. 48. unde si loco ∂p scribatur $X \partial p$ deducimus hanc aequationem

$$\left.\begin{matrix}\int X\partial p\int\partial q\int\partial r\int\partial s-\int X\partial p\int\partial q\int\partial r.\int\partial s+\int X\partial p\int\partial q.\int\partial s\int\partial r\\ -\int X\partial p.\int\partial s\int\partial r\int\partial q+\int\partial s\int\partial r\int\partial q\int X\partial p\end{matrix}\right\}=0,$$

cujus postremum membrum est ipsa nostra formula reducenda Z; hincque adeo colligimus

$$Z=\left\{\begin{matrix}\int\partial s\int\partial r\int\partial q.\int X\partial p-\int\partial s\int\partial r.\int X\partial p\int\partial q\\ +\int\partial s.\int X\partial p\int\partial q\int\partial r-\int X\partial p\int\partial q\int\partial r\int\partial s,\end{matrix}\right.$$

quae ergo reducta praebet

$$Z=\left\{\begin{matrix}\int\partial s\int q\partial r.\int X\partial p-\int r\partial s.\int Xq\partial p+s\int X\partial q\int r\partial q\\ -\int X\partial p\int\partial q\int s\partial r.\end{matrix}\right.$$

Corollarium.

§. 62. Si ponatur $s=r=q=p$, tum prodibit ista resolutio

$$Z=\left\{\begin{matrix}\frac{1}{6}p^3\int X\partial p-\frac{1}{2}pp\int Xp\partial p+\frac{1}{2}p\int Xpp\partial p\\ -\frac{1}{6}\int Xp^3\partial p.\end{matrix}\right.$$

IV. Resolutio
hujus formulae implicatae.

$$Z=\int\partial t\int\partial s\int\partial r\int\partial q\int X\partial p.$$

§. 63. Pro hoc casu sumamus theorema 5. §. 49. unde si loco ∂p scribatur $X\partial p$, prodibit ista aequatio

$$\left.\begin{matrix}\int X\partial p\int\partial q\int\partial r\int\partial s\int\partial t-\int X\partial p\int\partial q\int\partial r\int\partial s.\int\partial t\\ +\int X\partial p\int\partial q\int\partial r.\int\partial t\int\partial s-\int X\partial p\int\partial q.\int\partial t\int\partial s\int\partial r\\ +\int X\partial p.\int\partial t\int\partial s\int\partial r\partial q-\int\partial t\int\partial s\int\partial r\int\partial q\int X\partial p\end{matrix}\right\}=0,$$

cujus postremum membrum est ipsa nostra forma reducenda Z, unde ergo prodit

$$Z=\begin{cases}\int\partial t\int\partial s\int\partial r\int\partial q\,.\int X\partial p-\int\partial t\int\partial s\int\partial r\,.\int X\partial p\int\partial q\\ +\int\partial t\int\partial s\,.\int X\partial p\int\partial q\int\partial r-\int\partial t\,.\int X\partial p\int\partial q\int\partial r\int\partial s\\ +\int X\partial p\int\partial q\int\partial r\int\partial s\int\partial t,\end{cases}$$

quae ergo reducta praebet

$$Z=\begin{cases}\int\partial t\int\partial s\int q\partial r\,.\int X\partial p-\int\partial t\int r\partial s\,.\int Xq\partial p\\ +\int s\partial t\,.\int X\partial p\int r\partial q-t\int X\partial p\int\partial q\int s\partial r\\ +\int X\partial p\int\partial q\int\partial r\int t\partial s.\end{cases}$$

Corollarium.

§. 64. Si hic sumatur $t=s=r=q=p$, tum prodibit ista resolutio

$$Z=\begin{cases}\tfrac{1}{24}p^4\int X\partial p-\tfrac{1}{6}p^3\int Xp\partial p+\tfrac{1}{4}pp\int Xpp\partial p\\ \tfrac{1}{6}p\int Xp^3\partial p+\tfrac{1}{24}\int Xp^4\partial p.\end{cases}$$

§. 65. Quo indoles harum resolutionum clarius perspiciatur, quoniam litterae p, q, r, s, t, functiones datas ipsius x denotant, ideoque omnes expressiones ex iis formatae pariter ut cognitae spectari possunt, statuamus brevitatis gratia

$$\partial p\int\partial q=\partial p';\ \partial p\int\partial q\int\partial r=\partial p'';\ \partial p\int\partial q\int\partial r\int\partial s=\partial p''';$$
$$\partial p\int\partial q\int\partial r\int\partial s\int\partial t=\partial p'''';\ \text{etc.}$$

hocque modo postrema resolutio ita referetur

$$Z=\int\partial t\int\partial s\int\partial r\int\partial q\,.\int X\partial p-\int\partial t\int\partial s\int\partial r\,.\int X\partial p'$$
$$+\int\partial t\int\partial s\,.\int X\partial p''-\int\partial t\,.\int X\partial p'''+\int X\partial p''''.$$

Quod si hic porro statuamus

$$\int\partial t\int\partial s=\int s\partial t=t';\ \int\partial t\int\partial s\int\partial r=t'';\ \int\partial t\int\partial s\int\partial r\int\partial q=t''';$$

tota resolutio hoc modo conncinne repraesentabitur

$$Z=t'''\int X\partial p-t''\int X\partial p'+t'\int X\partial p''-t\int X\partial p'''$$
$$+\int X\partial p'''',$$

quam repraesentationem etiam ad praecedentes resolutiones accommodasse juvabit.

§. 66. Cum igitur integratio formulae implicatae

$$Z = \int \partial t \int \partial s \int \partial r \int \partial q \int X \partial p$$

reducatur ad integrationem sequentium formularum integralium simplicium: $\int x \partial p$; $\int x \partial p'$; $\int x \partial p''$; $\int x \partial p'''$; $\int x \partial p''''$; quaestio hinc oritur non parum curiosa: quemadmodum ex his formulis simplicibus vicissim quantitates q, r, s et t concludi queant? quod sequenti modo facile praestabitur. Cum sit $\partial p' = \partial p \int \partial q$, erit $\int \partial q = q = \frac{\partial p'}{\partial p}$. Ponatur nunc porro $\frac{\partial p''}{\partial p} = q'$; $\frac{\partial p'''}{\partial p} = q''$; $\frac{\partial p''''}{\partial p} = q'''$; etc. quibus valoribus introductis habebimus

$$q' = \int \partial q \int \partial r; \quad q'' = \int \partial q \int \partial r \int \partial s;$$
$$q''' = \int \partial q \int \partial r \int \partial s \int \partial t; \text{ etc.}$$

Quoniam igitur hi valores q, q', q'', q''', q'''' sunt dati, ex prima statim colligimus $\int \partial r = \frac{\partial q'}{\partial q} = r$. Ponamus autem porro $\frac{\partial q''}{\partial q} = r'$; $\frac{\partial q'''}{\partial q} = r''$; etc. eruntque etiam hi valores, r, r', r'', etc. dati, quibus substitutis habebitur $r' = \int \partial r \int \partial s$; $r'' = \int \partial r \int \partial s \int \partial t$; ex quarum prima sequitur $\int \partial s = s = \frac{\partial r'}{\partial r}$. Quare si porro fiat $s' = \frac{\partial r''}{\partial r}$, erit quoque $s' = \int \partial s \int \partial t$, hincque $\int \partial t = t = \frac{\partial s'}{\partial s}$. Ex his clare intelligitur, quomodo hae formulae inveniri queant pro casibus adhuc magis complicatis.

§. 67. Superest, ut etiam de transformatione talium formularum integralium implicatarum pauca adjiciamus, quod totum negotium sequenti problemate includi potest.

Problema.

§. 68. *Proposita formula implicata terna signa summatoria involvente* $\int \partial p \int \partial q \int \partial r$, *investigare aliam similem formulam*

$$\int \partial P \int \partial Q \int \partial R,$$

illi aequalem.

Solutio.

Per theorema 2. supra allatum formula proposita ita est resoluta

$$\int \partial q \int \partial r = \int \partial q . \int \partial r - \int \partial r \int \partial q = q \int \partial r - \int q \partial r.$$

Simili modo pro formula quaesita erit

$$\int \partial Q \int \partial R = Q \int \partial R - \int Q \partial R,$$

requiritur igitur ut sit

$$q \partial p \int \partial r - \partial p \int q \partial r = Q \partial P \int \partial R - \partial P \int Q \partial R,$$

quae aequalitas adimpleretur, sumendo $P = p$, $Q = q$ et $R = r$; verum permutandis membris statuamus

$$Q \partial P \int \partial R = -\partial p \int q \partial r \text{ et } \partial P \int Q \partial R = -q \partial p \int \partial r,$$

atque ex priore aequatione deducimus $Q \partial P = -\partial p$, ideoque $\partial P = \frac{\partial p}{Q}$, tum vero $\partial R = q \partial r$; ex altera vero aequatione habemus $\partial P = -q \partial p$ et $Q \partial R = \partial r$. Cum igitur esset $\partial P = -\frac{\partial p}{Q}$, erit $Q = \frac{1}{q}$, hincque porro $\partial R = q \partial r$, unde ob $Q = \frac{1}{q}$, erit $\partial Q = \frac{-\partial q}{qq}$. Consequenter formula integralis quaesita proposita $\int \partial p \int \partial q \int \partial r$ aequalis erit

$$\int q \partial p \int \frac{\partial q}{qq} \int q \partial r,$$

unde patet perpetuo loco formulae $\int \partial p \int \partial q \int \partial r$, scribi posse istam: $\int q \partial p \int \frac{\partial q}{qq} \int q \partial r$.

Corollarium 1.

§. 69. Quando igitur plura signa integralia sibi invicem fuerint involuta, veluti si habeamus $\int \partial p \int \partial q \int \partial r \int \partial s$, ista transformatio in quibusvis ternis signis se mutuo sequentibus institui poterit, unde in hac formula proposita duplex transformatio adhiberi poterit; prior scilicet in ternis signis prioribus praebebit

$$\int q \partial p \int \frac{\partial q}{q q} \int q \partial r \int \partial s,$$

at vero in ternis posterioribus haec transformatio adhibita dabit

$$\int \partial p \int r \partial q \int \frac{\partial r}{r r} \int r \partial s,$$

Corollarium 2.

§. 70. Hinc porro ope ejusdem transformationis aliae insuper fieri possunt, veluti ex postrema forma

$$\int \partial p \int r \partial q \int \frac{\partial r}{r r} \int r \partial s,$$

ut in ternis prioribus signis res expediri queat, loco $r \partial q$ scribamus ∂v, ut habeamus

$$\int \partial p \int \partial v \int \frac{\partial r}{r r} \int r \partial s,$$

quae transformatur in hanc

$$\int v \partial p \int \frac{\partial v}{v v} \int \frac{v \partial r}{r r} \int r \partial s,$$

quae omnes formulae ipsi propositae sunt prorsus aequales.

§. 71. Ut rem exemplo illustremus, sumamus esse $p = x^\alpha$; $q = x^\beta$; $r = x^\gamma$, ita ut formula proposita sit

$$\alpha \beta \gamma \int x^{\alpha-1} \partial x \int x^{\beta-1} \partial x \int x^{\gamma-1} \partial x = \frac{\alpha \beta x^{\gamma+\beta+\alpha}}{(\gamma+\beta)(\gamma+\beta+\alpha)}.$$

Jam pro transformatione erit primo

$\int q \partial r = \frac{\gamma x^{\beta+\gamma}}{\beta+\gamma}$, ideoque ob $\frac{\partial q}{q q} = \frac{\beta \partial x}{x^{\beta+1}}$, erit

$$\int \frac{\partial q}{q q} \int q \partial r = \frac{\beta x^{\gamma}}{\beta+\gamma},$$

quod ductum in $q \partial p$ et integratum producit

$$\frac{\alpha \beta x^{\alpha+\beta+\gamma}}{(\beta+\gamma)(\alpha+\beta+\gamma)}.$$

Patet igitur hanc transformationem latissime patere, atque ad omnes formulas implicatas accommodari posse eo pluribus diversis modis, quo plura signa integralia invicem involvantur.

§. 72. Haud abs re fore judico resolutiones supra traditas ad summationem serierum potestatum reciprocarum applicare, quod fiet si loco X sumamus fractionem $\frac{x}{1-x}$, tum vero pro singulis elementis ∂p, ∂q, ∂r, ∂s, scribamus $\frac{\partial x}{x}$, unde corollaria subnexa in usum vocari poterunt, ubi scilicet erit $p = l x$.

§. 73. Cum sit per seriem infinitam

$$X = x + x x + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \text{etc.}$$

erit

$$\int X \partial p = \int \frac{X \partial x}{x} = x + \tfrac{1}{2} x x + \tfrac{1}{3} x^3 + \tfrac{1}{4} x^4 + \tfrac{1}{5} x^5 + \tfrac{1}{6} x^6 + \text{etc.}$$

quam seriem constat exprimere logarithmum fractionis $\frac{1}{1-x}$, quandoquidem est

$$\int \frac{X \partial x}{x} = -l(1-x) = l \frac{1}{1-x}.$$

§. 74. Multiplicetur haec series porro per $\frac{\partial x}{x}$ et integretur, prodibitque

$$\int\frac{\partial x}{x}\int\frac{X\partial x}{x}=x+\tfrac{1}{4}xx+\tfrac{1}{9}x^3+\tfrac{1}{16}x^4+\tfrac{1}{25}x^5+\text{etc.}$$

at vero hujus formulae integralis resolutio supra §. 57. data praebet

$$\int\frac{\partial x}{x}\int\frac{X\partial x}{x}=lx\int\frac{\partial x}{1-x}-\int\frac{\partial x\, lx}{1-x},$$

quae quidem integralia ita accipi supponuntur, ut posito $x=0$ evanescant; hic autem imprimis notetur, casu quo sumitur $x=1$, ob $l\,1=0$, hujus seriei

$$1+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16}+\tfrac{1}{25}+\text{etc.}$$

summam fore $-\int\frac{\partial x\, lx}{1-x}$, cujus valorem olim primus inveni esse $=\frac{\pi\pi}{6}$.

§. 75. Ducamus superiorem seriem denuo in $\frac{\partial x}{x}$ et integrando obtinebimus:

$$\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{1-x}=x+\frac{1}{2^3}xx+\frac{1}{3^3}x^3+\frac{1}{4^3}+x^4+\frac{1}{5^3}x^5+\text{etc.}$$

Formula autem haec implicata per §. 59. ita resolvitur

$$\tfrac{1}{2}(lx)^2\int\frac{\partial x}{1-x}-lx\int\frac{\partial x\, lx}{1-x}+\tfrac{1}{2}\int\frac{\partial x(lx)^2}{1-x}.$$

Casu igitur quo $x=1$, summa seriei reciprocae cuborum

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\text{etc.}$$

erit $=\frac{1}{2}\int\frac{\partial x(lx)^2}{1-x}$.

§ 76. Simili modo superiorem seriem per $\frac{\partial x}{x}$ multiplicemus et integremus, tum prodibit

$$\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{1-x}=x+\frac{1}{2^4}xx+\frac{1}{3^4}x^3+\frac{1}{4^4}x^4+\text{etc.}$$

At vero haec formula implicata per §. 61. reducitur ad hanc formam

$$\tfrac{1}{6}(lx)^3\int\frac{\partial x}{1-x}-\tfrac{1}{2}(lx)^2\int\frac{\partial x\, lx}{1-x}+\tfrac{1}{2}lx\int\frac{\partial x(lx)^2}{1-x}$$
$$-\tfrac{1}{6}\int\frac{\partial x(lx)^3}{1-x}.$$

Pro casu ergo quo $x = 1$ hujus seriei reciprocae biquadratorum summa erit $-\frac{1}{6}\int\frac{\partial x\,(l\,x)^3}{1-x}$, cujus valorem olim ostendi esse $\frac{\pi^4}{90}$.

§. 77. Multiplicatione denuo per $\frac{\partial x}{x}$ instituta et integratione peracta habebimus:

$$\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{1-x} = x + \frac{1}{2^5}xx + \frac{1}{3^5}x^3 + \frac{1}{4^5}x^4 + \frac{1}{5^5}x^5 + \text{etc.}$$

quae formula implicata per §. 63. reducitur ad hanc formam

$$\frac{1}{24}(l\,x)^4\int\frac{\partial x}{x} - \frac{1}{6}(l\,x)^3\int\frac{\partial x\,l\,x}{1-x} + \frac{1}{4}(l\,x)^2\int\frac{\partial x\,(l\,x)^2}{1-x} - \frac{1}{6}\,l\,x\int\frac{\partial x\,(l\,x)^3}{1-x} + \frac{1}{24}\int\frac{\partial x\,(l\,x)^4}{1-x}.$$

Hinc ergo casu $x = 1$ hujus seriei reciprocae potestatum quintarum summa erit $\frac{1}{24}\int\frac{\partial x\,(l\,x)^4}{1-x}$.

§. 78. Colligamus omnes istas series pro casu $x = 1$, earumque summae sequenti modo per formulam integralem simplicem exprimentur:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \text{etc.} = \int\frac{\partial x}{1-x} = \infty,$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.} = -\int\frac{\partial x\,l\,x}{1-x} = \frac{\pi\pi}{6},$$

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \text{etc.} = \frac{1}{2}\int\frac{\partial x\,(l\,x)^2}{1-x},$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.} = -\frac{1}{6}\int\frac{\partial x\,(l\,x)^3}{1-x} = \frac{\pi^4}{90},$$

$$1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} + \text{etc.} = \frac{1}{24}\int\frac{\partial x\,(l\,x)^4}{1-x},$$

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \text{etc.} = -\frac{1}{120}\int\frac{\partial x\,(l\,x)^5}{1-x} = \frac{\pi^6}{945},$$

$$1 + \frac{1}{2^7} + \frac{1}{3^7} + \frac{1}{4^7} + \frac{1}{5^7} + \text{etc.} = \frac{1}{720}\int\frac{\partial x\,(l\,x)^6}{1-x},$$

$$1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \text{etc.} = -\frac{1}{5040}\int\frac{\partial x\,(l\,x)^7}{1-x} = \frac{\pi^8}{9450},$$

etc. etc.

§. 79. In genere igitur hujus seriei

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \text{etc.}$$

in infinitum continuatae summa ita exprimetur

$$\pm \frac{1}{1.\,2.\,3.\,\ldots\,(n-1)} \int \frac{\partial x\,(l\,x)^{n-1}}{1-x},$$

ubi signum superius $+$ valet, quando exponens n est impar, inferius vero, quando est par. Istas summationes, jam pridem quidem repertas, ideo hic afferre visum est, quod non ita pridem Celeberr. Lorgna easdem has summationes per formulas continuo magis implicatas expressas exhibuit, cum sine dubio istae formulae integrales simplices longe praeferendae videantur.