DE INSIGNIBVS PROPRIETATIBVS FORMVLARVM INTEGRALIVM PRAETER BINAS VARIABILES ETIAM EARVM DIFFERENTIALIA CVIVSCVNQVE ORDINIS INVOLVENTIVM.

Auctore

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§. 1.

Si Z fuerit functio quaecunque, non solum binas variabiles x et y, sed etiam earum differentialia cuiuscunque ordinis inuoluens, ea saltem a specie differentialium liberari potest ope sequentium positionum: $\partial y = p \partial x$; $\partial p = q \partial x$; $\partial q = r \partial x$; $\partial r = s \partial x$; $\partial s = t \partial x$; etc. tum enim his valoribus substitutis quantitas Z, si fuerit finita, euadet functio quantitatum finitarum x, y, p, q, r, s, t, etc. Ita si fuerit

$$Z = \frac{(\partial x^2 + \partial y^2)^{\frac{3}{2}}}{\partial x \partial \partial y - \partial y \partial \partial x},$$

quae est formula notissima pro radio osculi, ob $\partial y = p \partial x$ et $\partial \partial y = p \partial \partial x + \partial p \partial x = p \partial \partial x + q \partial x^2$, primo numerator hanc induet formam: $\partial x^3 (1 + pp)^{\frac{3}{2}}$, deinde vero denominator euadet $= q \partial x^3$, sicque ista quantitas erit

$$Z = \frac{(1 + pp)^{\frac{3}{2}}}{q}.$$

§. 2. Quodsi nunc talis functio Z differentietur, eius differentiale ex tot constabit partibus, quot in ea insunt litterarum x, y, p, q, r, s, etc., ideoque tali forma exprimetur:

$$\partial Z = M \partial x + N \partial y + P \partial p + Q \partial q + R \partial r + S \partial s + \text{etc.}$$

Hic autem, ne multitudo litterarum M, N, P, Q, R, etc. in calculo molestiam creet, eas, quoniam omnes pendent a natura functionis Z, per sequentes characteres vsu iam satis receptos repraesentabo: $M = (\frac{\partial Z}{\partial x})$; $N = (\frac{\partial Z}{\partial y})$; $P = (\frac{\partial Z}{\partial p})$; $Q = (\frac{\partial Z}{\partial q})$; $R = (\frac{\partial Z}{\partial r})$; $S = (\frac{\partial Z}{\partial s})$; etc. hocque modo, nullas litteras peregrinas introducendo, erit

$$\partial Z = \partial x (\tfrac{\partial Z}{\partial x}) + \partial y (\tfrac{\partial Z}{\partial y}) + \partial p (\tfrac{\partial Z}{\partial p}) + \partial q (\tfrac{\partial Z}{\partial q}) + \text{etc.}$$

ac si porro loco differentialium ∂y, ∂p, ∂q, ∂r, etc. valores supra assignatos adhibeamus, prodibit

$$\partial Z = \partial x (\tfrac{\partial Z}{\partial x}) + p \partial x (\tfrac{\partial Z}{\partial y}) + q \partial x (\tfrac{\partial Z}{\partial p}) + r \partial x (\tfrac{\partial Z}{\partial q}) + \text{etc.}$$

§. 3. Hinc ergo si statuamus:

$$V = (\tfrac{\partial Z}{\partial x}) + p (\tfrac{\partial Z}{\partial y}) + q (\tfrac{\partial Z}{\partial p}) + r (\tfrac{\partial Z}{\partial q}) + \text{etc.}$$

quae erit quantitas finita, pariterque certa functio ipsarum x, y, p, q, r, etc. ab indole functionis Z pendens, erit $\partial Z = V \partial x$, ideoque integrando $Z = \int V \partial x$, in qua integratione omnes litterae x, y, p, q, r, etc. tanquam variabiles insunt. Vbi probe notetur, si V fuerit talis functio,

tio, qualem descripsimus, tum formulam differentialem $V \partial x$ semper integrationem admittere, etiamsi binae variabiles x et y nullo modo a se inuicem pendeant, cum contra, si loco V alia quaecunque functio quantitatum x, y, p, q, r, etc. acciperetur, integratio locum habere non posset, nisi certa quaedam relatio inter binas variabiles x et y statueretur.

§. 4. His constitutis cum sit

$$V = \left(\frac{\partial Z}{\partial x}\right) + p\left(\frac{\partial Z}{\partial y}\right) + q\left(\frac{\partial Z}{\partial p}\right) + r\left(\frac{\partial Z}{\partial q}\right) + \text{etc.}$$

perpendamus valores differentiales ipsius V, qui oriuntur, si vel sola quantitas x, vel sola y, vel sola p, vel sola q, etc. pro variabili habeatur, quos valores simili ratione per hos characteres: $\left(\frac{\partial V}{\partial x}\right)$; $\left(\frac{\partial V}{\partial y}\right)$; $\left(\frac{\partial V}{\partial p}\right)$; $\left(\frac{\partial V}{\partial q}\right)$; etc. designemus. Ac primo quidem si sola quantitas x vt variabilis tractetur, iisdem characteribus adhibendis reperietur:

$$\left(\frac{\partial V}{\partial x}\right) = \left(\frac{\partial \partial Z}{\partial x^2}\right) + p\left(\frac{\partial \partial Z}{\partial y \partial x}\right) + q\left(\frac{\partial \partial Z}{\partial p \partial x}\right) + r\left(\frac{\partial \partial Z}{\partial q \partial x}\right) + s\left(\frac{\partial \partial Z}{\partial r \partial x}\right) + \text{etc.}$$

vbi scilicet, vti iam satis est vsu receptum, formula $\left(\frac{\partial \partial Z}{\partial x^2}\right)$ indicat, functionem Z bis differentiandam esse, sola x pro variabili assumta; at formula $\left(\frac{\partial \partial Z}{\partial x \partial y}\right)$ indicat, functionem Z etiam bis ita esse differentiandam, vt in altera differentiatione sola quantitas x, in altera vero sola y variabilis sumatur. Demonstratum autem est eundem valorem prodire, siue in prima operatione x, in secunda vero y, siue inuerso modo, in prima y in altera vero x variabilis statuatur; quod idem etiam de reliquis formulis duplicem differentiationem innuentibus est tenendum.

§. 5. Si iam in hac postrema expressione valorem $\left(\frac{\partial Z}{\partial x}\right)$ littera T designemus, hinc fiet $\left(\frac{\partial \partial Z}{\partial x^2}\right) = \frac{\partial T}{\partial x}$; tum vero ro $\left(\frac{\partial\partial Z}{\partial x\partial y}\right) = \frac{\partial T}{\partial y}$; $\left(\frac{\partial\partial Z}{\partial x\partial p}\right) = \frac{\partial T}{\partial p}$; etc. hisque formulis introductis erit

$$\left(\frac{\partial V}{\partial x}\right) = \left(\frac{\partial T}{\partial x}\right) + p\left(\frac{\partial T}{\partial y}\right) + q\left(\frac{\partial T}{\partial p}\right) + r\left(\frac{\partial T}{\partial q}\right) + \text{etc.}$$

At vero si quantitas ista T per variabilitatem omnium litterarum x, y, p, q, r, etc. differentietur, erit, vt supra iam vidimus, eius differentiale plenum:

$$\partial T = \partial x\left(\frac{\partial T}{\partial x}\right) + p\,\partial x\left(\frac{\partial T}{\partial y}\right) + q\,\partial x\left(\frac{\partial T}{\partial p}\right) + r\,\partial x\left(\frac{\partial T}{\partial q}\right) + \text{etc.}$$

vnde patet fore $\partial T = \partial x\left(\frac{\partial V}{\partial x}\right)$, ita vt integrando fit

$$T = \left(\frac{\partial Z}{\partial x}\right) = \int \partial x\left(\frac{\partial V}{\partial x}\right).$$

Hinc discimus, si formula $V\partial x$ integrationem admittat, semper etiam hanc formulam: $\partial x\left(\frac{\partial V}{\partial x}\right)$, integrationem esse admissuram; quam proprietatem hoc Theoremate I. referamus.

Si fuerit $\int V\partial x = Z$, *tum etiam semper erit*
$\int \partial x\left(\frac{\partial V}{\partial x}\right) = \left(\frac{\partial Z}{\partial x}\right)$, *siue*
$\partial x\left(\frac{\partial V}{\partial x}\right) = \partial.\left(\frac{\partial Z}{\partial x}\right)$.

§. 6. Nunc quantitatis V id consideremus differentiale, quod ex sola variabili y enascitur, ac reperietur:

$$\left(\frac{\partial V}{\partial y}\right) = \left(\frac{\partial\partial Z}{\partial x\partial y}\right) + p\left(\frac{\partial\partial Z}{\partial y^2}\right) + q\left(\frac{\partial\partial Z}{\partial p\partial y}\right) + r\left(\frac{\partial\partial Z}{\partial q\partial y}\right) + \text{etc.}$$

vnde si hic ponamus $\left(\frac{\partial Z}{\partial y}\right) = T$, erit

$$\left(\frac{\partial V}{\partial y}\right) = \left(\frac{\partial T}{\partial x}\right) + p\left(\frac{\partial T}{\partial y}\right) + q\left(\frac{\partial T}{\partial p}\right) + r\left(\frac{\partial T}{\partial q}\right) + \text{etc.}$$

hinc igitur vt supra patet fore

$$\partial x\left(\frac{\partial V}{\partial y}\right) = \partial T = \partial.\left(\frac{\partial Z}{\partial y}\right),$$

ex quo integrando erit

$$\int \partial x \left(\frac{\partial V}{\partial y}\right) = T = \left(\frac{\partial Z}{\partial y}\right);$$

vnde deducitur sequens Theorema 2.

Si fuerit $\int V \partial x = Z$, *tum semper erit*

$$\int \partial x \left(\frac{\partial V}{\partial y}\right) = \left(\frac{\partial Z}{\partial y}\right), \textit{siue}$$

$$\partial x \left(\frac{\partial V}{\partial y}\right) = \partial . \left(\frac{\partial Z}{\partial y}\right).$$

§. 7. Progrediamur autem vlterius, et differentiale ipsius V, ex sola variabilitate ipsius p oriundum, contemplemur, ac reperiemus

$$\left(\frac{\partial V}{\partial p}\right) = \left(\frac{\partial \partial Z}{\partial x \partial p}\right) + p\left(\frac{\partial \partial Z}{\partial y \partial p}\right) + q\left(\frac{\partial \partial Z}{\partial p^2}\right) + r\left(\frac{\partial \partial Z}{\partial q \partial p}\right) + \text{etc.}$$
$$+ \left(\frac{\partial Z}{\partial y}\right)$$

Hinc iam si ponamus $\left(\frac{\partial Z}{\partial p}\right) = T$, erit

$$\left(\frac{\partial V}{\partial p}\right) = \left(\frac{\partial T}{\partial x}\right) + p\left(\frac{\partial T}{\partial y}\right) + q\left(\frac{\partial T}{\partial p}\right) + r\left(\frac{\partial T}{\partial q}\right) + \text{etc.} + \left(\frac{\partial Z}{\partial y}\right),$$

vnde ergo sequitur fore

$$\partial x \left(\frac{\partial V}{\partial p}\right) = \partial T + \partial x \left(\frac{\partial Z}{\partial y}\right),$$

quod nobis suppeditat istud Theorema 3.

Si fuerit $\int V . \partial x = Z$, *tum etiam semper erit*

$$\int \partial x \left(\frac{\partial V}{\partial p}\right) = \left(\frac{\partial Z}{\partial p}\right) + \int \partial x \left(\frac{\partial Z}{\partial y}\right), \textit{siue}$$

$$\partial x \left(\frac{\partial V}{\partial p}\right) - \partial x \left(\frac{\partial Z}{\partial y}\right) = \partial . \left(\frac{\partial Z}{\partial p}\right).$$

§. 8. Sumta nunc sola quantitate q pro variabili simili modo orietur

$$\left(\frac{\partial V}{\partial q}\right) = \left(\frac{\partial \partial Z}{\partial x \partial q}\right) + p\left(\frac{\partial \partial Z}{\partial y \partial q}\right) + q\left(\frac{\partial \partial Z}{\partial p \partial q}\right) + r\left(\frac{\partial \partial Z}{\partial q^2}\right) + \text{etc.}$$
$$+ \left(\frac{\partial Z}{\partial p}\right)$$

vnde

vnde ſi hic ponatur $\left(\frac{\partial Z}{\partial q}\right) = T$, erit

$$\left(\frac{\partial V}{\partial q}\right) = \left(\frac{\partial T}{\partial x}\right) + p\left(\frac{\partial T}{\partial y}\right) + q\left(\frac{\partial T}{\partial p}\right) + r\left(\frac{\partial T}{\partial q}\right) + \text{etc.} + \left(\frac{\partial Z}{\partial p}\right).$$

ſicque per ∂x multiplicando fiet

$$\partial x\left(\frac{\partial V}{\partial q}\right) = \partial T + \partial x\left(\frac{\partial Z}{\partial p}\right).$$

Hinc orietur iſtud Theorema quartum:

Si fuerit $\int V \partial x = Z$, *tum ſemper erit*

$$\int \partial x\left(\frac{\partial V}{\partial q}\right) = \left(\frac{\partial Z}{\partial q}\right) + \int \partial x\left(\frac{\partial Z}{\partial p}\right), \textit{ ſiue}$$

$$\partial x\left(\frac{\partial V}{\partial q}\right) - \partial x\left(\frac{\partial Z}{\partial p}\right) = \partial.\left(\frac{\partial Z}{\partial q}\right).$$

§. 9. Sumatur iam ſola quantitas r pro variabili ac prodibit

$$\left(\frac{\partial V}{\partial r}\right) = \left(\frac{\partial \partial Z}{\partial x \partial r}\right) + p\left(\frac{\partial \partial Z}{\partial y \partial r}\right) + q\left(\frac{\partial \partial Z}{\partial p \partial r}\right)$$
$$+ r\left(\frac{\partial \partial Z}{\partial q \partial r}\right) + \text{etc.} + \left(\frac{\partial Z}{\partial q}\right),$$

vnde ſi ponatur $\left(\frac{\partial Z}{\partial r}\right) = T$, erit

$$\left(\frac{\partial V}{\partial r}\right) = \left(\frac{\partial T}{\partial x}\right) + p\left(\frac{\partial Z}{\partial y}\right) + q\left(\frac{\partial T}{\partial p}\right) + r\left(\frac{\partial T}{\partial q}\right) + \text{etc.} + \left(\frac{\partial Z}{\partial q}\right).$$

Hinc igitur vt ſupra patet fore

$$\partial x\left(\frac{\partial V}{\partial r}\right) = \partial T + \partial x\left(\frac{\partial Z}{\partial q}\right),$$

ſicque orietur ſequens Theorema quintum:

Si fuerit $\int V \partial x = Z$, *tum etiam ſemper erit*

$$\int \partial x\left(\frac{\partial V}{\partial r}\right) = \left(\frac{\partial Z}{\partial r}\right) + \int \partial x\left(\frac{\partial Z}{\partial q}\right), \textit{ ſiue}$$

$$\partial x\left(\frac{\partial V}{\partial r}\right) - \partial x\left(\frac{\partial Z}{\partial q}\right) = \partial.\left(\frac{\partial Z}{\partial r}\right).$$

§. 10. Haec iam ita ſunt manifeſta, vt ſuperfluum foret iſta theoremata vlterius proſequi. Ante autem quam repetitas differentiationes proſequamur, haec theoremata nobis

bis inseruire possunt, ad criterium illud generale demonstrandum, quo primus ostendi formulam $\int V \partial x$ semper admittere integrationem, quoties fuerit:

$$0 = \left(\frac{\partial V}{\partial y}\right) - \frac{1}{\partial x} \partial . \left(\frac{\partial V}{\partial p}\right) + \frac{1}{\partial x^2} \partial \partial . \left(\frac{\partial V}{\partial q}\right) - \frac{1}{\partial x^3} \partial^3 . \left(\frac{\partial V}{\partial r}\right) + \frac{1}{\partial x^4} \partial^4 . \left(\frac{\partial V}{\partial s}\right) - \text{etc.}$$

§. 11. Ad hanc autem regulam demonstrandam, posito $\int V \partial x = Z$, per gradus progrediamur, prouti functio Z continuo plures continet litterarum x, y, p, q, r, etc. Ac primo quidem contineat functio Z tantum binas variabiles x et y, exclusis omnibus differentialibus, ita vt fit

$$\left(\frac{\partial Z}{\partial p}\right) = 0; \left(\frac{\partial Z}{\partial q}\right) = 0; \left(\frac{\partial Z}{\partial r}\right) = V, \text{ etc.}$$

Hinc iam ex theoremate tertio erit $\left(\frac{\partial V}{\partial p}\right) = \left(\frac{\partial Z}{\partial y}\right)$, theorema autem secundum nobis praebet $\partial x \left(\frac{\partial V}{\partial y}\right) = \partial . \left(\frac{\partial Z}{\partial y}\right)$. Cum igitur inde fit $\left(\frac{\partial Z}{\partial y}\right) = \left(\frac{\partial V}{\partial p}\right)$, hoc valore substituto fiet

$$\partial x \left(\frac{\partial V}{\partial y}\right) = \partial . \left(\frac{\partial V}{\partial p}\right),$$

ideoque per ∂x diuidendo orietur haec aequatio:

$$0 = \left(\frac{\partial V}{\partial y}\right) - \frac{1}{\partial x} \partial . \left(\frac{\partial V}{\partial p}\right),$$

prorsus vt criterium meum postulat. Quoniam enim ex Z litterae p, q, r, etc. excluduntur, ob $\partial Z = V \partial x$ functio V neque litteram q, neque r, neque s, etc. continere potest, vnde etiam formulae $\left(\frac{\partial V}{\partial q}\right)$; $\left(\frac{\partial V}{\partial r}\right)$; etc. euanescunt.

§. 12. Contineat nunc functio Z, praeter litteras x et y, etiam p, vnde ob $\partial Z = V \partial x$ et $\partial p = q \partial x$, quantitas V etiamnunc q inuoluet, sequentes vero litterae r, s, t, etc. excludentur. Cum igitur iam fit $\left(\frac{\partial Z}{\partial q}\right) = 0$ multoque magis $\left(\frac{\partial Z}{\partial r}\right) = 0$; $\left(\frac{\partial Z}{\partial s}\right) = 0$; etc. theorema quartum nobis

nobis praebebit:

$$\partial x\left(\frac{\partial V}{\partial q}\right) - \partial x\left(\frac{\partial Z}{\partial p}\right) = 0,$$

vnde fit $\left(\frac{\partial Z}{\partial p}\right) = \left(\frac{\partial V}{\partial q}\right)$, qui valor in tertio theoremate substitutus dat

$$\partial x\left(\frac{\partial V}{\partial p}\right) - \partial x\left(\frac{\partial Z}{\partial y}\right) = \partial.\left(\frac{\partial V}{\partial q}\right),$$

vnde ergo erit

$$\frac{\partial Z}{\partial y} = \left(\frac{\partial V}{\partial p}\right) - \frac{1}{\partial x}\partial.\left(\frac{\partial V}{\partial q}\right),$$

qui valor in theoremate secundo substitutus praebet:

$$\partial x\left(\frac{\partial V}{\partial y}\right) = \partial.\left(\frac{\partial V}{\partial p}\right) - \frac{1}{\partial x}\partial\partial.\left(\frac{\partial V}{\partial q}\right),$$

vnde sequitur, prorsus vt nostrum criterium postulat,

$$0 = \left(\frac{\partial V}{\partial y}\right) - \frac{1}{\partial x}\partial.\left(\frac{\partial V}{\partial p}\right) + \frac{1}{\partial x^2}\partial\partial.\left(\frac{\partial V}{\partial q}\right).$$

§. 13. Inuoluat nunc functio Z etiam litteram q, et quantitas V etiamnunc continebit litteram r, ob $\partial q = r\partial x$: sequentes vero inde excludentur. Cum igitur sit $\left(\frac{\partial Z}{\partial r}\right) = 0$, theorema quintum nobis praebet

$$\partial x\left(\frac{\partial V}{\partial r}\right) - \partial x\left(\frac{\partial Z}{\partial q}\right) = 0,$$

vnde fit $\left(\frac{\partial Z}{\partial q}\right) = \left(\frac{\partial V}{\partial r}\right)$, qui valor in theoremate quarto substitutus suppeditat hanc aequationem:

$$\partial x\left(\frac{\partial V}{\partial q}\right) - \partial x\left(\frac{\partial Z}{\partial p}\right) = \partial.\left(\frac{\partial V}{\partial r}\right),$$

vnde colligitur:

$$\left(\frac{\partial Z}{\partial p}\right) = \left(\frac{\partial V}{\partial q}\right) - \frac{1}{\partial x}\partial.\left(\frac{\partial V}{\partial r}\right).$$

Substituatur hic valor in theoremate tertio, fietque

$$\partial x\left(\frac{\partial V}{\partial p}\right) - \partial x\left(\frac{\partial Z}{\partial y}\right) = \partial.\left(\frac{\partial V}{\partial q}\right) - \frac{1}{\partial x}\partial\partial.\left(\frac{\partial V}{\partial r}\right),$$

vnde fit

$$\left(\frac{\partial Z}{\partial y}\right) = \left(\frac{\partial V}{\partial p}\right) - \frac{1}{\partial x}\partial.\left(\frac{\partial V}{\partial q}\right) + \frac{1}{\partial x^2}\partial^2.\left(\frac{\partial V}{\partial r}\right),$$

qui

qui valor in fecundo theoremate fubftitutus praebet hanc aequationem:

$$o = \left(\frac{\partial V}{\partial y}\right) - \frac{1}{\partial x}\partial . \left(\frac{\partial V}{\partial p}\right) + \frac{1}{\partial x^2}\partial\partial . \left(\frac{\partial V}{\partial q}\right) - \frac{1}{\partial x^3}\partial^3 . \left(\frac{\partial V}{\partial r}\right).$$

§. 14. Hoc igitur modo criterium fupra memoratum, quod primum ex contemplatione maximorum et minimorum, via maxime indirecta, concluferam, omni rigore eft demonftratum; atque haec demonftratio non multum difcrepat ab ea, quam fagaciffimus nofter Profeffor Lexell exhibuit, (Nov. Commentar. Acad. Scientiar. Petropol. Tomo XV. pag. 127). Nunc igitur formulas differentiales fupra ex functione Z deductas per vlteriores differentiationes euoluamus, quandoquidem hinc innumerabilia alia theoremata, iis quae dedimus fimilia, deriuari poffunt.

Euolutio formulae
$$\left(\frac{\partial V}{\partial x}\right) = \left(\frac{\partial\partial Z}{\partial x^2}\right) + p\left(\frac{\partial\partial Z}{\partial y\,\partial x}\right) + q\left(\frac{\partial\partial Z}{\partial p\,\partial x}\right) + r\left(\frac{\partial\partial Z}{\partial q\,\partial x}\right) + \text{etc.}$$
per vlteriorem differentiationem.

§. 15. Sumamus primo folam x pro variabili, ac facta differentiatione prodibit

$$\left(\frac{\partial\partial V}{\partial x^2}\right) = \left(\frac{\partial^3 Z}{\partial x^3}\right) + p\left(\frac{\partial^3 Z}{\partial y\,\partial x^2}\right) + q\left(\frac{\partial^3 Z}{\partial p\,\partial x^2}\right) + r\left(\frac{\partial^3 Z}{\partial q\,\partial x^2}\right) + \text{etc.}$$

vbi fi ftatuamus $\left(\frac{\partial\partial Z}{\partial x^2}\right) = T$, erit

$$\left(\frac{\partial\partial V}{\partial x^2}\right) = \left(\frac{\partial T}{\partial x}\right) + p\left(\frac{\partial T}{\partial y}\right) + q\left(\frac{\partial T}{\partial p}\right) + r\left(\frac{\partial T}{\partial q}\right) + \text{etc.}$$

vnde manifefto erit

$$\partial x\left(\frac{\partial\partial V}{\partial x^2}\right) = \partial T = \partial . \left(\frac{\partial\partial Z}{\partial x^2}\right),$$

atque hinc nafcitur fequens theorema:

Si fuerit $\int V\partial x = Z$, *tum femper erit*
$$\int\partial x\left(\frac{\partial\partial V}{\partial x^2}\right) = \left(\frac{\partial\partial Z}{\partial x^2}\right), \textit{ fiue } \partial x\left(\frac{\partial\partial V}{\partial x^2}\right) = \partial . \left(\frac{\partial\partial Z}{\partial x^2}\right).$$

§. 16. Sumatur nunc pro eadem formula fola y pro variabili, ac reperietur

$$\left(\frac{\partial\partial V}{\partial x\,\partial y}\right)=\left(\frac{\partial^3 Z}{\partial y\,\partial x^2}\right)+p\left(\frac{\partial^3 Z}{\partial y^2\,\partial x}\right)+q\left(\frac{\partial^3 Z}{\partial p\,\partial x\,\partial y}\right)+r\left(\frac{\partial^3 Z}{\partial q\,\partial x\,\partial y}\right)+\text{etc.}$$

vbi ſi ſtatuamus $\left(\frac{\partial\partial Z}{\partial x\,\partial y}\right)=T$, erit

$$\left(\frac{\partial\partial V}{\partial x\,\partial y}\right)=\left(\frac{\partial T}{\partial x}\right)+p\left(\frac{\partial T}{\partial y}\right)+q\left(\frac{\partial T}{\partial p}\right)+r\left(\frac{\partial T}{\partial q}\right)+\text{etc.}$$

vnde manifeſto erit

$$\partial x\left(\frac{\partial\partial V}{\partial x\,\partial y}\right)=\partial T=\partial.\left(\frac{\partial\partial Z}{\partial x\,\partial y}\right),$$

ſicque adepti ſumus ſequens theorema:

Si fuerit $\int V\partial x=Z$, *tum ſemper erit*

$$\int\partial x\left(\frac{\partial\partial V}{\partial x\,\partial y}\right)=\left(\frac{\partial\partial Z}{\partial x\,\partial y}\right),\ \textit{ſiue}\ \partial x\left(\frac{\partial\partial V}{\partial x\,\partial y}\right)=\partial.\left(\frac{\partial\partial Z}{\partial x\,\partial y}\right).$$

§. 17. At ſi ſola p variabilis capiatur, tum erit

$$\left(\frac{\partial\partial V}{\partial x\,\partial p}\right)=\left(\frac{\partial^3 Z}{\partial x^2\,\partial p}\right)+p\left(\frac{\partial^3 Z}{\partial x\,\partial y\,\partial p}\right)+q\left(\frac{\partial^3 Z}{\partial x\,\partial p^2}\right)+r\left(\frac{\partial^3 Z}{\partial q\,\partial x\,\partial p}\right)+\text{etc.}$$
$$+\left(\frac{\partial\partial Z}{\partial x\,\partial y}\right).$$

Hinc ergo ſi ponatur $\left(\frac{\partial\partial Z}{\partial x\,\partial p}\right)=T$, erit

$$\partial x\left(\frac{\partial\partial V}{\partial x\,\partial p}\right)=\partial T+\partial x\left(\frac{\partial\partial Z}{\partial x\,\partial y}\right),$$

hincque formatur ſequens theorema:

Si fuerit $\int V\partial x=Z$, *tum ſemper erit*

$$\int\partial x\left(\frac{\partial\partial V}{\partial x\,\partial p}\right)=\left(\frac{\partial\partial Z}{\partial x\,\partial p}\right)+\int\partial x\left(\frac{\partial\partial Z}{\partial x\,\partial y}\right),\ \textit{ſiue}$$
$$\partial x\left(\frac{\partial\partial V}{\partial x\,\partial p}\right)-\partial x\left(\frac{\partial\partial Z}{\partial x\,\partial y}\right)=\partial.\left(\frac{\partial\partial Z}{\partial x\,\partial p}\right).$$

§. 18. Sit nunc ſola littera q variabilis, eritque

$$\left(\frac{\partial\partial V}{\partial x\,\partial q}\right)=\left(\frac{\partial^3 Z}{\partial x^2\,\partial q}\right)+p\left(\frac{\partial^3 Z}{\partial x\,\partial y\,\partial q}\right)+q\left(\frac{\partial^3 Z}{\partial x\,\partial p\,\partial q}\right)+r\left(\frac{\partial^3 Z}{\partial x\,\partial q^2}\right)+\text{etc.}$$
$$+\left(\frac{\partial\partial Z}{\partial p\,\partial x}\right)$$

hinc ergo ſi ponatur $\left(\frac{\partial\partial Z}{\partial x\,\partial q}\right)=T$, erit

$$\partial x\left(\frac{\partial\partial V}{\partial x\,\partial q}\right)=\partial T+\partial x\left(\frac{\partial\partial Z}{\partial p\,\partial x}\right),$$

hincque formatur ſequens theorema:

Si

Si fuerit $\int V \partial x = Z$, *tum semper erit*

$$\int \partial x \left(\frac{\partial \partial V}{\partial x \partial q}\right) = \left(\frac{\partial \partial Z}{\partial x \partial q}\right) + \int \partial x \left(\frac{\partial \partial Z}{\partial p \partial x}\right), \text{ siue}$$

$$\partial x \left(\frac{\partial \partial V}{\partial x \partial q}\right) - \partial x \left(\frac{\partial \partial Z}{\partial p \partial x}\right) = \partial \cdot \frac{\partial \partial Z}{\partial x \partial q}.$$

Euolutio formulae

$$\left(\frac{\partial V}{\partial y}\right) = \left(\frac{\partial \partial Z}{\partial x \partial y}\right) + p\left(\frac{\partial \partial Z}{\partial y^2}\right) + q\left(\frac{\partial \partial Z}{\partial p \partial y}\right) + r\left(\frac{\partial \partial Z}{\partial q \partial y}\right) + \text{etc.}$$

per vlteriorem differentiationem.

§. 19. Hanc euolutionem iam multo concinnius absoluere licebit. Cum enim forma propofita ita repraefentari poffit, vt fit $\partial x \left(\frac{\partial V}{\partial y}\right) = \partial \cdot \left(\frac{\partial Z}{\partial y}\right)$, fingulas differentiationes in hac forma inftituere poterimus. Ita fi fola x variabilis fumatur, erit $\partial x \left(\frac{\partial \partial V}{\partial x \partial y}\right) = \partial \cdot \left(\frac{\partial \partial Z}{\partial x \partial y}\right)$, quod iam eft theorema 5. praecedentis euolutionis. Simili modo fi fola y variabilis fumatur, prodibit $\partial x \left(\frac{\partial \partial V}{\partial y^2}\right) = \partial \cdot \left(\frac{\partial \partial Z}{\partial y^2}\right)$, quod eft nouum theorema ad hanc euolutionem pertinens, vnde fit $\int \partial x \left(\frac{\partial \partial V}{\partial y^2}\right) = \left(\frac{\partial \partial Z}{\partial y^2}\right)$. Hinc patet fi fuerit $\int V \partial x = Z$, tum femper fore $\int \partial x \left(\frac{\partial \partial V}{\partial y^2}\right) = \left(\frac{\partial \partial Z}{\partial y^2}\right)$. At fi fola p variabilis accipiatur, tum quadam circumfpectione opus eft, quoniam hoc cafu non erit $\partial x \left(\frac{\partial \partial V}{\partial y \partial p}\right) = \partial \cdot \left(\frac{\partial \partial Z}{\partial x \partial y \partial p}\right)$, fed infuper aliquod membrum accedet. Quoniam enim formula $\partial \cdot \left(\frac{\partial Z}{\partial y}\right)$ euoluta continet partem $p \partial x \left(\frac{\partial \partial Z}{\partial y^2}\right)$, huius differentiatio praebet $\partial x \left(\frac{\partial \partial Z}{\partial y^2}\right)$, quod ergo infuper adiici oportet, ita vt fit

$$\partial x \left(\frac{\partial \partial V}{\partial y \partial p}\right) = \partial \cdot \left(\frac{\partial \partial Z}{\partial y \partial p}\right) + \partial x \left(\frac{\partial \partial Z}{\partial y^2}\right),$$

confequenter fumtis integralibus erit

$$\int \partial x \left(\frac{\partial \partial V}{\partial y \partial p}\right) = \left(\frac{\partial \partial Z}{\partial y \partial p}\right) + \int \partial x \left(\frac{\partial \partial Z}{\partial y^2}\right),$$

ficque integratio formulae $\int \partial x \left(\frac{\partial \partial V}{\partial y \partial p}\right)$ infuper inuoluit formulam integralem $\int \partial x \left(\frac{\partial \partial Z}{\partial y^2}\right)$.

§. 20. Sumatur nunc ſola q pro variabili, et quia formula $\partial . (\frac{\partial z}{\partial y})$ continet terminum $q\,\partial x (\frac{\partial z}{\partial y \partial p})$, variabilitas ipſius q producet terminum $\partial x (\frac{\partial \partial z}{\partial y \partial p})$, ſicque orietur iſta aequatio:

$$\partial x (\frac{\partial \partial v}{\partial y \partial q}) = \partial . (\frac{\partial \partial z}{\partial y \partial q}) + \partial x (\frac{\partial \partial z}{\partial y \partial p}).$$

Eodem modo patet ſi ſola littera r variabilis accipiatur, tum fore

$$\partial x (\frac{\partial \partial v}{\partial y \partial r}) = \partial . (\frac{\partial \partial z}{\partial y \partial r}) + \partial x (\frac{\partial \partial z}{\partial y \partial q}).$$

Ac ſi ſola s variabilis accipiatur, tum erit

$$\partial x (\frac{\partial \partial v}{\partial y \partial s}) = \partial . (\frac{\partial \partial z}{\partial y \partial s}) + \partial x (\frac{\partial \partial z}{\partial y \partial r}),$$

ſicque porro

Euolutio formulae

$$(\frac{\partial v}{\partial p}) = (\frac{\partial \partial z}{\partial x \partial p}) + p (\frac{\partial \partial z}{\partial y \partial p}) + q (\frac{\partial \partial z}{\partial p^2}) + r (\frac{\partial \partial z}{\partial q \partial p}) + \text{etc.} + (\frac{\partial z}{\partial y})$$

quae reducitur ad hanc formam:

$$\partial x (\frac{\partial v}{\partial p}) = \partial . (\frac{\partial z}{\partial p}) + \partial x (\frac{\partial z}{\partial y}).$$

§. 21. Quodſi hic vel ſola x vel ſola y variabilis accipiatur, haec forma ſimpliciter differentiata ad quaeſitum perducit: priori ſcilicet caſu prodit

$$\partial x (\frac{\partial \partial v}{\partial x \partial p}) = \partial . (\frac{\partial \partial z}{\partial x \partial p}) + \partial x (\frac{\partial \partial z}{\partial x \partial y});$$

poſteriore vero erit

$$\partial x (\frac{\partial \partial v}{\partial y \partial p}) = \partial . (\frac{\partial \partial z}{\partial y \partial p}) + \partial x (\frac{\partial \partial z}{\partial y^2}),$$

haecque duae formulae iam ante prodierunt.

§. 22. Sin autem littera p variabilis ſtatuatur, quoniam formula $\partial . (\frac{\partial z}{\partial p})$ continet partem $p\,\partial x (\frac{\partial \partial z}{\partial y \partial p})$, huius differentiatio praebet $\partial x (\frac{\partial \partial z}{\partial y \partial p})$, quod ergo ad differentialia, ex reliquis membris oriunda, inſuper addi debet; hoc ergo modo prodibit iſta aequalitas:

∂x

$$\partial x\left(\frac{\partial\partial v}{\partial p^2}\right)=\partial\cdot\left(\frac{\partial\partial z}{\partial p^2}\right)+2\,\partial x\left(\frac{\partial\partial z}{\partial y\,\partial p}\right),$$

vnde intelligitur ob ∂p^2, quod in prima formula occurrit, poſtremum terminum duplicari debere. Theorematibus autem hinc deducendis non immoramur, quandoquidem deinceps theoremata multo generaliora producere licebit.

§. 23. Sumamus nunc ſolam q variabilem, et quoniam in formula $\partial\cdot\left(\frac{\partial z}{\partial p}\right)$ euoluta occurrit terminus $q\,\partial x\left(\frac{\partial\partial z}{\partial p^2}\right)$, ex hoc per differentiationem naſcitur terminus $\partial x\left(\frac{\partial\partial z}{\partial p^2}\right)$. Hinc ergo facta tota differentiatione peruenіemus ad hanc aequationem:

$$\partial x\left(\frac{\partial\partial v}{\partial p\,\partial q}\right)=\partial\cdot\left(\frac{\partial\partial z}{\partial p\,\partial q}\right)+\partial x\left(\frac{\partial\partial z}{\partial y\,\partial q}\right)+\partial x\left(\frac{\partial\partial z}{\partial p^2}\right),$$

vbi patet, ob bina elementa ∂p et ∂q, inſuper duos terminos adiici oportere, id quod etiam eueniet, ſi ſola r pro variabili ſumatur; nam quia formula $\partial\cdot\left(\frac{\partial z}{\partial p}\right)$ continet terminum $r\,\partial x\left(\frac{\partial\partial z}{\partial p\,\partial q}\right)$, ex hoc per differentiationem prodibit $\partial x\left(\frac{\partial\partial z}{\partial p\,\partial q}\right)$, quem ad reliquas partes inſuper adiici oportet; hocque modo impetrabimus hanc aequationem:

$$\partial x\left(\frac{\partial\partial v}{\partial p\,\partial r}\right)=\partial\cdot\left(\frac{\partial\partial z}{\partial p\,\partial r}\right)+\partial x\left(\frac{\partial\partial z}{\partial y\,\partial r}\right)+\partial x\left(\frac{\partial\partial z}{\partial p\,\partial q}\right),$$

vbi iterum ob elementa ∂p et ∂r duo membra acceſſerunt.

Euolutio formulae

$$\left(\frac{\partial v}{\partial q}\right)=\left(\frac{\partial\partial z}{\partial x\,\partial q}\right)+p\left(\frac{\partial\partial z}{\partial y\,\partial q}\right)+q\left(\frac{\partial\partial z}{\partial p\,\partial q}\right)+r\left(\frac{\partial\partial z}{\partial q^2}\right)+\text{etc.}+\left(\frac{\partial z}{\partial p}\right),$$

quae reducta eſt ad hanc:

$$\partial x\left(\frac{\partial v}{\partial q}\right)=\partial\cdot\left(\frac{\partial z}{\partial q}\right)+\partial x\left(\frac{\partial z}{\partial p}\right).$$

§. 24. Si hic vel x vel y ſolum variabile capiatur, nihil in differentiatione de nouo accedit, eritque caſu priore

$$\partial x\left(\frac{\partial\partial v}{\partial x\,\partial q}\right)=\partial\cdot\left(\frac{\partial\partial z}{\partial x\,\partial q}\right)+\partial x\left(\frac{\partial\partial z}{\partial x\,\partial p}\right),$$

 po-

pofteriore vero cafu

$$\partial x\left(\frac{\partial\partial V}{\partial y\partial q}\right)=\partial.\left(\frac{\partial\partial Z}{\partial y\partial q}\right)+\partial x\left(\frac{\partial\partial Z}{\partial y\partial p}\right).$$

In reliquis autem differentiationibus elementum ∂p fuppeditat, praeter differentiationem folitam, infuper membrum $\partial x\left(\frac{\partial\partial Z}{\partial y\partial q}\right)$, at vero elementum ∂q producit $\partial x\left(\frac{\partial\partial Z}{\partial p\partial q}\right)$; elementum porro ∂r producit $\partial x\left(\frac{\partial\partial Z}{\partial q^2}\right)$, elementum ∂s vero praebet $\partial x\left(\frac{\partial\partial Z}{\partial q\partial r}\right)$ etc. quibus obferuatis obtinebuntur fequentes aequationes:

$$\text{I. } \partial x\left(\frac{\partial\partial V}{\partial p\partial q}\right)=\partial.\left(\frac{\partial\partial Z}{\partial p\partial q}\right)+\partial x\left(\frac{\partial\partial Z}{\partial p^2}\right)+\partial x\left(\frac{\partial\partial Z}{\partial y\partial q}\right),$$

$$\text{II. } \partial x\left(\frac{\partial\partial V}{\partial q^2}\right)=\partial.\left(\frac{\partial\partial Z}{\partial q^2}\right)+2\,\partial x\left(\frac{\partial\partial Z}{\partial p\partial q}\right),$$

$$\text{III. } \partial x\left(\frac{\partial\partial V}{\partial q\partial r}\right)=\partial.\left(\frac{\partial\partial Z}{\partial q\partial r}\right)+\partial x\left(\frac{\partial\partial Z}{\partial p\partial r}\right)+\partial x\left(\frac{\partial\partial Z}{\partial q^2}\right),$$

$$\text{IV. } \partial x\left(\frac{\partial\partial V}{\partial q\partial s}\right)=\partial.\left(\frac{\partial\partial Z}{\partial q\partial s}\right)+\partial x\left(\frac{\partial\partial Z}{\partial p\partial s}\right)+\partial x\left(\frac{\partial\partial Z}{\partial q\partial r}\right).$$

§. 25. Ex his iam abunde perfpicitur, perpetuo, quoties vel fola x, vel fola y variabilis accipitur, differentiationem more confueto inftitui debere, nihilque infuper effe adiiciendum; fin autem reliquae litterae p, q, r, s, etc. variabiles accipiantur, tum pro quolibet elemento fiue ∂p, fiue ∂q, fiue ∂r, etc. praeterea vnum nouum terminum accedere debere. Hinc igitur pro folis elementis ∂x et ∂y iam fequens theorema latiffime patens conftitui poteft:

Theorema generale. I.

Si fuerit $\int V\partial x = Z$, *tum femper erit*

$$\int\partial x\left(\frac{\partial^{\alpha+\beta}V}{\partial x^{\alpha}\partial y^{\beta}}\right)=\left(\frac{\partial^{\alpha+\beta}Z}{\partial x^{\alpha}\partial y^{\beta}}\right).$$

Euo-

Euolutio harum formularum, ſi ſola p pro variabili accipiatur.

§. 26. Quemadmodum iam vidimus, cum ſit

$$\partial x\left(\frac{\partial V}{\partial p}\right)=\partial.\left(\frac{\partial Z}{\partial p}\right)+\partial x\left(\frac{\partial Z}{\partial y}\right),$$

tum fore

$$\partial x\left(\frac{\partial\partial V}{\partial p^2}\right)=\partial.\left(\frac{\partial\partial Z}{\partial p^2}\right)+2\,\partial x\left(\frac{\partial\partial Z}{\partial y\,\partial p}\right);$$

ita ſi porro differentiemus, ex ſola variabilitate ipſius p prodibit:

$$\text{I}^{\circ}.\ \partial x\left(\frac{\partial^3 V}{\partial p^3}\right)=\partial.\left(\frac{\partial^3 Z}{\partial p^3}\right)+3\,\partial x\left(\frac{\partial^3 Z}{\partial y\,\partial p^2}\right),$$

$$\text{II}^{\circ}.\ \partial x\left(\frac{\partial^4 V}{\partial p^4}\right)=\partial.\left(\frac{\partial^4 Z}{\partial p^4}\right)+4\,\partial x\left(\frac{\partial^4 Z}{\partial y\,\partial p^3}\right),$$

$$\text{III}^{\circ}.\ \partial x\left(\frac{\partial^5 V}{\partial p^5}\right)=\partial.\left(\frac{\partial^5 Z}{\partial p^5}\right)+5\,\partial x\left(\frac{\partial^5 Z}{\partial y\,\partial p^4}\right),$$

vnde generaliter habebimus

$$\partial x\left(\frac{\partial^\gamma V}{\partial p^\gamma}\right)=\partial.\left(\frac{\partial^\gamma Z}{\partial p^\gamma}\right)+\gamma\,\partial x\left(\frac{\partial^\gamma Z}{\partial y\,\partial p^{\gamma-1}}\right);$$

hincque deducimus ſequens

Theorema generale 2.

Si fuerit $\int V\partial x=Z$, tum ſemper erit

$$\int\partial x\left(\frac{\partial^{\alpha+\beta+\gamma}V}{\partial x^\alpha\,\partial y^\beta\,\partial p^\gamma}\right)=\left(\frac{\partial^{\alpha+\beta+\gamma}Z}{\partial x^\alpha\,\partial y^\beta\,\partial p^\gamma}\right)+\gamma\int\partial x\left(\frac{\partial^{\alpha+\beta+\gamma}Z}{\partial x^\alpha\,\partial y^{\beta+1}\,\partial p^{\gamma-1}}\right).$$

§. 27. Quodſi iam vlterius quantitas Q pro variabili ſumatur, et differentiatio continuo repetatur, inueſtigationem ſequenti modo ſuſcipiamus. Quoniam elementa ∂x et ∂y nihil turbant, proficiſcamur a formula ſupra inuenta:

$$\partial x\left(\frac{\partial^\gamma V}{\partial p^\gamma}\right)=\partial.\left(\frac{\partial^\gamma Z}{\partial p^\gamma}\right)+\gamma\,\partial x\left(\frac{\partial^\gamma Z}{\partial y\,\partial p^{\gamma-1}}\right),$$

vnde

vnde variabilitas folius q primo dabit:

$$\partial x\left(\frac{\partial^{\gamma+1} V}{\partial p^{\gamma} \partial q}\right)=\partial\cdot\left(\frac{\partial^{\gamma+1} Z}{\partial p^{\gamma}\partial q}\right)+\gamma\,\partial x\left(\frac{\partial^{\gamma+1} Z}{\partial y\,\partial p^{\gamma-1}\,\partial q}\right)$$
$$+\partial x\left(\frac{\partial^{\gamma+1} Z}{\partial p^{\gamma+1}}\right).$$

§. 28. Quodfi iam hanc formam vlterius fecundum ∂q differentiemus, perueniemus ad hanc aequationem:

$$\partial x\left(\frac{\partial^{\gamma+2} V}{\partial p^{\gamma} \partial q^{2}}\right)=\partial\cdot\left(\frac{\partial^{\gamma+2} Z}{\partial p^{\gamma}\partial q^{2}}\right)+\gamma\,\partial x\left(\frac{\partial^{\gamma+2} Z}{\partial y\,\partial p^{\gamma-1}\,\partial q^{2}}\right)$$
$$+2\,\partial x\left(\frac{\partial^{\gamma+2} Z}{\partial p^{\gamma+1}\,\partial q}\right),$$

et denuo differentiando prodibit:

$$\partial x\left(\frac{\partial^{\gamma+3} V}{\partial p^{\gamma} \partial q^{3}}\right)=\partial\cdot\left(\frac{\partial^{\gamma+3} Z}{\partial p^{\gamma}\partial q^{3}}\right)+\gamma\,\partial x\left(\frac{\partial^{\gamma+3} Z}{\partial y\,\partial p^{\gamma-1}\,\partial q^{3}}\right)$$
$$+3\,\partial x\left(\frac{\partial^{\gamma+3} Z}{\partial p^{\gamma+1}\,\partial q^{2}}\right),$$

haecque fufficiunt ad conftituendum fequens

Theorema generale 3.

Si fuerit $\int V\partial x=Z$, *tum femper erit:*

$$\int\partial x\left(\frac{\partial^{\alpha+\beta+\gamma+\delta} V}{\partial x^{\alpha}\partial y^{\beta}\partial p^{\gamma}\partial q^{\delta}}\right)=\left(\frac{\partial^{\alpha+\beta+\gamma+\delta} Z}{\partial x^{\alpha}\partial y^{\beta}\partial p^{\gamma}\partial q^{\delta}}\right)$$
$$+\gamma\int\partial x\left(\frac{\partial^{\alpha+\beta+\gamma+\delta} Z}{\partial x^{\alpha}\,\partial y^{\beta+1}\,\partial p^{\gamma-1}\,\partial q^{\delta}}\right)$$
$$+\delta\int\partial x\left(\frac{\partial^{\alpha+\beta+\gamma+\delta} Z}{\partial x^{\alpha}\,\partial y^{\beta}\,\partial p^{\gamma+1}\,\partial q^{\delta-1}}\right).$$

§. 29.

§. 29. Iam pluribus ambagibus opus non erit ad sequens theorema generalissimum constituendum:

Theorema generalissimum.

Si fuerit $\int V \partial x = Z$, *tum semper erit*

$$\int \partial x \left(\frac{\partial^{\alpha+\beta+\gamma+\delta+\varepsilon+\zeta} V}{\partial x^{\alpha} \partial y^{\beta} \partial p^{\gamma} \partial q^{\delta} \partial r^{\varepsilon} \partial s^{\zeta}} \right) = \left(\frac{\partial^{\alpha+\beta+\gamma+\delta+\varepsilon+\zeta} Z}{\partial x^{\alpha} \partial y^{\beta} \partial p^{\gamma} \partial q^{\delta} \partial r^{\varepsilon} \partial s^{\zeta}} \right)$$

$$+ \gamma \int \partial x \left(\frac{\partial^{\alpha+\beta+\gamma+\delta+\varepsilon+\zeta} Z}{\partial x^{\alpha} \partial y^{\beta+1} \partial p^{\gamma-1} \partial q^{\delta} \partial r^{\varepsilon} \partial s^{\zeta}} \right)$$

$$+ \delta \int \partial x \left(\frac{\partial^{\alpha+\beta+\gamma+\delta+\varepsilon+\zeta} Z}{\partial x^{\alpha} \partial y^{\beta} \partial p^{\gamma+1} \partial q^{\delta-1} \partial r^{\varepsilon} \partial s^{\zeta}} \right)$$

$$+ \varepsilon \int \partial x \left(\frac{\partial^{\alpha+\beta+\gamma+\delta+\varepsilon+\zeta} Z}{\partial x^{\alpha} \partial y^{\beta} \partial p^{\gamma} \partial q^{\delta+1} \partial r^{\varepsilon-1} \partial s^{\zeta}} \right)$$

$$+ \zeta \int \partial x \left(\frac{\partial^{\alpha+\beta+\gamma+\delta+\varepsilon+\zeta} Z}{\partial x^{\alpha} \partial y^{\beta} \partial p^{\gamma} \partial q^{\delta} \partial r^{\varepsilon+1} \partial s^{\zeta-1}} \right).$$