

INTEGRATIO
 FORMVLAE DIFFERENTIALIS
 MAXIME IRRATIONALIS,
 QVAM TAMEN PER LOGARITHMOS ET ARCVS
 CIRCVLARES EXPEDIRE LICET.

Auctore
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Conuentui exhib. die 26 Mart. 1777.

Problema.

Proposita hæc formula differentiali:

$$\partial V = \frac{\partial z (1 - z z)^2}{(1 + z z) \sqrt{(1 + 6 z z + z^4)^3}}$$

eius integrale, per logarithmos et arcus circulares expressum, inuenire.

Solutio.

§. 1. Ponatur breuitatis gratia $\sqrt{(1 + 6 z z + z^4)} = v$,
 vt formula proposita fit $\partial V = \frac{\partial z (1 - z z)^2}{(1 + z z) v^3}$ et nunc loco z binæ
 variables p et q in calculum introducantur, ponendo $p = \frac{1+z}{v}$
 et $q = \frac{1-z}{v}$, eritque $p^2 + q^2 = 2$ ideoque $p^3 \partial p + q^3 \partial q = 0$.

Porro

Porro vero erit $p^2 + q^2 = \frac{2z}{v}$ et $p^2 - q^2 = \frac{2z}{v}$, hincque fiet

$\frac{2z}{v} = z$, existente $\frac{2z}{v} = p^2 + q^2$ et $\frac{2z}{v} = p^2 - q^2$, hincque fiet

Cum igitur sit $\frac{z}{v} = \frac{p^2 + q^2}{2}$, erit $\frac{\partial z}{\partial v} = \frac{2(p^2 + q^2)}{(p^2 + q^2)^2}$,

ubi ob $\frac{p^2 + q^2}{2} = z$ erit $\frac{\partial z}{\partial v} = \frac{2}{v} v v (q \partial p - p \partial q)$. Deinde

vero, quia est $\frac{\partial p}{\partial v} = \frac{p}{v}$, erit $q \partial p - p \partial q = \frac{q \partial p}{v} - \frac{p \partial q}{v}$.

Simili modo, posto $\frac{\partial q}{\partial v} = \frac{q}{v}$, erit $q \partial p - p \partial q = \frac{q \partial p}{v} - \frac{p \partial q}{v}$,

propterea quod $p^4 + q^4 = 2$; ficque elementum ∂z duplici

modo, scilicet per ∂p et per ∂q habebimus expressum, erit

que primo $\partial z = -\frac{v \partial q}{p^3}$, tum vero $\partial z = +\frac{v \partial p}{q^3}$, quam

duplicem expressionem per $\partial z = v v \partial \omega$ repraesentemus, exi-

stente vel $\partial \omega = \frac{\partial q}{p^3}$, vel $\partial \omega = +\frac{\partial p}{q^3}$.

§. 3. Deinde vero ob $1 + z = p v$ et $1 - z = q v$

erit $1 - z z = p q v v$, ideoque $(1 - z z)^2 = p^2 q^2 v^4$; ficque

numerator nostrae formulae erit $\partial z (1 - z z)^2 = v^6 p p q q \partial \omega$.

Pro denominatore autem habebimus $1 + z z = \frac{1}{2}(p p + q q) v v$,

ita vt iam totus denominator sit $\frac{1}{2} v^5 (p p + q q)$, quocirca

ipsa formula nostra proposita ita repraesentabitur:

$\partial V = \frac{2 v p p q q \partial \omega}{p p + q q} = \frac{4 p p q q \partial \omega}{(p + q)(p p + q q)}$

Multiplicemus autem porro supra et infra per $p - q$, vt

prodeat ista forma: $\partial V = \frac{4(p - q) p p q q \partial \omega}{p^4 - q^4}$

§. 4. Quoniam nunc numerator ex duabus partibus

constat, utramque seorsim evoluamus. Pars igitur prior, quae

est $\frac{4 p^3 q q \partial \omega}{p^4 - q^4}$, si loco $\partial \omega$ valorem priorem supra datum scri-

bamus, scilicet $\partial \omega = \frac{\partial q}{p^3}$, erit $\frac{4 q q \partial q}{p^4 - q^4}$, quomobrem

fi hic in denominatore pro p scribamus eius valorem $2 - q^2$,
 ista pars erit per solam variabilem q ita expressa: $-\frac{2qq\partial q}{1-q^4}$
 Simili modo altera nostrae formulae pars $-\frac{4ppq\partial\omega}{p^4-q^4}$, si loco
 $\partial\omega$ scribamus valorem $+\frac{\partial p}{q^3}$, induet hanc formam: $-\frac{4pp\partial p}{p^4-q^4}$
 Hic igitur loco q^2 scribatur $2-p^2$, ac pars ista iam per so-
 lam variabilem p exprimetur, fietque $= +\frac{2pp\partial p}{1-p^4}$, conse-
 quenter ipsa formula proposita reduca est ad has partes:

$$\partial V = \frac{2pp\partial p}{1-p^4} - \frac{2qq\partial q}{1-q^4}$$

quae non solum sunt rationales, sed etiam binas variables
 p et q penitus separatas inuoluunt.

§. 5. Ad integrale igitur inueniendum notetur esse
 $\frac{2pp}{1-p^4} = \frac{1}{1-pp} - \frac{1}{1+pp}$, unde erit prioris partis integrale

$$\int \frac{2pp\partial p}{1-p^4} = \int \frac{\partial p}{1-pp} - \int \frac{\partial p}{1+pp} = \frac{1}{2} \log \frac{1+p}{1-p} - A \text{ tang. } p,$$

eodemque modo altera pars erit

$$\int \frac{2qq\partial q}{1-q^4} = \frac{1}{2} \log \frac{1+q}{1-q} - A \text{ tang. } q,$$

quamobrem totum integrale quaesitum erit

$$V = \frac{1}{2} \log \frac{1+p}{1-p} - \frac{1}{2} \log \frac{1+q}{1-q} + A \text{ tang. } q - A \text{ tang. } p,$$

§. 6. Resituamus nunc loco p et q valores assum-
 tos, scilicet $p = \frac{1+z}{v}$ et $q = \frac{1-z}{v}$, eritque

$$V = \frac{1}{2} \log \frac{v+1+z}{v-1-z} - \frac{1}{2} \log \frac{v+1-z}{v-1+z} + A \text{ tang. } \frac{1-z}{v} - A \text{ tang. } \frac{1+z}{v},$$

vbi cum sit

$$A \text{ tang. } a - A \text{ tang. } b = A \text{ tang. } \frac{a-b}{1+ab}, \text{ erit}$$

$$A \text{ tang. } \frac{1-z}{v} - A \text{ tang. } \frac{1+z}{v} = -A \text{ tang. } \frac{2vz}{vv+1-zz}$$

Deinde etiam logarithmi inuicem combinari possunt et re-
 sultabit

$$V = \frac{1}{2} \log \frac{(v+1+z)(v-1-z)}{(v-1-z)(v+1-z)} - A \text{ tang. } \frac{2vz}{vv+1-zz}$$

Quin

Quin etiam utile erit logarithmos hoc modo iterum separare, vt fit

$$V = \frac{1}{2} l \frac{v+1+z}{v+1-z} + \frac{1}{2} l \frac{v-1+z}{v-1-z} - A \operatorname{tang.} \frac{2vz}{vv+1-zz}$$

§. 7. Hadenus constantem per integrationem addendam negleximus; eam igitur nunc ita definiamus, vt posito $z=0$ ipsum integrale V euanescat. Hunc in finem consideremus z tanquam minimum, et quia $v = (1 + 6zz + z^4)^{\frac{1}{2}}$, erit $v = 1 + \frac{3}{2}zz$; euidens autem est huius particulae minimae $\frac{3}{2}zz$ rationem tantum in posteriore logarithmo esse habendam, quoniam in eo occurrit $v-1$. Hoc igitur valore substituto erit nunc nostrum integrale

$$V = \frac{1}{2} l \frac{2+z}{2-z} + \frac{1}{2} l \frac{\frac{3}{2}zz+z}{\frac{3}{2}zz-z} - A \operatorname{tang.} \frac{2z}{2-zz}$$

§. 8. Hic iam quia z est quantitas minima, erit $\frac{1}{2} l \frac{2+z}{2-z} = \frac{z}{2}$; alter vero logarithmus erit $\frac{1}{2} l \frac{3z+2}{3z-2}$, vbi loco constantis adiaci debet $l-1$, vt haec altera pars fiat $\frac{1}{2} l \frac{2+3z}{2-3z}$, cuius valor erit $\frac{3z}{2}$, ita vt ambo logarithmi iundim praebeant $2z$. Deinde vero ex arcu circulari fit $A \operatorname{tang.} \frac{2z}{2-zz} = A \operatorname{tang.} z = z$, ita vt tota formula praebeat $V = 2z - z = z$, qui valor cum formula proposita egregie conspirat; posito enim z infinite paruo habetur $\partial V = \partial z$, ideoque $V = z$.

§. 9. Quoniam igitur constans addenda reperta est $l-1$, in superiori expressione loco $l(v-1-z)$ scribamus $l(1+z-v)$, vt iam totum integrale rite determinatum fit:

$$V = \frac{1}{2} l \frac{v+1+z}{v+1-z} + \frac{1}{2} l \frac{v+z-1}{1+z-v} - A \operatorname{tang.} \frac{2vz}{1+vv-zz}$$

qui valor euanescit sumto $z=0$.

Problema 2.

Proposita hac formula differentiali:

$$\partial V = \frac{\partial z (1 + z z)^2}{(1 - z z) \sqrt[4]{(1 - \beta z z + z^4)^3}},$$

eius integrale per logarithmos et arcus circulares inuestigare.

Solutio.

§. 10. Solutio huius problematis vix aliter erui posse videtur, nisi ex praecedente solutione deriuetur. Consideremus igitur formulam prioris problematis hac ratione repraesentatam:

$$\int \frac{\partial y (1 - y y)^2}{(1 + y y) \sqrt[4]{(1 + 6 y y + y^4)^3}} = U, \text{ et fac}$$

ta debita immutatione positoque $\sqrt[4]{(1 + 6 y y + y^4)} = u$, integrale ita erit expressum:

$$U = \frac{1}{2} l \frac{u+1+y}{u+1-y} + \frac{1}{2} l \frac{u-1+y}{u-1-y} - A \text{ tang. } \frac{2uy}{uu+1-y^2}.$$

§. 11. In hac forma ponamus $y = z \sqrt{-1}$, et statuamus formulam radicalem hinc natam

$$\sqrt[4]{(1 - 6 z z + z^4)} = v,$$

quo facto erit

$$U = \int \frac{\partial z \sqrt{-1} (1 + z z)^2}{(1 - z z) \sqrt[4]{(1 - 6 z z + z^4)^3}},$$

vnde patet esse $U = V \sqrt{-1}$, ita vt inuento valore U eruatur valor quaesitus $V = \frac{U}{\sqrt{-1}} = -U \cdot \sqrt{-1}$. Posito autem $y = z \sqrt{-1}$ et loco u scripto v , integrale U accipiet hanc formam:

$$U =$$

$$U = \frac{1}{2} l \frac{v+1+z\sqrt{-1}}{v+1-z\sqrt{-1}} + \frac{1}{2} l \frac{v-1+z\sqrt{-1}}{v-1-z\sqrt{-1}} - A \operatorname{tang.} \frac{2vz\sqrt{-1}}{vv+1+zz}$$

Totum igitur negotium, huc redit, vt isti logarithmi imaginarii cum arcu imaginario ad realitatem reducantur, id quod sequenti modo commodissime perficietur.

§. 12. In subsidium vocetur istud Lemma satis notum:

$$l a + b\sqrt{-1} = \frac{1}{2} l (a a + b b) + \sqrt{-1} A \operatorname{tang.} \frac{b}{a},$$

vbi ergo, si loco b scribamus $-b$, erit

$$l a - b\sqrt{-1} = \frac{1}{2} l (a a + b b) - \sqrt{-1} A \operatorname{tang.} \frac{b}{a},$$

quae forma a praecedente subtrahenda nobis dat

$$l \frac{a+b\sqrt{-1}}{a-b\sqrt{-1}} = 2\sqrt{-1} A \operatorname{tang.} \frac{b}{a},$$

atque si hic faciamus $b = c\sqrt{-1}$, erit

$$l \frac{a-c}{a+c} = 2\sqrt{-1} A \operatorname{tang.} \frac{c\sqrt{-1}}{a},$$

hincque vicissim

$$A \operatorname{tang.} \frac{c\sqrt{-1}}{a} = \frac{1}{2\sqrt{-1}} l \frac{a-c}{a+c} = \frac{1}{2} \sqrt{-1} l \frac{a+c}{a-c}$$

§. 13. Iam pro priore logarithmo imaginario erit $a = v+1$ et $b = z$, vnde habebimus:

$$l \frac{v+1+z\sqrt{-1}}{v+1-z\sqrt{-1}} = 2\sqrt{-1} A \operatorname{tang.} \frac{z}{v+1}$$

Pro altero vero logarithmo erit $a = v-1$ et $b = z$, hincque

$$l \frac{v-1+z\sqrt{-1}}{v-1-z\sqrt{-1}} = 2\sqrt{-1} A \operatorname{tang.} \frac{z}{v-1}$$

Denique pro arcu erit $c = 2vz$ et $a = vv+1+zz$, vnde colligitur:

$$A \operatorname{tang.} \frac{2vz\sqrt{-1}}{vv+1+zz} = \frac{1}{2} \sqrt{-1} l \frac{(v+z)^2+1}{(v-z)^2+1},$$

quibus valoribus substitutis erit

Q 2

U =

$$U = \sqrt{-1} A \operatorname{tang.} \frac{z}{v+1} + \sqrt{-1} A \operatorname{tang.} \frac{z}{v-1} - \frac{1}{2} \sqrt{-1} \int \frac{1+(v+z)^2}{1+(v-z)^2}$$

qui valor ductus in $-\sqrt{-1}$ praebet ipsum valorem quaesitum:

$$V = -\frac{1}{2} \int \frac{1+(v+z)^2}{1+(v-z)^2} + A \operatorname{tang.} \frac{z}{v+1} - A \operatorname{tang.} \frac{z}{v-1}$$

quae expressio, arcubus in vnum contrahis, transmutatur in hanc:

$$V = \frac{1}{2} \int \frac{1+(v-z)^2}{1+(v+z)^2} - A \operatorname{tang.} \frac{2vz}{1+zz-vv}$$

§. 14. Haec solutio eo magis est notatu digna, quod per imaginaria est traducta, atque adeo nulla via patere videtur eam directe inueniendi. Fortasse autem si forma integralis inuenta probe perpendatur, inde methodus excogitare poterit, cuius ope sine subsidio imaginariorum ista solutio directe elici queat, hocque argumentum utique dignum videtur, in quo Geometrae sagacitatem suam exercent.

§. 15. At vero, quoniam nulla via patet, integrale posterioris formulae directe inueniendi, operae pretium erit rursus ex aequatione integrali differentialem propositam elicere, visuri, num forsitan haec operatio nobis inferuire possit, aliam resolutionem detegendi, quam per imaginaria progrediendo, quem in finem sequens problema coronidis loco adiungamus.

Problema.

Inuenire differentiale huius expressionis:

$$V = \frac{1}{2} \int \frac{1+zz+vv-2vz}{1+zz+vv+2vz} - A \operatorname{tang.} \frac{2vz}{1+zz-vv}$$

existente $v = \sqrt[4]{(1-6zz-z^4)}$.

Solu-

Solutio.

§. 16. Ponatur

$$\frac{2vz}{1+zz+vv} = p \text{ et } \frac{2vz}{1+zz-vv} = q,$$

eritque

$$V = \frac{1}{2} \log \frac{1-p}{1+p} - A \text{ tang. } q,$$

vnde fit differentiando:

$$\partial V = -\frac{\partial p}{1-p^2} - \frac{\partial q}{1+q^2}.$$

Est vero

$$\partial p = \frac{2v\partial z(1-zz+vv) + 2z\partial v(1+zz-vv)}{(1+zz+vv)^2} \text{ et}$$

$$\partial q = \frac{2v\partial z(1-zz-vv) + 2z\partial v(1+zz+vv)}{(1+zz-vv)^2}.$$

$$\text{Deinde } 1 - pp = \frac{1+2zz+2vv+z^4-2vvzz+vv^4}{(1+zz+vv)^2},$$

fiue ob $v^4 = 1 - 6zz + z^4$, erit

$$1 - pp = \frac{2(1-zz)(1-zz+vv)}{(1+zz+vv)^2},$$

eodemque modo

$$1 + qq = \frac{2(1-zz)(1-zz-vv)}{(1+zz-vv)^2},$$

quibus substitutis fit

$$\partial V = -\frac{v\partial z(1-zz+vv) - z\partial v(1+zz-vv)}{(1-zz)(1-zz+vv)} - \frac{v\partial z(1-zz-vv) - z\partial v(1+zz+vv)}{(1-zz)(1-zz-vv)}.$$

§. 17. Reducatur haec expressio ad eundem denominatorem:

$$(1-zz)[(1-zz)^2 - v^4] = 4zz(1-zz),$$

fietque

$$\partial V = -\frac{2v\partial z[(1-zz)^2 - v^4] - z\partial v[(1-vv)^2 + (1+vv)^2 - 2z^4]}{4zz(1-zz)}.$$

Haec forma porro reducitur ad hanc:

$$\partial V = -\frac{2vz\partial z - \partial v(1-3zz)}{z(1-zz)},$$

quae si supra et infra per v^3 multiplicetur, ob

Q 3

$$v^4 =$$

$$v^4 = 1 - 6zz + z^4 \text{ et}$$

$$v^3 \partial v = -3z \partial z + z^3 \partial z = z \partial z (zz - 3),$$

abibit in sequentem formam:

$$\partial V = - \frac{2 \partial z (1 - 6zz + z^4) - \partial z (zz - 3)(1 - 3zz)}{(1 - zz) v^3},$$

qua euoluta prodit denique

$$\partial V = \frac{\partial z (1 + zz)^2}{(1 - zz) \sqrt{(1 - 6zz + z^4)^3}}$$

Haec igitur formula cum proposita formula differentiali superioris problematis profus conuenit, ita vt certi simus huius formulae integrale reuera esse

$$V = \frac{1}{2} \int \frac{1 + (v - z)^2}{1 + (v + z)^2} - A \text{ tang. } \frac{2zz}{1 + zz - vv},$$

etiamfi non pateat, quomodo hoc integrale methodo directa elici queat.