VLTERIOR DISQVISITIO
DE
FORMVLIS INTEGRALIBVS IMAGINARIIS.

Auctore

L. EVLERO

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§. 1.

Vniuersa Theoria Imaginariorum, vnde tot egregia incrementa nunc quidem in Analysin sunt illata, hoc potissimum nititur fundamento: quod si Z fuerit functio quaecunque ipsius z, eaque posito $z = x + y\sqrt{-1}$ abeat in hanc formam: $M + N\sqrt{-1}$, tum eadem functio Z, posito $z = x - y\sqrt{-1}$, euadat $= M - N\sqrt{-1}$; vbi quidem litterae M & N semper denotant quantitates reales. Hinc si proponatur ista formula differentialis: $Z\partial z$, cuius integrale fit $\int Z\partial z = V$, in eaque ponatur $z = x + y\sqrt{-1}$, vnde prodeat $Z = M + N\sqrt{-1}$, ipsum integrale erit huius formae: $V = P + Q\sqrt{-1}$. Cum enim fit

$$\partial V = Z\partial z = M\partial x - N\partial y + (N\partial x + M\partial y)\sqrt{-1},$$

 inte-

integrale erit

$$\int(M\partial x - N\partial y) + \sqrt{-1}\int(N\partial x + M\partial y) = P + Q\sqrt{-1}.$$

Necesse igitur est, vt posito $z = x - y\sqrt{-1}$ fiat

$$P - Q\sqrt{-1} = \int(M\partial x - N\partial y) - \sqrt{-1}\int(N\partial x + M\partial y).$$

Hinc autem manifestum est fore

$$P = \int(M\partial x - N\partial y) \text{ et } Q = \int(N\partial x + M\partial y).$$

Ex quo intelligitur, in huiusmodi substitutionibus semper partes reales et imaginarias seorsim inter se aequari debere.

§. 2. Haec euolutio nobis iam suppeditat insignes proprietates, quae inter quantitates M, N, P et Q intercedunt. Primo enim cum sit $P = \int(M\partial x - N\partial y)$, quoniam haec formula semper integrationem admittit, erit per criterium huiusmodi formularum generale $(\frac{\partial M}{\partial y}) = -(\frac{\partial N}{\partial x})$. Eodem autem modo, quia habemus $Q = \int(N\partial x + M\partial y)$, ob integrabilitatem huius formulae erit $(\frac{\partial M}{\partial x}) = (\frac{\partial N}{\partial y})$. Ecce ergo per talem substitutionem semper inueniuntur eiusmodi duae functionos M et N binarum variabilium x et y, his insignibus proprietatibus praeditae, vt sit tam $(\frac{\partial M}{\partial y}) = -(\frac{\partial N}{\partial x})$ quam $(\frac{\partial M}{\partial x}) = (\frac{\partial N}{\partial y})$.

§. 3. Similis proprietas etiam conuenit quantitatibus P et Q. Cum enim sit $\partial P = M\partial x - N\partial y$ et $\partial Q = M\partial y + N\partial x$, erit per similes characteres

$$(\frac{\partial P}{\partial x}) = M \text{ et } (\frac{\partial P}{\partial y}) = -N,$$

$$(\frac{\partial Q}{\partial x}) = N \text{ et } (\frac{\partial Q}{\partial y}) = M,$$

vnde

vnde manifeſtum eſt fore

$$\left(\frac{\partial P}{\partial x}\right) = \left(\frac{\partial Q}{\partial y}\right) \text{ et } \left(\frac{\partial P}{\partial y}\right) = -\left(\frac{\partial Q}{\partial x}\right).$$

Tales autem relationes eo magis ſunt notatu dignae, quod earum ratio minus perſpicitur, ſimulac pro Z functiones aliquanto magis complicatae accipiuntur.

§. 4. Non ita pridem contemplatus ſum, ſecundum haec principia, formulam integralem $\int \frac{z^{m-1}\partial z}{1 \pm z^n}$, vnde plures huiusmodi relationes non contemnendas ſum adeptus. Deinde etiam hanc ſpeculationem extendi ad formulam $\int \frac{\partial z}{\sqrt[n]{(1+z^n)}}$, cuius integrale cum ſemper per logarithmos et arcus circulares exprimere liceat, inde etiam pro litteris P et Q eiusmodi formulae prodire debent, quae ſimilem integrationem admittant, etiamſi vix vlla via pateat iſtam integrationem exſequendi. Hocque modo deductus ſum ad Theorema quoddam maxime memorabile, cuius demonſtratio propemodum vires Analyſeos ſuperare videbatur; interim tamen deinceps eius demonſtrationem elicui; quam ob rem conſtitui iſtud argumentum aliquanto generalius retractare.

§. 5. Conſiderabo igitur hic iſtam formulam integralem multo latius patentem: $\int \frac{z^{m-1}\partial z}{(a+bz^n)^\lambda} = V$, vbi, vt calculus commodius ſuccedat, loco z ſubſtituo iſtam formulam imaginariam: $z = v(\cos.\theta + \sqrt{-1}\sin.\theta)$, quippe quae omnia Imaginaria in ſe complectitur; tum vero hic angulum θ pro conſtante ſum habiturus, ita vt ſola v nobis ſit variabilis, vnde ergo ſtatim fit $\frac{\partial z}{z} = \frac{\partial v}{v}$. Cum igitur ſit

$$z^m = v^m (\text{coſ.}\, m\theta + \sqrt{-1}\, \text{ſin.}\, m\theta)$$

numerator huius formulae ſtatim fit

$$z^{m-1} \partial z = v^{m-1} (\text{coſ.}\, m\theta + \sqrt{-1}\, \text{ſin.}\, m\theta)\, \partial v.$$

Ex hac autem ſubſtitutione ſumamus prodire iſtum valorem integralem: $V = P + Q\sqrt{-1}$.

§. 6. Pro denominatore autem obtinebimus

$$a + b z^n = a + b v^n (\text{coſ.}\, n\theta + \sqrt{-1}\, \text{ſin.}\, n\theta)$$

cuius ergo pars realis eſt $a + b v^n \text{coſ.}\, n\theta$, pars imaginaria vero $b v^n \sqrt{-1}\, \text{ſin.}\, n\theta$; vnde ſi exponens eſſet numerus integer, imaginaria facile ex denominatore in numeratorem transferri poſſent, dum ſcilicet ſupra et infra multiplicaremus per

$$[a + b v^n (\text{coſ.}\, n\theta - \sqrt{-1}\, \text{ſin.}\, n\theta)]^\lambda.$$

Verum quia hi caſus nulla laborant difficultate, calculum potiſſimum ad exponentes fractos pro λ accipiendos accommodari conuenit.

§. 7. Hunc in finem loco variabilis v aliam in calculum introducamus s, cum certo angulo ϕ, ita vt ſit

$$a + b v^n \text{coſ.}\, n\theta = s\, \text{coſ.}\, \phi \text{ et}$$
$$b v^n \text{ſin.}\, n\theta = s\, \text{ſin.}\, \phi$$

vnde ergo certa relatio inter hanc nouam variabilem s et angulum ϕ definitur, ita vt vel ſola littera s vel ſolus angulus ϕ in calculum introduci queat. Euidens autem eſt, has duas quantitates per variabilem v ita definiri, vt ſit

$$\text{I}^{\circ}.\ ss = aa + 2abv^n \operatorname{cof.} n\theta + bbv^{2n},$$

$$\text{II}^{\circ}.\ \operatorname{tang.} \Phi = \frac{bv^n \operatorname{fin.} n\theta}{a + bv^n \operatorname{cof.} n\theta}.$$

§. 8. Hic autem ftatim intelligitur, ipfam quantitatem s loco ipfius v non commode in calculum introduci poffe, quandoquidem angulum Φ, cuius varia multipla occurrunt, nullo modo ex calculo eliminare liceret, vel faltem formulae inextricabiles in calculum implicarentur. Quamobrem conueniet totum calculum ad folam variabilem Φ reuocare, ita vt nobis incumbat, ambas quantitates v et s per iftam nouam variabilem Φ determinare.

§. 9. Ante autem quam hoc exfequamur, obferuemus, denominatorem noftrae formulae per binas variabiles affumtas s & Φ ita concinne expreffum iri, vt fiat

$$a + bz^n = s(\operatorname{cof.} \Phi + \sqrt{-1} \operatorname{fin.} \Phi)$$

hincque totus denominator

$$(a + bz^n)^\lambda = s^\lambda (\operatorname{cof.} \lambda\Phi + \sqrt{-1} \operatorname{fin.} \lambda\Phi).$$

Quodfi igitur fupra et infra per $\operatorname{cof.} \lambda\Phi - \sqrt{-1} \operatorname{fin.} \lambda\Phi$ multiplicemus, formula noftra propofita, retento adhuc numeratore, fequentem accipiet formam:

$$\int \frac{v^{m-1}\partial v(\operatorname{cof.} m\theta + \sqrt{-1} \operatorname{fin.} m\theta)(\operatorname{cof.} \lambda\Phi - \sqrt{-1} \operatorname{fin.} \lambda\Phi)}{s^\lambda} = W$$

quae contrahitur in hanc formam fatis fimplicem:

$$\int \frac{v^{m-1}\partial v}{s^\lambda}[\operatorname{cof.}(m\theta - \lambda\Phi) + \sqrt{-1} \operatorname{fin.}(m\theta - \lambda\Phi)],$$

cuius valor cum pofitus fit $= P + Q\sqrt{-1}$, realia ab ima-

imaginariis ſeparando erit

$$P = \int \frac{v^{m-1}\,\partial v \cos.(m\theta - \lambda\phi)}{s^{\lambda}} \text{ et}$$

$$Q = \int \frac{v^{m-1}\,\partial v \sin.(m\theta - \lambda\phi)}{s^{\lambda}}.$$

§. 10. Vt nunc hinc binas litteras v et s abigamus, recurramus ad binas poſitiones ante ſtabilitas:

$$\text{I. } a + b v^{n} \cos. n\theta = s \cos.\phi,$$

$$\text{II. } b v^{n} \sin. n\theta = s \sin.\phi.$$

Hic primo quantitas s eliminabitur per hanc combinationem: I. $\sin.\phi$ — II. $\cos.\phi$, vnde fit $a \sin.\phi = b v^{n} \sin.(n\theta - \phi)$, ideoque $v^{n} = \frac{a \sin.\phi}{b \sin.(n\theta - \phi)}$ ſicque iam valorem litterae v per angulum ϕ ſumus adepti. Porro vero haec combinatio I. $\sin. n\theta$ — II. $\cos. n\theta$ praebet $a \sin. n\theta = s \sin.(n\theta - \phi)$, vnde fit $s = \frac{a \sin. n\theta}{\sin.(n\theta - \phi)}$, ex quo valore nanciſcimur

$$P = \frac{1}{a^{\lambda} \sin. n\theta^{\lambda}} \int v^{m-1}\,\partial v \sin.(n\theta - \phi)^{\lambda} \cos.(m\theta - \lambda\phi) \text{ et}$$

$$Q = \frac{1}{a^{\lambda} \sin. n\theta^{\lambda}} \int v^{m-1}\,\partial v \sin.(n\theta - \phi)^{\lambda} \sin.(m\theta - \lambda\phi).$$

§. 11. Quoniam porro inuenimus $v^{n} = \frac{a \sin.\phi}{b \sin.(n\theta - \phi)}$, ſumtis logarithmis habebimus

$$n\,l\,v = l\,a \sin.\phi - l\,b \sin.(n\theta - \phi)$$

hincque differentiando

$$\frac{n\,\partial v}{v} = \frac{\partial\phi \cos.\phi}{\sin.\phi} + \frac{\partial\phi \cos.(n\theta - \phi)}{\sin.(n\theta - \phi)} = \frac{\partial\phi \sin. n\theta}{\sin.\phi \sin.(n\theta - \phi)}.$$

Deinde

Deinde vero erit $v^m = \left(\frac{a\,\text{fin.}\,\Phi}{b\,\text{fin.}(n\theta-\Phi)}\right)^{\frac{m}{n}}$. His igitur valoribus fubftitutis ad fequentes formulas integrales deducemur:

$$P = \frac{1}{n a^{\lambda}\,\text{fin.}\,n\theta^{\lambda-1}} \int \left(\frac{a\,\text{fin.}\,\Phi}{b\,\text{fin.}(n\theta-\Phi)}\right)^{\frac{m}{n}} \frac{\partial\Phi\,\text{fin.}(n\theta-\Phi)^{\lambda}\,\text{cof.}(m\theta-\lambda\Phi)}{\text{fin.}\,\Phi\,\text{fin.}\,(n\theta-\Phi)}$$

$$Q = \frac{1}{n a^{\lambda}\,\text{fin.}\,n\theta^{\lambda-1}} \int \left(\frac{a\,\text{fin.}\,\Phi}{b\,\text{fin.}(n\theta-\Phi)}\right)^{\frac{m}{n}} \frac{\partial\Phi\,\text{fin.}(n\theta-\Phi)^{\lambda}\,\text{fin.}(m\theta-\lambda\Phi)}{\text{fin.}\,\Phi\,\text{fin.}\,(n\theta-\Phi)}$$

Quod fi iam breuitatis gratia ponamus $n\theta - \Phi = \psi$, vt fit $\Phi + \psi = n\theta$, ideoque $\partial\Phi + \partial\psi = 0$, ambae formulae concinnius fequenti modo repraefentari poterunt:

$$P = \frac{a^{\frac{m}{n}-\lambda}}{n\, b^{\frac{m}{n}}\,\text{fin.}\,n\theta^{\lambda-1}} \int \partial\Phi\,\text{fin.}\,\Phi^{\frac{m-n}{n}}\,\text{fin.}\,\psi^{\lambda-\frac{m}{n}-1}\,\text{cof.}(m\theta-\lambda\Phi),$$

$$Q = \frac{a^{\frac{m}{n}-\lambda}}{n\, b^{\frac{m}{n}}\,\text{fin.}\,n\theta^{\lambda-1}} \int \partial\Phi\,\text{fin.}\,\Phi^{\frac{m-n}{n}}\,\text{fin.}\,\psi^{\lambda-\frac{m}{n}-1}\,\text{fin.}(m\theta-\lambda\Phi);$$

§. 12. En ergo deducti fumus ad binas formulas integrales, quarum integratio, quantumuis, ob exponentem fractum $\frac{m}{n}$, videatur difficilis, tamen femper pendet a formula principali propofita $\int \frac{z^{m-1}\,\partial z}{(a \pm bz^n)^{\lambda}}$, cuius ergo integrale, fi vel algebraice, vel faltem per logarithmos et arcus circulares affignari queat, etiam certo affirmare poterimus, ambas formulas hic inuentas fecundum eandem legem integrari poffe. Hic quidem primo fe offert cafus $m = n$, quo adeo integrale algebraice exhiberi poteſt; verum quia hoc caſu $\frac{m}{n}$ non amplius eſt fractio, eum praetereamus.

§. 13. Imprimis autem hic occurrit caſus maxime memorabilis, quo $\lambda = \frac{m}{n}$, quippe quo integrationem per logarithmos & arcus circulares expedire licet. Si enim pro noſtra formula integrali $\int \frac{\partial z}{z} \cdot \frac{z^m}{(a + b z^n)^{\frac{m}{n}}}$ ſtatuamus $\frac{z}{(a + b z^n)^{\frac{1}{n}}} = t$, vt formula integranda fit $\int t^m \frac{\partial z}{z}$, erit $t^n = \frac{z^n}{a + b z^n}$, vnde colligitur $z^n = \frac{a t^n}{b t^n - 1}$, hincque differentiando ſumtis logarithmis, erit

$$\frac{\partial z}{z} = \frac{\partial t}{t} - \frac{b t^{n-1} \partial t}{b t^n - 1} = \frac{-\partial t}{t (b t^n - 1)},$$

ita vt formula noſtra integranda fit $-\int \frac{t^{m-1} \partial t}{b t^n - 1}$, quae cum ſit rationalis, ſemper per logarithmos & arcus circulares integrari poteſt, quod ergo etiam de binis noſtris formulis P et Q erit tenendum.

§. 14. Statuamus igitur in noſtris formulis ſupra inuentis $\lambda = \frac{m}{n}$, eaeque transmutabuntur in ſequentes:

$$P = \frac{1}{n b^{\frac{m}{n}} \text{ſin.}\, n \theta^{\lambda - 1}} \cdot \int \frac{\partial \Phi \text{ ſin.}\, \Phi^{\frac{m}{n} - 1} \text{ coſ.}\, (m \theta - \frac{m}{n} \Phi)}{\text{ſin.}\, \psi},$$

Q =

$$Q = \frac{1}{n\, b^{\frac{m}{n}}\ \text{fin.}\ n\, \theta^{\lambda - 1}} \cdot \int \frac{\partial \Phi\ \text{fin.}\ \Phi^{\frac{m}{n} - 1}\ \text{fin.}\ (m\,\theta - \frac{m}{n}\,\Phi)}{\text{fin.}\ \psi},$$

vbi breuitatis gratia loco coëfficientis conſtantis ſcribatur C, et cum ex indole formulae propoſitae ſemper ſit $m < n$, has formulas ita ſuccinctius exhibere licet:

$$P = C \int \frac{\partial \Phi\ \text{cof.}\ (m\,\theta - \frac{m}{n}\,\Phi)}{\text{fin.}\ \psi\ \text{fin.}\ \Phi^{\frac{n-m}{n}}} \text{ et}$$

$$Q = C \int \frac{\partial \Phi\ \text{fin.}\ (m\,\theta - \frac{m}{n}\,\Phi)}{\text{fin.}\ \psi\ \text{fin.}\ \Phi^{\frac{n-m}{n}}},$$

quae ergo formulae, quicunque numeri pro m et n accipiantur, ſemper a logarithmis et arcubus circularibus pendere ſunt cenſendae.

§. 15. Quodſi binae formulae cof. $(m\,\theta - \frac{m}{n}\,\Phi)$ et fin. $(m\,\theta - \frac{m}{n}\,\Phi)$, euoluantur, ambae formulae integrales inuentae commode in vnam contrahi poterunt, quae hanc habebit formam:

$$C \int \partial \Phi \frac{\alpha\ \text{fin.}\ \frac{m}{n}\,\Phi + \beta\ \text{cof.}\ \frac{m}{n}\,\Phi}{\text{fin.}\ \psi\ \text{fin.}\ \Phi^{\frac{n-m}{n}}},$$

quae praefata lege integrationem admittet, quicunque valores litteris α et β tribuantur. Deinde quia $\psi = n\,\theta - \Phi$, facta euolutione loco fin. ψ, eiusue multipli cuiusque, ſcribi poterit γ fin. $\Phi + \delta$ cof. Φ, ſicque nunc formula noſtra erit

$$\int \frac{\partial \Phi}{\text{fin.}\ \Phi^{\frac{n-m}{n}}} \cdot \frac{\alpha\ \text{fin.}\ \frac{m}{n}\,\Phi + \beta\ \text{cof.}\ \frac{m}{n}\,\Phi}{\gamma\ \text{fin.}\ \Phi + \delta\ \text{cof.}\ \Phi},$$

 vbi

vbi litterae α, β, γ, δ pro lubitu accipi poſſunt; quamobrem ſi ad fractiones tollendas ſtatuamus $\Phi = n\,\omega$, vt ſit $\frac{m}{n}\Phi = m\,\omega$, ad ſequens perducimur theorema notatu digniſſimum.

Theorema.

Si litterae m *et* n *denotent numeros integros quoscunque, integratio huius formulae:*

$$\int \frac{\partial\,\omega}{(\text{ſin.}\,n\,\omega)^{\frac{n-m}{n}}} \cdot \frac{\alpha\,\text{ſin.}\,m\,\omega + \beta\,\text{coſ.}\,m\,\omega}{\gamma\,\text{ſin.}\,n\,\omega + \delta\,\text{coſ.}\,n\,\omega},$$

ſemper ad logarithmos et arcus circulares reduci poteſt, quicunque etiam valores litteris α, β, γ, δ *tribuantur.*

§. 16. Quam ardua huius Theorematis demonſtratio ſit, clarius intelligemus, ſi hanc formulam ab angulis ad quantitates ordinarias revocemus. Ponamus igitur tang. $\omega = t$, erit $\partial\,\omega = \frac{\partial t}{1+tt}$; deinde ſi brev. gr. vncias poteſtatum binomii hoc modo deſignemus, vt ſit

$$(1+x)^{\lambda} = 1 + (\tfrac{\lambda}{1})\,x + (\tfrac{\lambda}{2})\,x^2 + (\tfrac{\lambda}{3})\,x^3 + \text{etc.}$$

ſinus et coſinus angulorum multiplorum ipſius ω ſequenti modo per t exprimentur:

$$\text{ſin.}\,m\,\omega = \frac{(\frac{m}{1})\,t - (\frac{m}{3})\,t^3 + (\frac{m}{5})\,t^5 - (\frac{m}{7})\,t^7 + \text{etc.}}{(1+tt)^{\frac{m}{2}}}$$

$$\text{coſ.}\,m\,\omega = \frac{1 - (\frac{m}{2})\,t^2 + (\frac{m}{4})\,t^4 - (\frac{m}{6})\,t^6 + \text{etc.}}{(1+tt)^{\frac{m}{2}}}$$

Ponamus autem porro breuitatis ergo

$$(\tfrac{m}{1})\,t - (\tfrac{m}{3})\,t^3 + (\tfrac{m}{5})\,t^5 - \text{etc.} = M$$
$$1 - (\tfrac{m}{2})\,t^2 + (\tfrac{m}{4})\,t^4 - \text{etc.} = \mathfrak{M}$$

ſimili

ſimilique modo etiam ponamus:

$$\left(\tfrac{n}{1}\right)t - \left(\tfrac{n}{3}\right)t^3 + \left(\tfrac{n}{5}\right)t^5 - \text{etc.} = \mathrm{N}$$
$$1 - \left(\tfrac{n}{2}\right)t^2 + \left(\tfrac{n}{4}\right)t^4 - \text{etc.} = \mathfrak{N}$$

vt habeamus

$$\text{ſin.}\, m\,\omega = \frac{\mathrm{M}}{(1+tt)^{\frac{m}{2}}},\quad \text{coſ.}\, m\,\omega = \frac{\mathfrak{M}}{(1+tt)^{\frac{m}{2}}},$$
$$\text{ſin.}\, n\,\omega = \frac{\mathrm{N}}{(1+tt)^{\frac{n}{2}}},\quad \text{coſ.}\, n\,\omega = \frac{\mathfrak{N}}{(1+tt)^{\frac{m}{2}}},$$

quibus valoribus ſubſtitutis formula noſtra integralis ſequentem induet formam:

$$\int \frac{(1+tt)^{n-m-1}\,\partial t\,(\alpha\,\mathrm{M} + \beta\,\mathfrak{M})}{\mathrm{N}^{\frac{n-m}{n}}\,(\gamma\,\mathrm{N} + \delta\,\mathfrak{N})}.$$

Vbi omnia quidem ſunt rationalia, praeter formulam $\mathrm{N}^{\frac{m}{n}}$, quae autem, quia abit in $\left[\left(\tfrac{n}{1}\right)t - \left(\tfrac{n}{3}\right)t^3 + \left(\tfrac{n}{5}\right)t^5 - \text{etc.}\right]^{\frac{m}{n}}$, ſtatim atque n binarium ſuperat, tantopere ſit irrationalis, vt nulla plane via pateat irrationalitatem tollendi, ſi tantum fuerit $n = 3$; multo minus, ſi exponens n magis increſcat, vllo modo reductionem ad rationalitatem ſperare licebit. Interim tamen ſequentem demonſtrationem mihi eruere contigit.

Demonſtratio ſuperioris Theorematis.

§. 17. Ante omnia hic in ſubſidium vocari convenit formulas illas imaginarias, quibus iam ſaepius cum egregio ſucceſſu ſum vſus, quibus pono

$$\cos.\omega + \sqrt{-1}\,\sin.\omega = p \text{ et } \cos.\omega - \sqrt{-1}\,\sin.\omega = q,$$
eritque $pq = 1$ et $\frac{\partial p}{p} = \partial\omega\sqrt{-1}$. Deinde vero hinc pro finibus et cofinibus angulorum multiplorum habebimus:

$$\sin.m\omega = \frac{p^m - q^m}{2\sqrt{-1}} = \frac{p^{2m} - 1}{2p^m\sqrt{-1}} \text{ et}$$

$$\cos.m\omega = \frac{p^m + q^m}{2} = \frac{p^{2m} + 1}{2p^m},$$

fimilique modo

$$\sin.n\omega = \frac{p^{2n} - 1}{2p^n\sqrt{-1}} \text{ et } \cos.n\omega = \frac{p^{2n} + 1}{2p^n}.$$

§. 18. Quo fubftitutio horum valorum magis fublevetur, obferuaffe juuabit, coëfficientes conftantes nihil ad integrabilitatem conferre, ideoque vel omitti, vel fub alia forma referri poffe. Hanc ob rem ftatuemus

$$\partial\omega = \frac{\partial p}{p} \text{ et } \sin.n\omega = \frac{p^{2n} - 1}{p^n};$$

deinde vero, mutata conftantium forma, poni poterit

$$\alpha\sin.m\omega + \beta\cos.m\omega = \frac{\alpha' p^{2m} + \beta'}{p^m},$$

eodemque modo

$$\gamma\sin.n\omega + \delta\cos.n\omega = \frac{\gamma' p^{2n} + \delta'}{p^n}.$$

His igitur valoribus fubftitutis formula noftra hanc induet formam:

$$\int \frac{p^{2n-2m-1}\,\partial p}{(p^{2n}-1)^{\frac{n-m}{n}}} \cdot \frac{\alpha' p^{2m} + \beta'}{\gamma' p^{2n} + \delta'}.$$

§. 19. Haec iam formula vltro ſe ſcindit in duas partes, quas ita ſeorſim repraeſentemus:

$$\alpha \int \frac{p^{2n-1}\,\partial p}{(p^{2n}-1)^{\frac{n-m}{n}}(\gamma' p^{2n}+\delta')} + \beta' \int \frac{p^{2n-2m-1}\,\partial p}{(p^{2n}-1)^{\frac{n-m}{n}}(\gamma' p^{2n}+\delta')}$$

et nunc non amplius difficile erit, vtramque harum formularum ſeorſim ad rationalitatem reducere. In priore enim tantum opus eſt ſtatuere $p^{2n}-1 = x^{2n}$; tum enim erit $p^{2n-1}\,\partial p = x^{2n-1}\,\partial x$ et $(p^{2n}-1)^{\frac{n-m}{n}} = x^{2n-2m}$, ſicque formula prior accipiet hanc formam: $\alpha \int \frac{x^{2m-1}\,\partial x}{\gamma' x^{2n} + \gamma' + \delta'}$, cuius ergo integrale per logarithmos et arcus circulares exhibere licet.

§. 20. Quod vero ad alteram formulam attinet reductio etiam ſe facile offeret, ſi ipſa formula hoc modo repraeſentetur:

$$\beta' \int \frac{\partial p}{p} \cdot \frac{p^{2n-2m}}{(p^{2n}-1)^{\frac{n-m}{n}}(\gamma' p^{2n}+\delta')},\ \text{ſiue}$$

$$\beta' \int \frac{\partial p}{p} \left(\frac{p^2}{(p^{2n}-1)^{\frac{1}{n}}}\right)^{n-m} \cdot \frac{1}{\gamma' p^{2n}+\delta'}.$$

Si enim hic ponatur $\frac{p^2}{(p^{2n}-1)^{\frac{1}{n}}} = y^2$, fiet $\frac{p^{2n}}{p^{2n}-1} = y^{2n}$,

hinc-

hincque $p^{2n} = \frac{y^{2n}}{y^{2n}-1}$, vnde ſumtis logarithmis et differentiando prodit $\frac{\partial p}{p} = \frac{-\partial y}{y(y^{2n}-1)}$, quibus valoribus ſubſtitutis iſta formula euadet

$$-\beta'\int \frac{y^{2n-2m-1}\,\partial y}{(\gamma'+\delta')\,y^{2n}-\delta'},$$

quae ergo pariter eſt rationalis.

§. 21. Hoc igitur modo veritas noſtri theorematis ſatis firmiter eſt demonſtrata, atque iſte caſus ita eſt comparatus, vt tota formula ope vnius ſubſtitutionis nullo modo rationalis reddi queat, quae circumſtantia eo magis eſt notatu digna, quod vulgo ſtatui ſolet, omnes formulas differentiales, quantumuis fuerint irrationales, ſi earum integralia per logarithmos et arcus circulares exhiberi poſſunt, eas ſemper ope certae ſubſtitutionis ad rationalitatem perduci poſſe. Nunc igitur videmus hoc effatum ita reſtringi debere, vt tantum ad ſingulas partes totius formulae propoſitae extendatur, quandoquidem fieri poteſt, vt quaelibet pars peculiarem ſubſtitutionem poſtulet.

§. 22. Quod ſi hanc demonſtrationem attentius perpendamus, facile videre licebit, eam ad formulas multo latius patentes extendi poſſe. Apparebit enim ſequentem formulam multo generaliorem ſemper ad rationalitatem perduci poſſe, id quod in ſequente theoremate clarius explicemus.

Theo-

Theorema maxime generale.

§. 23. *Si litterae P et Q denotent functiones quascunque rationales formae* x^n, *iftius formulae:*

$$\int \frac{\partial x\,(\mathrm{P}\,x^{m-1}+\mathrm{Q}\,x^{n-1})}{(a+b\,x^n)^{\frac{m}{n}}},$$

integrale femper per logarithmos et arcus circulares exprimetur.

Demonftratio.

Secetur, vt fupra fecimus, ifta formula etiam in duas partes, quae fint

$$\int \frac{\mathrm{P}\,x^{m-1}\,\partial x}{(a+b\,x^n)^{\frac{m}{n}}} \text{ et } \int \frac{\mathrm{Q}\,x^{n-1}\,\partial x}{(a+b\,x^n)^{\frac{m}{n}}},$$

atque ftatim patet pofteriorem partem rationalem reddi, ponendo $a+b\,x^n=r^n$; tum enim erit $(a+b\,x^n)^{\frac{m}{n}}=r^m$, tum vero $x^n=\frac{r^n-a}{b}$ et $x^{n-1}\,\partial x=\frac{r^{n-1}\,\partial r}{b}$. Quia nunc Q eft functio rationalis ipfius x^n, facta hac fubftitutione prodibit certe functio rationalis ipfius r^n, ficque pars pofterior accipiet hanc formam: $\frac{1}{b}\int \mathrm{Q}\,r^{n-m-1}\,\partial r$.

Quo prior pars ad rationalitatem reuocetur, ftatuatur $\frac{x}{(a+b\,x^n)^n}=s$, vt fiat $\frac{x^m}{(a+b\,x^n)^{\frac{m}{n}}}=s^m$, tum vero erit $x^n=\frac{a\,s^n}{1-b\,s^n}$, qui ergo valor fi in P loco x^n fubſtituatur, manifeſto dabit functionem rationalem ipſius s^n; deinde vero hinc fit $\frac{\partial x}{x} = \frac{\partial s}{s(1 - b s^n)}$, ex quibus valoribus oritur pars noſtra prior $= \int \frac{P s^{m-1} \partial s}{1 - b s^m}$, quae ergo formula etiam eſt rationalis.

Quin etiam in angulis Theorema multo generalius proponi poteſt, quod ita ſe habebit:

Theorema generale.

§. 24. *Si litterae P et Q denotent functiones quascunque rationales binarum formularum ſin. $2 n \omega$ et coſ. $2 n \omega$, ſequens formula integralis ſemper per logarithmos et arcus circulares expediri poterit:*

$$\int (P \sin m\omega + Q \cos m\omega)\, \partial . (\sin n\omega)^{\frac{m}{n}},$$

vbi notetur eſſe

$$\partial (\sin n\omega)^{\frac{m}{n}} = \frac{m\, \partial\omega \cos n\omega}{(\sin n\omega)^{\frac{n-m}{n}}},$$

cuius demonſtratio ſimili modo ſuccedet vt ſupra, dum pariter ad binas partes peruenietur, quarum vtramque certa ſubſtitutione rationalem reddere licebit.

§. 25. Pluribus fortaſſe diſplicebit, quod reſolutio poſtremae formulae per ſubſtitutiones imaginarias peragatur, cum tamen hic nobis ſit propoſitum imaginaria a realibus ſeparare; plurimum igitur optandum eſſet vt hoc negotium um fine imaginariis abſolui poſſet; verum equidem fateri cogor, me neutiquam perſpicere, quomodo hoc praeſtari queat. Ceterum quia reductio Imaginariorum ad realia iam ſatis eſt exculta, tale remedium non adeo deſiderandum videtur. Quin potius hic nouus ſe prodit vſus Imaginariorum in ipſa reſolutione formularum integralium, dum eiusmodi formulae integrabiles exhiberi poſſunt, quarum integralia fine auxilio Imaginariorum eruere non licet.