

VARIAE SPECVLATIONES
 SVPER AREA
 TRIANGVLORVM SPHAERICORVM.

Auctore

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Conuentui exhib. die 29 Ianuar. 1778.

§. I.

Primus, qui aream trianguli sphaerici definire docuit, erat, teste Wallisio, Albertus Girard, qui demonstrauit, aream trianguli sphaerici semper proportionalem esse excessui summae ternorum angulorum super duobus reſtis, atque adeo ipsam aream inueniri, si iste excessus, in arcum circuli maximi conuersus, per radium sphaerae multiplicetur. Quemadmodum autem area trianguli sphaerici ex eius lateribus sit determinanda, inuestigationem multo difficiliorem postulat. Inueni autem iam olim egregium theorema, quo ista determinatio facile institui potest, quod ita se habet: *Si latera trianguli sphaerici denotentur litteris a, b, c, area vero eiusdem trianguli ponatur = Δ, tum semper erit*

$$\operatorname{cof.} \frac{1}{2} \Delta = \frac{1 + \operatorname{cof.} a + \operatorname{cof.} b + \operatorname{cof.} c}{4 \operatorname{cof.} \frac{1}{2} a \operatorname{cof.} \frac{1}{2} b \operatorname{cof.} \frac{1}{2} c},$$

cuius veritas non nisi per longas ambages, siue ex theoremate

mate Girardi, siue immediate per calculum integralem ostendi potest. Vtramque igitur demonstrationem hic in medium attulisse operae erit pretium.

Problema.

Tab. I. *Si trianguli sphaerici AZB super basi AB extructi*
 Fig. 4. *bina latera AZ et BZ suis differentialibus augeantur, vt inde oriatur triangulum AzB, inuestigare augmentum quod hinc areae trianguli AZB accessit.*

Solutio.

§. 2. Posita basi huius trianguli $AB = a$ vocentur eius latera $AZ = x$ et $BZ = y$, ita vt latera trianguli aucti futura sint $Az = x + \partial x$ et $Bz = y + \partial y$. Porro vero vocentur anguli $BAZ = \Phi$ et $ABZ = \Psi$, vt prodeant anguli elementares $ZAz = \partial \Phi$ et $ZBz = \partial \Psi$, quibus positis constat trianguli elementaris ZAz aream esse $= \partial \Phi (1 - \cos. x)$, trianguli vero $ZBz = \partial \Psi (1 - \cos. y)$. Quoniam igitur haec duo triangula elementaria exhibent augmentum areae trianguli Δ , habebimus hanc aequationem:

$$\partial \Delta = \partial \Phi (1 - \cos. x) + \partial \Psi (1 - \cos. y).$$

§. 3. Nunc igitur angulos Φ et Ψ ex calculo eliminemus, eorumque loco ipsa latera x et y introducamus ope praeceptorum Trigonometricorum, quae nobis praebent

$$\cos. \Phi = \frac{\cos. y - \cos. a \cos. x}{\sin. a \sin. x} \quad \text{et} \quad \cos. \Psi = \frac{\cos. x - \cos. a \cos. y}{\sin. a \sin. y}$$

hinc igitur per differentiationem colligimus

$$-\partial \Phi \sin. \Phi = \frac{\partial x \cos. a - \partial x \cos. x \cos. y - \partial y \sin. x \sin. y}{\sin. a \sin. x^2}$$

eodemque modo erit

$$-\partial \Psi \sin. \Psi = \frac{\partial y \cos. a - \partial y \cos. x \cos. y - \partial x \sin. x \sin. y}{\sin. a \sin. y^2}$$

At

At vero cum fit $\text{cof. } \Phi = \frac{\text{cof. } y - \text{cof. } a \text{ cof. } x}{\text{fin. } a \text{ fin. } x}$, erit

$$\text{fin. } \Phi = \frac{\sqrt{(1 - \text{cof. } a^2 - \text{cof. } x^2 - \text{cof. } y^2 + 2 \text{cof. } a \text{ cof. } x \text{ cof. } y)}}{\text{fin. } a \text{ fin. } x}$$

fimilique modo erit

$$\text{fin. } \Psi = \frac{\sqrt{(1 - \text{cof. } a^2 - \text{cof. } x^2 - \text{cof. } y^2 + 2 \text{cof. } a \text{ cof. } x \text{ cof. } y)}}{\text{fin. } a \text{ fin. } y}$$

Quoniam hae ambae formulae radicales sunt eadem, ponamus brevitatis gratia

$$\sqrt{(1 - \text{cof. } a^2 - \text{cof. } x^2 - \text{cof. } y^2 + 2 \text{cof. } a \text{ cof. } x \text{ cof. } y)} = v,$$

vt habeamus

$$\text{fin. } \Phi = \frac{v}{\text{fin. } a \text{ fin. } x} \text{ et } \text{fin. } \Psi = \frac{v}{\text{fin. } a \text{ fin. } y}$$

§. 4. His igitur valoribus substitutis nanciscemur hos valores differentiales :

$$\partial \Phi = \frac{\partial x \text{ cof. } a + \partial x \text{ cof. } x \text{ cof. } y + \partial y \text{ fin. } x \text{ fin. } y}{v \text{ fin. } x} \text{ et}$$

$$\partial \Psi = \frac{\partial y \text{ cof. } a + \partial y \text{ cof. } x \text{ cof. } y + \partial x \text{ fin. } x \text{ fin. } y}{v \text{ fin. } y}$$

Hinc igitur incrementum areae quaesitum erit

$$\partial \Delta = \frac{\left. \begin{aligned} & -\text{cof. } a [\partial x \text{ fin. } y (1 - \text{cof. } x) + \partial y \text{ fin. } x (1 - \text{cof. } y)] \\ & + \partial x \text{ fin. } y [\text{cof. } x \text{ cof. } y (1 - \text{cof. } x) + \text{fin. } x^2 (1 - \text{cof. } y)] \\ & + \partial y \text{ fin. } x [\text{cof. } x \text{ cof. } y (1 - \text{cof. } y) + \text{fin. } y^2 (1 - \text{cof. } x)] \end{aligned} \right\}}{v \text{ fin. } x \text{ fin. } y}$$

quod evolutum induet hanc formam :

$$v \partial \Delta = \left. \begin{aligned} & + \partial x \text{ fin. } x (1 - \text{cof. } y) + \frac{\partial x \text{ cof. } x \text{ cof. } y (1 - \text{cof. } x)}{\text{fin. } x} - \frac{\partial x \text{ cof. } a (1 - \text{cof. } x)}{\text{fin. } x} \\ & + \partial y \text{ fin. } y (1 - \text{cof. } x) + \frac{\partial y \text{ cof. } x \text{ cof. } y (1 - \text{cof. } y)}{\text{fin. } y} - \frac{\partial y \text{ cof. } a (1 - \text{cof. } y)}{\text{fin. } y} \end{aligned} \right\}$$

Hic iam notetur esse

$$\frac{1 - \text{cof. } x}{\text{fin. } x} = \text{tang. } \frac{1}{2} x \text{ et } \frac{1 - \text{cof. } y}{\text{fin. } y} = \text{tang. } \frac{1}{2} y,$$

hinc ergo termini elementum ∂x inuoluentes erunt

$$\partial x \sin. x (1 - \cos. y) + \partial x \cos. x \cos. y \operatorname{tang.} \frac{1}{2} x - \partial x \cos. a \operatorname{tang.} \frac{1}{2} x.$$

Quoniam autem non solum est $\operatorname{tang.} \frac{1}{2} x = \frac{1 - \cos. x}{\sin. x}$, sed etiam $\operatorname{tang.} \frac{1}{2} x = \frac{\sin. x}{1 + \cos. x}$, in primo membro loco $\sin. x$ scribatur $(1 + \cos. x) \operatorname{tang.} \frac{1}{2} x$, ut ∂x vbique multiplicatum fit per $\operatorname{tang.} \frac{1}{2} x$, sicque istud membrum reducetur ad hanc formam:

$$\partial x \operatorname{tang.} \frac{1}{2} x (1 + \cos. x - \cos. y - \cos. a).$$

Eodem modo alterum membrum erit

$$\partial y \operatorname{tang.} \frac{1}{2} y (1 + \cos. y - \cos. x - \cos. a),$$

sicque tota nostra aequatio ita erit expressa:

$$v \partial \Delta = \partial x \operatorname{tang.} \frac{1}{2} x (1 + \cos. x - \cos. y - \cos. a) \\ + \partial y \operatorname{tang.} \frac{1}{2} y (1 + \cos. y - \cos. x - \cos. a).$$

§. 5. Quod si iam breuitatis gratia ponamus $\cos. a + \cos. x + \cos. y = s$, erit

$$v \partial \Delta = \partial x \operatorname{tang.} \frac{1}{2} x (1 - s + 2 \cos. x) \\ + \partial y \operatorname{tang.} \frac{1}{2} y (1 - s + 2 \cos. y),$$

quae aequatio hoc modo repraesentari potest:

$$v \partial \Delta = (1 - s) (\partial x \operatorname{tang.} \frac{1}{2} x + \partial y \operatorname{tang.} \frac{1}{2} y) \\ + 2 \partial x \cos. x \operatorname{tang.} \frac{1}{2} x + 2 \partial y \cos. y \operatorname{tang.} \frac{1}{2} y.$$

Cum nunc sit $\operatorname{tang.} \frac{1}{2} x = \frac{1 - \cos. x}{\sin. x}$, erit

$$\operatorname{tang.} \frac{1}{2} x \cos. x = \frac{\cos. x - \cos. x^2}{\sin. x} = \frac{\cos. x - 1 + \sin. x^2}{\sin. x} \\ = \sin. x - \operatorname{tang.} \frac{1}{2} x.$$

Eodem modo erit $\operatorname{tang.} \frac{1}{2} y \cos. y = \sin. y - \operatorname{tang.} \frac{1}{2} y$, hisque valoribus substitutis orietur haec aequatio:

$$v \partial \Delta = - (1 + s) (\partial x \operatorname{tang.} \frac{1}{2} x + \partial y \operatorname{tang.} \frac{1}{2} y) \\ + 2 \partial x \sin. x + 2 \partial y \sin. y.$$

§. 6.

§. 6. Haec postrema forma ideo notatu maxime est digna, quod membrum dextrum absolute fit integrabile, si diuidatur per $1+s$. Facta enim hac diuisione nostra aequatio erit

$$\frac{v \partial \Delta}{1+s} = -\partial x \operatorname{tang} \frac{1}{2} x - \partial y \operatorname{tang} \frac{1}{2} y + \frac{2 \partial x \sin x + 2 \partial y \sin y}{1 + \cos a + \cos x + \cos y};$$

vbi notetur esse $\int \partial x \operatorname{tang} \frac{1}{2} x = -2l \cos \frac{1}{2} x$, similique modo $\int \partial y \operatorname{tang} \frac{1}{2} y = -2l \cos \frac{1}{2} y$, ac denique

$$2 \int \frac{\partial x \sin x + \partial y \sin y}{1 + \cos a + \cos x + \cos y} = -2l(1 + \cos a + \cos x + \cos y) \\ = -2l(1+s),$$

sic igitur per integrationem reperimus

$$\int \frac{v \partial \Delta}{1+s} = +2l \cos \frac{1}{2} x + 2l \cos \frac{1}{2} y - 2l(1+s) \\ = +2 \int \frac{\cos \frac{1}{2} x \cos \frac{1}{2} y}{1+s},$$

at vero hoc modo membrum sinistrum non est integrabile, cui ergo sequenti modo remedium afferetur.

§. 7. Cum enim posuerimus $s = \cos a + \cos x + \cos y$, statuamus insuper $\cos \frac{1}{2} a \cos \frac{1}{2} x \cos \frac{1}{2} y = q$, vt habeamus hanc aequationem:

$$\int \frac{v \partial \Delta}{1+s} = +2 \int \frac{q}{(1+s) \cos \frac{1}{2} a},$$

vbi porro fiat $\frac{q}{1+s} = p$, ita vt fit

$$\int \frac{v \partial \Delta}{1+s} = +2 \int \frac{p}{\cos \frac{1}{2} a},$$

quae aequatio denuo differentiata praebet $\frac{v \partial \Delta}{1+s} = +\frac{2 \partial p}{p}$, vnde conficitur $\partial \Delta = +\frac{2 \partial p (1+s)}{p v}$, quae ergo formula integrationem admittet, si modo fuerit $\frac{v}{1+s}$ functio quaedam ipsius p , id quod

quod iam certo afferere possumus, propterea quod $\partial \Delta$ designat differentiale ipsius areae trianguli.

§. 8. Ad hoc ostendendum observasse iuvabit esse.

$$\begin{aligned} vv + (1+s)^2 &= 2(1 + \text{cof. } a + \text{cof. } x + \text{cof. } y + \text{cof. } a \text{ cof. } x \\ &\quad + \text{cof. } a \text{ cof. } y + \text{cof. } x \text{ cof. } y + \text{cof. } a \text{ cof. } x \text{ cof. } y) \\ &= 2(1 + \text{cof. } a)(1 + \text{cof. } x)(1 + \text{cof. } y). \end{aligned}$$

Constat autem esse $1 + \text{cof. } a = 2 \text{ cof. } \frac{1}{2} a^2$;

$1 + \text{cof. } x = 2 \text{ cof. } \frac{1}{2} x^2$ et $1 + \text{cof. } y = 2 \text{ cof. } \frac{1}{2} y^2$,
quare cum posuerimus $q = \text{cof. } \frac{1}{2} a \text{ cof. } \frac{1}{2} x \text{ cof. } \frac{1}{2} y$, erit

$$vv + (1+s)^2 = 16qq \text{ ideoque } v = \sqrt{[16qq - (1+s)^2]}$$

hincque porro

$$\frac{v}{(1+s)} = \sqrt{\left(\frac{16qq}{(1+s)^2} - 1\right)}.$$

§. 9. Cum igitur aequatio nostra differentialis fuisset $\frac{v \partial \Delta}{1+s} = + \frac{2 \partial p}{p}$, ob $p = \frac{q}{1+s}$ ea induet hanc formam:

$$\partial \Delta \sqrt{16pp - 1} = \frac{2 \partial p}{p}, \text{ ideoque } \partial \Delta = \frac{2 \partial p}{p \sqrt{16pp - 1}}.$$

Fiat iam $p = \frac{r}{4}$, vt habeatur $\partial \Delta = - \frac{2 \partial r}{\sqrt{16 - rr}}$, vnde integrando colligimus $\Delta = C + 2 \text{ Arc. cof. } \frac{r}{4}$ et loco r valore substituto, qui est

$$r = \frac{1}{p} = \frac{1+s}{q} = \frac{1 + \text{cof. } a + \text{cof. } x + \text{cof. } y}{\text{cof. } \frac{1}{2} a \text{ cof. } \frac{1}{2} x \text{ cof. } \frac{1}{2} y}$$

nostra aequatio integralis erit

$$\Delta = C + 2 \text{ Arc. cof. } \frac{1 + \text{cof. } a + \text{cof. } x + \text{cof. } y}{4 \text{ cof. } \frac{1}{2} a \text{ cof. } \frac{1}{2} x \text{ cof. } \frac{1}{2} y}.$$

§. 10. Nunc ergo totum negotium eo redit, ut valor constantis per integrationem ingressae C indagetur, quem scilicet ex casu quodam cognito erui oportet; manifestum autem est aream trianguli evanescere debere, quando alterum binorum crurum x vel y evanescit. Ponamus igitur esse $y = 0$, tum vero necesse est, ut fiat $x = 0$, hoc ergo casu constituto nostra aequatio erit

$$0 = C + 2 \text{Arc. cof. } \frac{2 + 2 \text{cof. } a}{4 \text{cof. } \frac{1}{2} a^2}.$$

Quoniam vero $4 \text{cof. } \frac{1}{2} a^2 = 2 + 2 \text{cof. } \frac{1}{2} a$ et $\text{Arc. cof. } 1 = c$, evidens est statui debere $C = 0$, ita ut habeamus

$$\Delta = 2 \text{Arc. cof. } \frac{1 + \text{cof. } a + \text{cof. } x + \text{cof. } y}{4 \text{cof. } \frac{1}{2} a \text{cof. } \frac{1}{2} x \text{cof. } \frac{1}{2} y},$$

unde concluditur

$$\text{cof. } \frac{1}{2} \Delta = \frac{1 + \text{cof. } a + \text{cof. } x + \text{cof. } y}{4 \text{cof. } \frac{1}{2} a \text{cof. } \frac{1}{2} x \text{cof. } \frac{1}{2} y},$$

quae ipsa expressio cum theoremate supra memorato egregie conuenit, si modo loco x et y scribantur litterae b et c .

Alia demonstratio Geometrica theorematis initio allati.

§. 11. Sit igitur ABC triangulum sphaericum propositum, cuius latera vocentur a, b, c , et anguli iis oppositi α, β, γ , area vero, quam quaerimus, designemus characterem Δ . Cum igitur ex theoremate Girardi sit

$$\Delta = \alpha + \beta + \gamma - 180^\circ, \text{ erit } \text{cof. } \Delta = -\text{cof. } (\alpha + \beta + \gamma).$$

Nunc vero ex compositione angulorum constat esse

$$\text{fin. } (\alpha + \beta) = \text{fin. } a \text{cof. } \beta + \text{cof. } a \text{fin. } \beta \text{ et}$$

Tab. I.
Fig. 5.

$$\text{cof. } (\alpha + \beta) = \text{cof. } a \text{ cof. } \beta - \text{fin. } a \text{ fin. } \beta,$$

vnde colligitur

$$\begin{aligned} \text{cof. } (\alpha + \beta + \gamma) &= \text{cof. } (\alpha + \beta) \text{ cof. } \gamma - \text{fin. } (\alpha + \beta) \text{ fin. } \gamma \\ &= \text{cof. } a \text{ cof. } \beta \text{ cof. } \gamma - \text{cof. } a \text{ fin. } \beta \text{ fin. } \gamma \\ &\quad - \text{cof. } \beta \text{ fin. } a \text{ fin. } \gamma - \text{cof. } \gamma \text{ fin. } a \text{ fin. } \beta, \end{aligned}$$

consequenter habebimus

$$\begin{aligned} \text{cof. } \Delta &= + \text{cof. } a \text{ fin. } \beta \text{ fin. } \gamma + \text{cof. } \beta \text{ fin. } a \text{ fin. } \gamma + \text{cof. } \gamma \text{ fin. } a \text{ fin. } \beta \\ &\quad - \text{cof. } a \text{ cof. } \beta \text{ cof. } \gamma. \end{aligned}$$

§. 12. Ex Trigonometria sphaerica autem nouimus

effe

$$\text{cof. } \alpha = \frac{\text{cof. } a - \text{cof. } b \text{ cof. } c}{\text{fin. } b \text{ fin. } c},$$

$$\text{cof. } \beta = \frac{\text{cof. } b - \text{cof. } a \text{ cof. } c}{\text{fin. } a \text{ fin. } c} \text{ et}$$

$$\text{cof. } \gamma = \frac{\text{cof. } c - \text{cof. } a \text{ cof. } b}{\text{fin. } a \text{ fin. } b},$$

hincque colligimus porro

$$\text{fin. } \alpha = \frac{\sqrt{(1 - \text{cof. } a^2 - \text{cof. } b^2 - \text{cof. } c^2 + 2 \text{cof. } a \text{ cof. } b \text{ cof. } c)}}{\text{fin. } b \text{ fin. } c},$$

$$\text{fin. } \beta = \frac{\sqrt{(1 - \text{cof. } a^2 - \text{cof. } b^2 - \text{cof. } c^2 + 2 \text{cof. } a \text{ cof. } b \text{ cof. } c)}}{\text{fin. } a \text{ fin. } c} \text{ et}$$

$$\text{fin. } \gamma = \frac{\sqrt{(1 - \text{cof. } a^2 - \text{cof. } b^2 - \text{cof. } c^2 + 2 \text{cof. } a \text{ cof. } b \text{ cof. } c)}}{\text{fin. } a \text{ fin. } b}.$$

Ponamus igitur breuitatis gratia

$$\sqrt{(1 - \text{cof. } a^2 - \text{cof. } b^2 - \text{cof. } c^2 + 2 \text{cof. } a \text{ cof. } b \text{ cof. } c)} = v,$$

ita vt fit

$$\text{fin. } \alpha = \frac{v}{\text{fin. } b \text{ fin. } c}; \text{ fin. } \beta = \frac{v}{\text{fin. } a \text{ fin. } c} \text{ et fin. } \gamma = \frac{v}{\text{fin. } a \text{ fin. } b};$$

quibus valoribus substitutis fiet

$$\begin{aligned} \text{cof. } \Delta &= \frac{v v (\text{cof. } a - \text{cof. } b \text{ cof. } c)}{\text{fin. } a^2 \text{ fin. } b^2 \text{ fin. } c^2} + \frac{v v (\text{cof. } b - \text{cof. } a \text{ cof. } c)}{\text{fin. } a^2 \text{ fin. } b^2 \text{ fin. } c^2} \\ &\quad + \frac{v v (\text{cof. } c - \text{cof. } a \text{ cof. } b)}{\text{fin. } a^2 \text{ fin. } b^2 \text{ fin. } c^2} - \frac{(\text{cof. } a - \text{cof. } b \text{ cof. } c)(\text{cof. } b - \text{cof. } a \text{ cof. } c)(\text{cof. } c - \text{cof. } a \text{ cof. } b)}{\text{fin. } a^2 \text{ fin. } b^2 \text{ fin. } c^2}, \end{aligned}$$

fic.

ficque erit

$$\begin{aligned} & \text{fin. } a^2 \text{ fin. } b^2 \text{ fin. } c^2 \text{ cof. } \Delta \\ & = vv(\text{cof. } a + \text{cof. } b + \text{cof. } c - \text{cof. } a \text{ cof. } b - \text{cof. } a \text{ cof. } c - \text{cof. } b \text{ cof. } c) \\ & - \text{cof. } a \text{ cof. } b \text{ cof. } c + \text{cof. } a^2 \text{ cof. } b^2 + \text{cof. } a^2 \text{ cof. } c^2 + \text{cof. } b^2 \text{ cof. } c^2 \\ & - \text{cof. } a \text{ cof. } b \text{ cof. } c (\text{cof. } a^2 + \text{cof. } b^2 + \text{cof. } c^2) + \text{cof. } a^2 \text{ cof. } b^2 \text{ cof. } c^2. \end{aligned}$$

§. 13. Quo nunc has formulas non parum complicatas commodius tractare liceat, ponamus primo brevitatis gratia $\text{cof. } a = A$; $\text{cof. } b = B$; $\text{cof. } c = C$, ut habeamus

$$\begin{aligned} (1-A^2)(1-B^2)(1-C^2)\text{cof. } \Delta &= vv(A+B+C-AB-AC-BC) \\ & - ABC + AAB + AAC + BBCC \\ & - ABC(AA+BB+CC) + AABCC, \end{aligned}$$

vbi iam erit

$$vv = 1 - A^2 - B^2 - C^2 + 2ABC.$$

§. 14. Quoniam hic ternae litterae A, B, C aequaliter in calculum ingrediuntur, ita ut tanquam radices cuiuspiam aequationis cubicae spectari queant; ad calculum contrahendum non parum conferet statui

$$\begin{aligned} A + B + C &= P \\ AB + AC + BC &= Q \\ ABC &= R, \end{aligned}$$

hincque facile colligitur fore

$$AA + BB + CC = PP - 2Q$$

ideoque

$$vv = 1 - PP + 2Q + 2R.$$

Dein-

Deinde notetur formulam $(1 - A^2)(1 - B^2)(1 - C^2)$ esse productum ex his duabus formulis:

$$(1 + A)(1 + B)(1 + C) = 1 + P + Q + R,$$

et ex

$$(1 - A)(1 - B)(1 - C) = 1 - P + Q - R,$$

ficque nostra aequatio hanc induet formam:

$$\begin{aligned} (1 + P + Q + R)(1 - P + Q - R) \operatorname{cof.} \Delta \\ = (1 - PP + 2Q + 2R)(P - Q) - R + QQ - 2PR - R(PP - 2Q) \\ + RR \end{aligned}$$

cuius membrum dextrum euolutum dat

$$P - Q - R - QQ + 2PQ + RR - P^3 + PPQ - PPR,$$

quod per $1 - P + Q - R$ diuisum praebet quotientem $P - Q - R + PP$, consequenter nostra aequatio hanc induet formam:

$$(1 + P + Q + R) \operatorname{cof.} \Delta = P - Q - R + PP.$$

§. 15. Haecenus igitur deducti sumus ad hanc aequationem: $\operatorname{cof.} \Delta = \frac{P - Q - R + PP}{1 + P + Q + R}$, vnde porro colligimus

$$1 + \operatorname{cof.} \Delta = \frac{1 + PP}{1 + P + Q + R} = 2 \operatorname{cof.} \frac{1}{2} \Delta,$$

consequenter habebimus

$$\operatorname{cof.} \frac{1}{2} \Delta = \frac{1 + P}{\sqrt{2}(1 + P + Q + R)}.$$

Cum igitur fit

$$\begin{aligned} 1 + P + Q + R &= (1 + A)(1 + B)(1 + C) \\ &= (1 + \operatorname{cof.} a)(1 + \operatorname{cof.} b)(1 + \operatorname{cof.} c), \end{aligned}$$

angulis dimidiis introductis erit

$$1 + P$$

$$1 + P + Q + R = 8 \operatorname{cof.} \frac{1}{2} a \operatorname{cof.} \frac{1}{2} b \operatorname{cof.} \frac{1}{2} c.$$

Quare cum fit

$$1 + P = 1 + \operatorname{cof.} a + \operatorname{cof.} b + \operatorname{cof.} c,$$

hinc tandem impetramus istum valorem:

$$\operatorname{cof.} \frac{1}{2} \Delta = \frac{1 + \operatorname{cof.} a + \operatorname{cof.} b + \operatorname{cof.} c}{4 \operatorname{cof.} \frac{1}{2} a \cdot \operatorname{cof.} \frac{1}{2} b \cdot \operatorname{cof.} \frac{1}{2} c},$$

quae est altera demonstratio theorematis initio commemorati.

§. 16. Haec cogitandi occasionem mihi dedit theoremata a Celeb. Professore Lexell in medium allatum, circa omnia triangula sphaerica eiusdem areae super eadem basi extruenda, quo acutissime demonstravit, omnes vertices horum triangulorum semper in quodam circulo minore sphaerae esse fitos; quae elegantissima proprietates non nisi per plures ambages ex nostro theoremate derivari potest; verum sequens consideratio viam planissimam ad hoc praestandum aperiet.

De eximiiis proprietatibus binorum circulorum parallelorum inter se aequalium in superficie sphaerica.

§. 17. Sint MN et mn duo huiusmodi circuli paralleli, et quia sumuntur inter se aequales, ab aequatore AB vtrique aequaliter erunt remoti, perinde atque ab utroque polo P et p. Hic prima proprietates, quae se offert, in hoc consistit, quod quilibet arcus circuli maximi Ee, inter hos duos parallelos interceptus, ad vtrumque aequaliter inclinatur, atque ab aequatore in O in duas partes

Tab. I.
Fig. 6.

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tes aequales fecetur. Si enim per punctum O ducatur meridianus OPp , binos parallelos secans in H et h , ob angulos HOE et hOe inter se aequales ambo trilinea HOE et hOe manifesto inter se erunt aequalia et similia, ideoque tam erit $OE = Oe$, quam angulus $OEH = Oeh$. Praeterea hic obseruasse iuuabit, si iste arcus Ee vsque ad semicirculum continuetur, eum iterum in circulum minorem mn incidere, scilicet in eius puncto, quod puncto E diametraliter opponitur.

Tab. I. §. 18. Ducatur nunc inter eosdem parallelos infra Fig. 7. per alius arcus circuli maximi Ff , ad vtrumque perinde inclinatus atque arcus Ee , et manifestum est non solum hos duos arcus Ee et Ff inter se esse aequales, sed etiam circulorum minorum arcus EF et ef . Quare cum in hoc quadrilineo $EFef$ non solum latera opposita, sed etiam anguli oppositi sint inter se aequales, istud quadrilineum uite vocari posset parallelogrammum sphaericum, propterea quod omnibus proprietatibus parallelogrammorum est praeditum. Euidens enim est, istud quadrilineum etiam ab vtraque diagonali Ef et Fe in duo trilinea aequalia secari, scilicet tam area trilinei efF quam EeF erit semissis areae parallelogrammi $EefF$.

Fig. 8. §. 19. Extrui nunc concipiatur super eodem arcu EF , tanquam basi, aliud huiusmodi parallelogrammum sphaericum $EF\zeta\epsilon$, atque facile intelligitur, areas horum duorum parallelogrammorum $EFfe$ et $EF\zeta\epsilon$ esse inter se aequales. Hic enim prorsus eodem modo, vti in plano, ambo trilinea Eee et $Ff\zeta$ inter se perfecte sunt aequalia, a quibus si trilineum commune $oe\zeta$ auferatur, quadrilinea
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residua $Eo\zeta\epsilon$ et $Foef$ erunt inter se aequalia; quibus si addatur trilineum EoF , ambo parallelogramma integra erunt etiam aequalia; sicque etiam euistum est, omnia parallelogramma sphaerica, inter binos circulos parallelos et aequales, super eadem basi EF exstructa, esse inter se aequalia.

§. 20. Cum igitur talia parallelogramma sphaerica a diagonalibus in duas partes aequales diuidantur, etiam omnia trilinea super eadem basi EF exstructa, et in altero parallelo mn terminata, areas habebunt inter se aequales; in hac scilicet figura quatuor habebuntur trilinea inter se aequalia, scilicet: 1. EfF ; 2. EeF ; 3. $E\zeta F$; 4. $E\epsilon F$.

§. 21. Haec autem trilinea ideo non vocamus triangula, quia eorum basis EF non est arcus circuli maximi, quemadmodum in triangulis sphaericis statui solet. Facile autem haec trilinea in triangula sphaerica conuertuntur, si ab E ad F ducatur arcus circuli maximi $E\alpha F$, quo praedictis trilineis, idem augmentum $EF\alpha E$ accedit, ita vt nunc etiam omnia triangula sphaerica super eadem basi $E\alpha F$ exstructa, quorum vertices in alterum parallelum mn incidunt areas habeant aequales, si modo termini baseos E et F in altero parallelo illi opposito MN fuerint assumti; sicque iam clare euistum est, si super basi quacunq; innumera constituantur triangula sphaerica, quorum areae sint inter se aequales, eorum vertices semper fitos esse in circulo quodam sphaerae minore. Hoc obseruato problema clarissimi Professoris Lexell sequenti modo facillime resolui poterit.

Problema.

Tab. I. *In superficie sphaerica super data basi EF omnia tri-*
 Fig. 9. *angula sphaerica exstruere, quorum area fit data = Δ, ubi*
quidem Δ designat arcum circuli maximi, qui per radium
sphaerae multiplicatus producat aream praescriptam.

Solutio.

§. 22. Sit igitur EF ipsa basis proposita = a , et totum negotium huc redit, vt inueniantur poli P et p , qui quaesito satisfaciant; his enim inuentis si ex polo p interuallo $pe = PE$ describatur circulus minor ef , omnium triangulorum super basi EF exstruendorum, et in circulo minori ef terminatorum areae erunt inter se aequales; tantumque superest, vt ex area proposita Δ positio polorum P et p determinetur.

§. 23. Cum igitur EF fit arcus circuli maximi = a , ponatur $EP = FP = x$ et angulus $EPF = \omega$, eritque ex Trigonometria sphaerica $\cos. \omega = \frac{\cos. a - \cos. x^2}{\sin. x^2}$, ideoque

$$1 - \cos. \omega = \frac{1 - \cos. a}{\sin. x^2} = 2 \sin. \frac{1}{2} \omega^2.$$

Quare cum fit $1 - \cos. a = 2 \sin. \frac{1}{2} a^2$, erit $\sin. \frac{1}{2} \omega = \frac{\sin. \frac{1}{2} a}{\sin. x}$,

hincque vicissim $\sin. x = \frac{\sin. \frac{1}{2} \omega}{\sin. \frac{1}{2} a}$. Ex cognito autem angulo

ω innotescit tota area segmenti sphaerici inter binos semicirculos PEp et PFp , quippe quae erit = 2ω . Scilicet
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arcus circuli maximi $= 2\omega$ per radium sphaerae $= 1$ multiplicatus dabit aream huius segmenti.

§. 24. Quaeramus nunc etiam aream trianguli EPF, quem in finem vocetur angulus $PEF = PFE = \Phi$, ita ut summa trium angulorum huius trianguli sit $= \omega + 2\Phi$, unde area huius trianguli erit $= \omega + 2\Phi - \pi$. Quare si etiam ab e ad f arcus circuli maximi $e\omega f$ ducatur, erit quoque area trianguli sphaerici $pe\omega f = \omega + 2\Phi - \pi$. Hinc ergo area quadrilateri sphaerici $EFfe$ inter arcus circulo-
rum maximorum Ee ; Ff ; EF et $e\omega f$ comprehensi erit $2\omega - 2(\omega + 2\Phi - \pi) = 2\pi - 4\Phi$, cuius semiffis manifesto praebet aream trianguli sphaerici EFe .

§. 25. Cum igitur punctum e sit etiam in circulo minori ef , erit triangulum EeF vnum ex illis triangulis infinitis, quae super basi EF exstruere oportet, cuius area debet esse $= \Delta$, sicque adepti sumus hanc aequationem $\Delta = \pi - 2\Phi$, unde colligimus angulum $\Phi = \frac{1}{2}\pi - \frac{1}{2}\Delta$. Cum igitur angulus Δ detur, super basi data EF exstruantur utrinque anguli aequales $FEP = EFP = 90 - \frac{1}{2}\Delta$. Sicque innotescet polus P , ideoque et ei oppositus p , ex quo si interuallo $pe = PE$ describatur circulus minor ef , omnia triangula super basi EF exstruenda et in peripheria circuli minoris ef terminata habebunt ipsam aream propositam $= \Delta$.

§. 26. Quo haec constructio facilius reddatur, ex polo P in medium basis Π ducatur arcus normalis $P\Pi$, et quia in triangulo $EP\Pi$ habetur latus $E\Pi = \frac{1}{2}a$, cum
H 3
angulo

angulo $\text{PE}\Pi = 90 - \frac{1}{2}\Delta$, hinc colligitur latus $\text{EP} = x$,
 cuius tangens est $\text{tang. } x = \frac{\text{tang. } \frac{1}{2}a}{\text{fin. } \frac{1}{2}\Delta}$. Nunc igitur inuen-
 ta quantitate arcuum EP et FP eorum intersecio dabit
 polum P , ex cuius opposito p circulus minor interuallo
 $pe = x$ descriptus praebet loca verticum omnium triangu-
 lorum super basi EF describendorum, quae constructio egre-
 gie conuenit cum ea, quam clarissimus Lexell inuenit.



VTRVM

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