INVESTIGATIO QVARVNDAM SERIERVM, QVAE AD RATIONEM PERIPHERIAE CIRCVLI AD DIAMETRVM VERO PROXIME DEFINIENDAM MAXIME SVNT ACCOMMODATAE.

Auctore

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§. 1.

Qui post *Ludolphum a Ceulen* veram rationem peripheriae ad diametrum proxime assignare susceperunt, usi sunt serie *Leibnitiana*, qua pro circulo, cuius radius $= 1$, arcus quicunque s per suam tangentem t ita exprimi solet, ut sit

$$s = t - \tfrac{1}{3}t^3 + \tfrac{1}{5}t^5 - \tfrac{1}{7}t^7 + \text{etc.},$$

quae eo magis convergit, quo minor tangens t accipiatur. Sed quia arcus s ad totam peripheriam, vel ad arcum quadrantis cognitam rationem tenere debet, pro arcu s vix minorem valorem assumere licet, quam 30 graduum quippe cuius tangens est $\frac{1}{\sqrt{3}}$, quo valore in serie substituto, si semiperipheria circuli per π designetur, erit $\pi = 6s$, unde deducitur haec series:

$$\pi = \sqrt{12} \times \left(1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \frac{1}{9.3^4} - \text{etc.}\right)$$

Hinc

Hinc patet, calculum huius seriei ante institui non posse, quam radix quadrata ex numero 12 ad tot figuras decimales fuerit extracta, ad quot valor ipsius π desideratur, quem stupendum laborem olim *Abrahamus Sharp* usque ad 72 figuras decimales; tum vero Professor Greshamiensis *Machin* ad 100 figuras est exsecutus. Multo maiorem autem laborem sollertissimus calculator Gallus *de Lagny* est exantlare coactus, qui ex eadem serie valorem ipsius π adeo usque ad 128 figuras decimales determinavit, qui labor certe plus quam Herculeus est censendus, cum tamen extractio radicis ex numero 12 tantum tanquam opus praeliminare sit spectandum, istam enim immensam fractionem decimalem demum opus erat continuo per 3 dividere, quo facto insuper singuli termini per numeros impares 3, 5, 7, 9, 11, etc. ordine dividi debebant. Cum igitur istius seriei quilibet terminus in hac forma contineatur: $\frac{+\sqrt{12}}{(2n+1)\,3^n}$, ubi n denotat numerum terminorum, tot terminos computari oportet, donec fiat $\frac{(2n+1)\,3^n}{\sqrt{12}} = 10^{128}$, sive, logarithmis vulgaribus sumendis, donec fiat $l\,(2n+1) + n\,l\,3 - \frac{1}{2}\,l\,12 = 128$; unde primam partem $l\,(2n+1)$ negligendo colligitur $n = \frac{128 + \frac{1}{2}\,l\,12}{l\,3}$, hincque prodit terminorum numerus aliquanto minor quam 269; ex quo utique maxime est mirandum, quemquam fuisse repertum, qui hunc stupendum laborem exsequi sit ausus.

§. 2. Iam dudum autem proposui methodum istum laborem plurimum sublevandi. Postquam scilicet ostendi, duos

duos arcus satis exiguos in hunc usum adhiberi posse, quorum quidem neuter ad peripheriam teneat rationem rationalem, quorum tamen summa talem rationem teneat. Tales arcus sunt: $A\,\text{tang.}\,\frac{1}{2} + A\,\text{tang.}\,\frac{1}{3} = A\,\text{tang.}\,1 = \frac{\pi}{4}$, ita ut $\pi = 4\,A\,\text{tang.}\,\frac{1}{2} + 4\,A\,\text{tang.}\,\frac{1}{3}$, quorum uterque per nostram seriem facile evolvitur, cum sit:

$$A\,\text{tang.}\,\tfrac{1}{2} = \tfrac{1}{2} - \tfrac{1}{3\cdot 2^3} + \tfrac{1}{5\cdot 2^5} - \tfrac{1}{7\cdot 2^7} + \text{etc. et}$$

$$A\,\text{tang.}\,\tfrac{1}{3} = \tfrac{1}{3} - \tfrac{1}{3\cdot 3^3} + \tfrac{1}{5\cdot 3^5} - \tfrac{1}{7\cdot 3^7} + \text{etc.}$$

ubi termini illius seriei fere in ratione quadrupla decrescunt, huius vero in ratione fere noncupla, ideoque multo magis convergunt, quam series ab Auctoribus memoratis usurpata. Praecipue vero notandum est hoc modo nullam extractionem radicis requiri, sicque fere maximam partem illius laboris evitari; praeterea etiam singuli termini harum novarum seriérum facillime in fractiones decimales convertuntur, quae, quia figurae certum ordinem, imprimis ab initio, servant, computus ad quotcunque figuras sine magno labore extenditur.

§. 3. Multo magis autem labor diminuetur, si adhuc minores arcus in subsidium vocentur. Cum enim sit

$$A\,\text{tang.}\,\tfrac{1}{2} = A\,\text{tang.}\,\tfrac{1}{3} + A\,\text{tang.}\,\tfrac{1}{7},$$

erit nunc

$$\pi = 8\,A\,\text{tang.}\,\tfrac{1}{3} + 4\,A\,\text{tang.}\,\tfrac{1}{7},$$

sicque in serie priore termini statim in ratione noncupla decrescunt, in posteriore vero adeo 49 vicibus evadunt minores. Vnicum autem, quod hic desiderari posset, in hoc consistit, quod non tam facile per 49 continua divisio instituatur, optandumque fuisset, ut ista divisio vel per potestatem denarii vel alius numeri simplicem ad 10 rationem tenentis, expediri posset.

§. 4.

§. 4. Incidi autem nuper in modum prorſus ſingularem, quo huic incommodo feliciſſimo ſucceſſu occurritur atque adeo ſeries praecedentes magis convergentes redduntur. Conſtat autem iſte modus in idonea transformatione ſeriei *Leibnitianae*, quae per ſequentes operationes procedit:

$$s = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} + \text{etc.}$$

$$stt = t^3 - \frac{t^5}{3} + \frac{t^7}{5} - \frac{t^9}{7} + \text{etc.}$$

ergo $s + stt = t + \frac{2}{3}t^3 - \frac{2}{3.5}t^5 + \frac{2}{5.7}t^7 - \text{etc.} = t + s'tt$

ergo $s' = \frac{2}{3}t - \frac{2}{3.5}t^3 + \frac{2}{5.7}t^5 - \frac{2}{7.9}t^7 + \text{etc.}$

hinc $s'tt = \frac{2}{1.3}t^3 - \frac{2}{3.5}t^5 + \frac{2}{5.7}t^7 - \text{etc.}$

$$s'(1+tt) = \frac{2}{3}t + \frac{2.4}{3.5}t^3 - \frac{2.4}{3.5.7}t^5 + \frac{2.4}{5.7.9}t^7 - \text{etc.} = \frac{2}{3}t + s''tt$$

ergo $s'' = \frac{2.4}{1.3.5}t - \frac{2.4}{3.5.7}t^3 + \frac{2.4}{5.7.9}t^5 - \text{etc.}$

$$s''tt = + \frac{2.4}{1.3.5}t^3 - \frac{2.4}{3.5.7}t^5 + \text{etc.}$$

$$s''(1+tt) = \frac{2.4}{3.5}t + \frac{2.4.6}{1.3.5.7}t^3 - \frac{2.4.6}{3.5.7.9}t^5 + \text{etc.} = \frac{2.4}{3.5}t + s'''tt$$

$$s''' = \frac{2.4.6}{3.5.7}t - \frac{2.4.6}{3.5.7.9}t^3 + \frac{2.4.6}{5.7.9.11}t^5 - \text{etc.}$$

$$s'''tt = + \frac{2.4.6}{1.3.5.7}t^3 - \frac{2.4.6}{3.5.7.9}t^5 + \text{etc.}$$

$$s'''(1+tt) = \frac{2.4.6}{3.5.7}t + \frac{2.4.6.8}{1.3.5.7.9}t^3 - \frac{2.4.6.8}{3.5.7.9.11}t^5 + \text{etc.}$$

etc. etc.

§. 5. Colligamus iam ſingulas ſubſtitutiones factas, quae ſunt:

$$s = \frac{t}{1+tt} + \frac{s'tt}{1+tt},$$

$$s' = \frac{2t}{3(1+tt)} + \frac{s''tt}{1+tt},$$

$$s'' = \frac{2 \cdot 4\, t}{3 \cdot 5 (1 + tt)} + \frac{s''' tt}{1 + tt},$$
$$s''' = \frac{2 \cdot 4 \cdot 6\, t}{3 \cdot 5 \cdot 7 (1 + tt)} + \frac{s'''' tt}{1 + tt},$$
etc.

Quod si iam valores posteriores in praecedentibus substituantur, pro arcu s sequens obtinebitur nova series:

$$s = \frac{t}{1+tt} + \frac{2}{3} \cdot \frac{t^3}{(1+tt)^2} + \frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{t^5}{(1+tt)^3} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} \cdot \frac{t^7}{(1+tt)^4} + \text{etc.}$$

quae ad sequentem formam commodiorem reducitur:

$$s = \frac{t}{1+tt}\left[1 + \frac{2}{3}\left(\frac{tt}{1+tt}\right) + \frac{2 \cdot 4}{3 \cdot 5}\left(\frac{tt}{1+tt}\right)^2 + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}\left(\frac{tt}{1+tt}\right)^3 + \text{etc.}\right]$$

ubi singuli termini adhuc facilius evolvuntur quam in serie praecedente, propterea quod ex quolibet termino sequens immediate determinari potest. Ita ex primo termino reperitur secundus, si ille per $\frac{2}{3}$ et per $\frac{tt}{1+tt}$ multiplicetur (Multiplicatio autem per $\frac{2}{3}$ fit, dum pars tertia subtrahitur). Secundus per $\frac{4}{5}\left(\frac{tt}{1+tt}\right)$ multiplicatus dat tertium; hic vero, per $\frac{6}{7}\left(\frac{tt}{1+tt}\right)$ multiplicatus, dat quartum, et ita porro. Facillime autem per fractiones $\frac{4}{5}, \frac{6}{7}, \frac{8}{9}$, etc. multiplicatur. Praeterea vero haud exiguum est lucrum, quod omnes termini sunt positivi, eorumque ergo sola additio arcum quaesitum s suppeditat.

§. 6. Ad hanc autem novam seriem primum methodo longe alia sum perductus, quam hic apposuisse operae erit pretium. Cum sit $s = \int \frac{\partial t}{1+tt}$, quaestionem hoc modo determinate sum contemplatus, ut scilicet quaereretur valor huius formulae integralis, si a termino $t = 0$ usque ad terminum $t = a$ extendatur, ita ut futurum sit $s = \text{A tang.}\, a$.

§. 7. Tum vero huius formulae denominatorem $1 + tt$ sub hac forma repraesento: $1 + aa - (aa - tt)$,

 hinc-

hincque porro ſub hac: $1 + aa(1 - \frac{aa-tt}{1+aa})$, quo facto fractio $\frac{1}{1+tt}$ evolvetur in hanc ſeriem:

$$\frac{1}{1+aa}[1 + \frac{aa-tt}{1+aa} + (\frac{aa-tt}{1+aa})^2 + (\frac{aa-tt}{1+aa})^3 + \text{etc.}]$$

ſicque erit

$$s = \frac{1}{1+aa}\int \partial t [1 + \frac{aa-tt}{1+aa} + (\frac{aa-tt}{1+aa})^2 + \text{etc.}],$$

poſtquam ſcilicet integratio a $t = 0$ usque ad $t = a$ fuerit extenſa; unde ſtatim patet, pro primo termino fore $\int \partial t = a$; pro ſecundo autem $\int \partial t (aa - tt) = \frac{2}{3} a^3$.

§. 8. At vero, quo facilius omnes termini ſequentes integrentur, ſequentem aequationem evolvi conveniet:

$$\int \partial t (aa-tt)^{n+1} = A \int \partial t (aa-tt)^n + Bt(aa-tt)^{n+1},$$

quae differentiata ac per $\partial t (aa - tt)^n$ diviſa praebet:

$$aa - tt = A + B(aa - tt) - 2(n+1)Btt,$$

ubi duplicis generis termini occurrunt, ſcilicet vel mere conſtantes, vel quadrato tt affecti, qui ſeorſim ſe mutuo tollere debent.

§. 9. Quoniam autem huius aequationis membrum primum et tertium continet factorem $aa - tt$, neceſſe eſt ut ſecundum cum quarto eundem factorem involvat, quod evenit, ſtatuendo $A = 2(n+1)Baa$, quo facto, ſi aequatio inſuper per $aa - tt$ dividatur, prodibit $1 = B(2n+3)$; unde colligitur: $B = \frac{1}{2n+3}$ hincque $A = \frac{2(n+1)}{2n+3} aa$, ſicque aequatio noſtra aſſumta iam erit:

$$\int \partial t (aa-tt)^{n+1} = \frac{2(n+1)}{2n+3} aa \int \partial t (aa-tt)^n + \frac{t}{2n+3}(aa-tt)^{n+1}.$$

Qua-

Quare ſi integralia a $t=0$ usque ad $t=a$ extendantur, poſtremum membrum ſponte abit in nihilum, ſicque habebimus hanc reductionem generalem:

$$\int \partial t\,(aa-tt)^{n+1} = \frac{2(n+1)aa}{2n+3}\int \partial t\,(aa-tt)^{n}.$$

§. 10. Iam ope huius reductionis ex quolibet termino noſtrae ſeriei facillime terminus ſequens aſſignari poterit. Quod ſi enim loco exponentis n ſucceſſive omnes valores 0, 1, 2, 3, 4, 5, etc. ponamus, ſequentia integralia nanciſcemur:

$$\int \partial t\,(aa-tt) = \tfrac{2}{3}a^3,$$

$$\int \partial t\,(aa-tt)^2 = \tfrac{2.4}{3.5}a^5,$$

$$\int \partial t\,(aa-tt)^3 = \tfrac{2.4.6}{3.5.7}a^7,$$

$$\int \partial t\,(aa-tt)^4 = \tfrac{2.4.6.8}{3.5.7.9}a^9,$$

etc.

§. 11. Quod ſi iam ſinguli hi valores in noſtra ſerie ſubſtituantur, integrale, quod quaerimus, ſequenti modo exprimetur:

$$s = \text{A tag.}\,a = \frac{1}{1+aa}\left(a + \frac{\frac{2}{3}a^3}{1+aa} + \frac{\frac{2.4}{3.5}a^5}{(1+aa)^2} + \frac{\frac{2.4.6}{3.5.7}a^7}{(1+aa)^3} + \text{etc.}\right)$$

unde, ſi loco a reſtituamus t, orietur ipſa ſeries methodo praecedente inventa, ſcilicet:

$$s = \text{A tag.}\,t = \frac{t}{1+tt}\left[1 + \tfrac{2}{3}\left(\tfrac{tt}{1+tt}\right) + \tfrac{2.4}{3.5}\left(\tfrac{tt}{1+tt}\right)^2 + \tfrac{2.4.6}{3.5.7}\left(\tfrac{tt}{1+tt}\right)^3 + \text{etc.}\right].$$

§. 12. Nunc igitur hanc novam ſeriem ad noſtrum inſtitutum propius accommodemus, et quoniam ſupra primo hanc habuimus aequationem: $\pi = 4\,\text{A tang.}\,\frac{1}{2} + 4\,\text{A tang.}\,\frac{1}{3}$, pro

pro priore parte, ubi $t = \frac{1}{2}$, obtinebimus hanc seriem:

$$A\ \text{tang.}\ \tfrac{1}{2} = \tfrac{2}{5}\left(1 + \tfrac{2}{3}\cdot\tfrac{1}{5} + \tfrac{2\cdot 4}{3\cdot 5}\cdot\tfrac{1}{5^2} + \tfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\tfrac{1}{5^3} + \text{etc.}\right)$$

pro altera autem parte, ubi $t = \frac{1}{3}$, erit

$$A\ \text{tang.}\ \tfrac{1}{3} = \tfrac{3}{10}\left(1 + \tfrac{2}{3}\cdot\tfrac{1}{10} + \tfrac{2\cdot 4}{3\cdot 5}\cdot\tfrac{1}{10^2} + \tfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\tfrac{1}{10^3} + \text{etc.}\right)$$

consequenter valor ipsius π per binas sequentes series exprimetur:

$$\pi = \left\{\begin{array}{l} + \tfrac{16}{10}\left(1 + \tfrac{2}{3}\left(\tfrac{2}{10}\right) + \tfrac{2\cdot 4}{3\cdot 5}\left(\tfrac{2}{10}\right)^2 + \tfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\tfrac{2}{10}\right)^3 + \text{etc.}\right) \\ + \tfrac{12}{10}\left(1 + \tfrac{2}{3}\left(\tfrac{1}{10}\right) + \tfrac{2\cdot 4}{3\cdot 5}\left(\tfrac{1}{10}\right)^2 + \tfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\tfrac{1}{10}\right)^3 + \text{etc.}\right) \end{array}\right\},$$

quae duae series manifesto multo minore labore per numeros evolvuntur, quam eae, quas supra dedimus, propterea quod hic in factoribus habemus ipsum denarium, atque hae series adeo magis convergunt.

§. 13. Lucrum autem adhuc multo erit maius, si forma $\pi = 8\ A\ \text{tang.}\ \frac{1}{3} + 4\ A\ \text{tang.}\ \frac{1}{7}$ per novam seriem evolvatur, cuius pars prior iam est evoluta; pro altera autem, ubi $t = \frac{1}{7}$, nunc habebimus:

$$A\ \text{tang.}\ \tfrac{1}{7} = \tfrac{7}{50}\left(1 + \tfrac{2}{3}\cdot\tfrac{1}{50} + \tfrac{2\cdot 4}{3\cdot 5}\cdot\tfrac{1}{50^2} + \tfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\tfrac{1}{50^3} + \text{etc.}\right).$$

Hinc igitur nanciscemur sequentes series pro valore semiperipheriae π indagando:

$$\pi = \left\{\begin{array}{l} + \tfrac{24}{10}\left(1 + \tfrac{2}{3}\left(\tfrac{1}{10}\right) + \tfrac{2\cdot 4}{3\cdot 5}\left(\tfrac{1}{10}\right)^2 + \tfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\tfrac{1}{10}\right)^3 + \text{etc.}\right) \\ + \tfrac{28}{50}\left(1 + \tfrac{2}{3}\left(\tfrac{2}{100}\right) + \tfrac{2\cdot 4}{3\cdot 5}\left(\tfrac{2}{100}\right)^2 + \tfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\tfrac{2}{100}\right)^3 + \text{etc.}\right) \end{array}\right\};$$

haeque duae series sunt aptissimae ad valorem ipsius π ad quotcunque figuras decimales exprimendum, propterea quod singuli termini ex praecedentibus facillime formantur atque adeo prioris seriei termini iam in ratione decupla, posterioris vero in quinquies decupla decrescunt. Vnde si quis hunc va-

lorem

lorem ad 128 figures definire vellet, pro priori ferie computare deberet terminos centum viginti octo, pofterioris vero feptuaginta quinque tantum.

§. 14. Quo ufus harum ferierum clarius appareat, utriusque feriei octo terminos priores in fractiones decimales evolvamus, eritque

Pro parte priore.

term.	I.	= 2,4 etc.
—	II.	= - 16 etc.
—	III.	= - - 128 etc.
—	IV.	= - - 109,714285,714285,714285,7142 etc.
—	V.	= - - - - 97,523809,523809,523809,523 etc.
—	VI.	= - - - - - 8,865800,865800,865800,865 etc.
—	VII.	= - - - - - - 81,838161,838161,838161,8 etc.
—	VIII.	= - - - - - - - 76,382284,382284,382284, etc.
Pars.	I.	= 2,574004427 | 231435 | 231435 | 231435 etc.

Pro parte pofteriore.

term.	I.	= 0,560 etc.
—	II.	= - - 7466666666666666666666666 etc.
—	III.	= - - - - 119466666666666666666666 etc.
—	IV.	= - - - - 2048000000000000000000 etc.
—	V.	= - - - - - - 3640888888888888888 etc.
—	VI.	= - - - - - - - 661979797979797979 etc.
—	VII.	- - - - - - - - - - 12221165501165501 etc.
—	VIII.	= - - - - - - - - - 228128422688422 etc.
Pars	II.	= 0,5675882184166513 1 | 412587 | 4125 etc.

Hinc

Hinc patet iſtas ſummas octo priorum terminorum, ob revolutiones periodicas in figuris occurrentes, ſine ullo labore ad quotcunque figuras continuari poſſe.

§. 15. Ex hoc ſchemate iam ſtatim verus valor ipſius π ad octo figuras usque aſſignari poterit. Cum enim octo priorum terminorum ſumma ſit

Partis prioris $= 2,57400443$
Partis poſterioris $= 0,56758822$

erit valor ipſius $\pi = 3,14159265$

ubi ne in ultima quidem figura erratur. Facile autem iſte calculus ad plures figuras extendi poteſt, propterea quod termini octavum ſubſequentes ex eo ipſo ſine difficultate computantur. Eſt enim

Pro parte priore.

$$\text{terminus IX.} = \tfrac{1}{10}\left(1 - \tfrac{1}{17}\right) \text{VIII.}$$
$$\text{——— X.} = \tfrac{1}{10}\left(1 - \tfrac{1}{19}\right) \text{IX.}$$
$$\text{——— XI.} = \tfrac{1}{10}\left(1 - \tfrac{1}{21}\right) \text{X.}$$
etc.

Pro parte poſteriore.

$$\text{terminus IX.} = \tfrac{2}{100}\left(1 - \tfrac{1}{17}\right) \text{VIII.}$$
$$\text{——— X.} = \tfrac{2}{100}\left(1 - \tfrac{1}{19}\right) \text{IX.}$$
$$\text{——— XI.} = \tfrac{2}{100}\left(1 - \tfrac{1}{21}\right) \text{X.}$$
etc.

§. 16. Quo uſus harum formularum magis elucescat, quaeramus valorem ipſius π usque ad 16 figuras, et calculus erit:

Pro

Pro parte priore.

I. - - -	VIII.	=	2,57400442723143523
term.	IX.	=	- - - - 718892088
—	X.	=	- - - - 68105566
—	XI.	=	- - - - - 6486244
—	XII.	=	- - - - - 620423
—	XIII.	=	- - - - - - 59561
—	XIV.	=	- - - - - - - 5735
—	XV.	=	- - - - - - - 554
—	XVI.	=	- - - - - - - - 54
Summa		=	2,57400443517313748.

Pro parte posteriore.

I. - - -	VIII.	=	0,56758821841665131
term.	IX.	=	- - - - - - - 429
—	X.	=	- - - - - - - - 8
Pars II.		=	0,56758821841665567
Pars I.		=	2,57400443517313748
hinc π		=	3,14159265358979315

§. 17. Possunt vero etiam aliae huiusmodi formulae pro π inveniri, quae adhuc magis convergant ac pariter per potestates denarii procedant. Cum enim in genere sit

$$\text{A tang.} \frac{\alpha}{a} = \text{A tang.} \frac{\beta}{b} + \text{A tang.} \frac{\alpha b - \beta a}{\alpha\beta + ab},$$

si sumamus $t = \frac{\alpha}{a}$, vel $\frac{\beta}{b}$, erit $\frac{tt}{1+tt} = \frac{\alpha\alpha}{\alpha\alpha + aa}$ vel $\frac{\beta\beta}{\beta\beta + bb}$; sumto vero $t = \frac{\alpha b - \beta a}{\alpha\beta + ab}$ fiet $\frac{tt}{1+tt} = \frac{(\alpha b - \beta a)^2}{(\alpha\alpha + aa)(\beta\beta + bb)}$. Vnde patet, si priores denominatores $\alpha\alpha + aa$ et $\beta\beta + bb$ fuerint potestates denarii, vel eo saltem reduci queant, quod eve-

evenit, quando alios factores non involvunt praeter 2 et 5, tum etiam tertium denominatorem certe ad potestatem denarii reduci posse.

§. 18. Quoniam igitur habuimus hanc formulam:

$$\pi = 8 \text{ A tang. } \tfrac{1}{3} + 4 \text{ A tang. } \tfrac{1}{7},$$

loco prioris arcus ope reductionis allatae duos alios introducamus, ponendo scilicet $\frac{\alpha}{a} = \frac{1}{3}$; et pro $\frac{\beta}{b}$ sumamus $\frac{1}{7}$, fietque tertius arcus $=$ A tang. $\frac{2}{11}$, ita ut fit

$$\text{A tang. } \tfrac{1}{3} = \text{A tang. } \tfrac{1}{7} + \text{A tang. } \tfrac{2}{11},$$

quo valore substituto formula nostra erit

$$\pi = 12 \text{ A tang. } \tfrac{1}{7} + 8 \text{ A tang. } \tfrac{2}{11},$$

cuius arcum priorem iam ante evolvimus. At vero ob $\frac{tt}{1+tt} = \frac{4}{125} = \frac{32}{1000}$ pro altero habebimus:

$$\text{A tang. } \tfrac{2}{11} = \tfrac{22}{125} \left[1 + \tfrac{2}{3}\left(\tfrac{32}{1000}\right) + \tfrac{2 \cdot 4}{3 \cdot 5}\left(\tfrac{32}{1000}\right)^2 + \tfrac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}\left(\tfrac{32}{1000}\right)^3 + \text{etc.}\right]$$

Verum hic continua multiplicatio per numerum 32 non satis ad calculum est idonea, praecipue autem haec series minus convergit quam quae ex $\frac{1}{7}$ est deducta.

§. 19. Hanc ob caussam penitus reiiciamus istum arcum, eiusque loco ope reductionis supra datae substituamus duos novos arcus, quorum alter fit $\frac{1}{7}$, statuendo $\frac{\alpha}{a} = \frac{2}{11}$ et $\frac{\beta}{b} = \frac{1}{7}$, hincque fiet $\frac{\alpha b - \beta a}{\alpha\beta + a b} = \frac{3}{79}$, ita ut fit

$$\text{A tang. } \tfrac{2}{11} = \text{A tang. } \tfrac{1}{7} + \text{A tang. } \tfrac{3}{79},$$

hincque

$$\pi = 20 \text{ A tang. } \tfrac{1}{7} + 8 \text{ A tang. } \tfrac{3}{79}.$$

Vbi

Vbi notetur, posito $t = \frac{3}{79}$ fore

$$\frac{tt}{1+tt} = \frac{9}{6250} = \frac{144}{100000},$$

quae fractio propemodum est $\frac{1}{700}$; unde patet, hanc seriem:

$$\text{A tang.}\ \tfrac{3}{79} = \tfrac{237}{6250}\left[1 + \tfrac{2}{3}\left(\tfrac{144}{100000}\right) + \tfrac{2 \cdot 4}{3 \cdot 5}\left(\tfrac{144}{100000}\right)^2 + \text{etc}\right]$$

maxime convergere eiusque terminos propemodum septingenties fieri minores.

§. 20. Ista igitur series maxime est notatu digna, propter insignem convergentiam, atque adeo plurimum operae pretium erit multiplicatione per 144 non deterreri, quippe quae, bis per 12 multiplicando, facile absolvi potest. Per 12 autem multiplicare vix difficilius est quam per 2. Evolvamus igitur ambos istos arcus per nostram novam seriem, atque impetrabimus sequentem formam:

$$\pi = \begin{cases} + \frac{28}{10}\left[1 + \frac{2}{3}\left(\frac{2}{100}\right) + \frac{2 \cdot 4}{3 \cdot 5}\left(\frac{2}{100}\right)^2 + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}\left(\frac{2}{100}\right)^3 + \text{etc.}\right] \\ + \frac{30336}{100000}\left[1 + \frac{2}{3}\left(\frac{144}{100000}\right) + \frac{2 \cdot 4}{3 \cdot 5}\left(\frac{144}{100000}\right)^2 + \text{etc}\right] \end{cases}$$

Hic igitur coëfficiens prioris seriei quinquies maior est quam supra, unde etiam singuli termini ibi exhibiti toties maiores sunt capiendi, unde summa octo priorum terminorum erit:

2, 837941092083 25 65 | 706293 | 706293 | 706 etc.

octavus autem terminus:

0, 000000000000114064 | 211344 | 211344 | 211 etc.

ex quo iam sequentes termini facile colliguntur.

§. 21. Quo autem pro altera serie calculus commodius institui possit, primo conveniet divisiones per 100000 prorsus praetermitti, ita ut ex quolibet termino sequens obtineatur, dum ille bis per 12 multiplicetur et a produ debita pars subtrahatur, nullo respectu habito ad loca phrarum decimalium; quandoquidem ex hoc capite aber ri nequit, dum satis constat quoties quilibet terminus i nor est praecedente. Talem calculum pro sex prioribus t minis hic exhibeamus:

term. I. =	0,30336
	364032
3.)	4368384
	1456128
term. II. =	2912256
	34947072
5.)	419364864
	838729728
term. III. =	3354918912
	40259026944
7.)	483108323328
	69015474761, 142857, 142857, 142 etc.
term. IV. =	414092848566, 857142, 857142, 857 etc.
	4969114182802, 285714, 285714, 285 etc.
9.)	59629370193627, 428571, 428571, 428 e
	6625485577069, 714285, 714285, 714 e
term. V. =	53003884616557, 714285, 714285, 714 e
	636046615398692, 571428, 571428, 571 et

11.) $= 763255938478431\text{o}, 857142, 857142, 857$ etc.
$= 69386903498039\text{1}, 896103, 896103, 8961$ etc.

term. VI. $= 69386903498039\text{1}, 896103, 896103, 8961$ etc.

unde ipſos terminos deſumamus et in unam ſummam colligamus:

term. I. $= 0,30336$
— II. $= \quad 2912256$
— III. $= \quad 33549189 12$
— IV. $= \quad 414092848566, 857142, 857142$
— V. $= \quad 5300388461 6557, 714285, 7$
— VI. $= \quad 693869034980391 8, 96$

Summa $= 0,3036515615065147812820577003918,961038,$
$961038, 961038$ etc.

ubi imprimis notatu dignum occurrit, quod ſumma quinque priorum terminorum abſolute exhiberi poteſt, dum ſcilicet fractio decimalis in figura 26^{ma} abrumpitur, haecque poſtrema formula pro π data ad calculum maxime videtur accommodata.

§. 22. Ex eodem principio, unde noſtram ſeriem deduximus, aliae ſimiles ſeries derivari poſſunt pariter maxime convergentes. Inchoando ſcilicet a ſerie vulgari:

$$\text{A tang. } t = t - \tfrac{1}{3}t^3 + \tfrac{1}{5}t^5 - \tfrac{1}{7}t^7 + \text{ etc.}$$

ponamus huius ſeriei iam n terminos actu eſſe collectos, quorum ſumma ſit

$$\Sigma = 1 - \tfrac{1}{3}t^3 + \tfrac{1}{5}t^5 - \ldots\ldots \pm \frac{t^{2n-1}}{2n-1}.$$

Sum-

Summam autem ſequentium terminorum ſtatuamus:

$$s = \frac{t^{2n+1}}{2n+1} - \frac{t^{2n+3}}{2n+3} + \frac{t^{2n+5}}{2n+5} - \text{etc.}$$

ita ut ſit A tang. $t = \Sigma \pm s$, ubi ergo numerus Σ tanquam iam inventus ſpectatur, alter vero s inveſtigari debeat.

§. 23. Ratiocinium igitur eodem modo inſtituamus, ut ſupra §. 4, quas operationes hic apponamus.

$$s = \frac{t^{2n+1}}{2n+1} - \frac{t^{2n+3}}{2n+3} + \frac{t^{2n+5}}{2n+5} - \text{etc.}$$

$$stt = \quad + \frac{t^{2n+3}}{2n+1} - \frac{t^{2n+5}}{2n+3} + \text{etc.}$$

$$s(1+tt) = \frac{t^{2n+1}}{2n+1} + \frac{2t^{2n+3}}{(2n+1)(2n+3)} - \frac{2t^{2n+5}}{(2n+3)(2n+5)} + \text{etc.}$$

$$= \frac{t^{2n+1}}{2n+1} + s'tt.$$

$$s(1+tt) = \frac{t^{2n+1}}{2n+1} + \frac{2t^{2n+3}}{(2n+1)(2n+3)} - \frac{2t^{2n+5}}{(2n+3)(2n+5)} + \text{etc.}$$

$$= \frac{t^{2n+1}}{2n+1} + s'tt, \text{ ergo}$$

$$s' = \frac{2t^{2n+1}}{(2n+1)(2n+3)} - \frac{2t^{2n+3}}{(2n+3)(2n+5)} + \text{etc.}$$

$$s'tt = \quad + \frac{2t^{2n+3}}{(2n+1)(2n+3)} - \text{etc.}$$

$$s'(1+tt) = \frac{2t^{2n+1}}{(2n+1)(2n+3)} + \frac{2 \cdot 4\, t^{2n+3}}{(2n+1)(2n+3)(2n+5)} - \text{etc.}$$

$$= \frac{2t^{2n+1}}{(2n+1)(2n+3)} + s''tt. \text{ etc.}$$

§. 24.

§. 24. Quod si iam valores introducti restituantur, facile patet tandem ad hanc seriem perventum iri:

$$s = \frac{t^{2n+1}}{(2n+1)(1+tt)} + \frac{2\,t^{2n+3}}{(2n+1)(2n+3)(1+tt)^2} + \frac{2\cdot 4\,t^{2n+5}}{(2n+1)(2n+3)(2n+5)(1+tt)^3} + \text{etc.}$$

quae expressio contrahitur in sequentem:

$$s = \frac{t^{2n+1}}{(2n+1)(1+tt)}\left(1 + \frac{2tt}{(2n+3)(1+tt)} + \frac{2\cdot 4\,t^4}{(2n+3)(2n+5)(1+tt)^2} + \text{etc.}\right)$$

haecque series utique aliquanto magis convergit quam praecedens, propterea quod denominatores multo maiores sunt quam numeratores; veruntamen formulae ante exhibitae his seriebus longissime anteferendae videntur, siquidem ad usum practicum respiciamus.

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