

DE NOVO GENERE
SERIERVM RATIONALIVM
ET VALDE CONVERGENTIVM
QVIBVS
RATIO PERIPHERIAE AD DIAMETRVM
EXPRIMI POTEST.

Auctore
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Conventui exhibita die 17 Junii 1779.

§. I.

Principium, unde hae series sunt deduciae, situm est in
hac formula binomiale: $4+x^4$, quam constat involvere
hos duos factores rationales: $2+2x+x^2$ et $2-2x+x^2$.
Hinc enim statim sequitur hanc formulam integralem:
 $\int \frac{dx}{4+x^4}$, quam signo \odot indicemus, reduci ad hanc:
 $\odot = \int \frac{dx}{2-2x+x^2}$, cuius integrale, ita sumum, ut evanes-
cat posito $x=0$, est A tang. $\frac{x}{2-x}$. Vbi observetur, casu
 $x=1$ esse $\odot=\frac{\pi}{4}$; at vero casu $x=\frac{1}{2}$ erit $\odot=A$ tang. $\frac{1}{3}$;
cum vero casu $x=\frac{1}{4}$ erit $\odot=A$ tang. $\frac{1}{7}$. Notum autem
est esse
 $2A$ tang. $\frac{1}{3} + A$ tang. $\frac{1}{7} = A$ tang. $1 = \frac{\pi}{4}$.

§. 2.

— 151 —

§. 2. Cum igitur formula integralis illa signo \textcircled{O} indicata tribus constet partibus, singulas seorsim evolvamus, quas brev. gr. sequentibus characteribus insigniamus:

$$\text{I. } \int \frac{\partial x}{4+x^4} = \mathfrak{H}; \text{ II. } \int \frac{x \partial x}{4+x^4} = 2; \text{ III. } \int \frac{x^2 x \partial x}{4+x^4} = \sigma;$$

ita ut sit

$$\textcircled{O} = 2\mathfrak{H} + 2 \cdot 2 + \sigma = A \tan \frac{\arctan x}{2}$$

Nunc igitur istas tres formulas integrales more solito in series infinitas evolvamus, inde formandas, quod fit

$$\frac{\text{I}}{4+x^4} = \frac{1}{4} \left(1 - \frac{x^4}{4} + \frac{x^8}{4^2} - \frac{x^{12}}{4^3} + \frac{x^{16}}{4^4} - \text{etc.} \right)$$

§. 3. Quod si iam primo istam seriem ducamus in ∂x et integreremus, prima formula \mathfrak{H} per frequentem seriem exprimetur:

$$\mathfrak{H} = \frac{x}{4} \left[1 - \frac{1}{5} \cdot \frac{x^4}{4} + \frac{1}{9} \left(\frac{x^4}{4} \right)^2 - \frac{1}{13} \left(\frac{x^4}{4} \right)^3 + \text{etc.} \right].$$

At vero illa series duxa in $x \partial x$ et integrata dabit

$$2 = \frac{x^2}{8} \left[1 - \frac{1}{3} \cdot \frac{x^4}{4} + \frac{1}{5} \left(\frac{x^4}{4} \right)^2 - \frac{1}{7} \left(\frac{x^4}{4} \right)^3 + \text{etc.} \right].$$

Denuo eadem series duxa in $x x \partial x$ et integrata praebet

$$\sigma = \frac{x^3}{4} \left[\frac{1}{3} - \frac{1}{7} \frac{x^4}{4} + \frac{1}{11} \left(\frac{x^4}{4} \right)^2 - \frac{1}{15} \left(\frac{x^4}{4} \right)^3 + \text{etc.} \right].$$

§. 4. Cum igitur sit $\textcircled{O} = 2\mathfrak{H} + 2 \cdot 2 + \sigma$, evolvamus seorsim casus initio memoratos, quibus est vel $x = 1$, vel $x = \frac{1}{2}$, vel $x = \frac{1}{4}$, quorum primo est $\frac{x^4}{4} = \frac{1}{4}$; secundo vero est $\frac{x^4}{4} = \frac{1}{64}$; tertio vero $\frac{x^4}{4} = \frac{1}{1024}$; unde patet binos casus posteriores maxime convergere; quin etiam ipsa prima, cuius termini in ratione quadruplica decrescent, iam magis convergit quam series Leibnitiana, summa parca cuius tangens est

est $\frac{1}{\sqrt{3}}$, praeterquam quod hic calculus nulla irrationalitate perturbatur.

Evolutio casus primi.

quo $x = 1$ et $\Theta = A \tan^{-1} x = \frac{\pi}{4}$.

§. 5. Cum igitur hic sit $\frac{x^4}{4} = \frac{1}{4}$, tres nostrae series principales \mathfrak{h} , \mathfrak{A} , \mathfrak{O} sequenti modo procedent:

$$\mathfrak{h} = \frac{1}{4} [1 - \frac{1}{5} \cdot \frac{1}{4} + \frac{1}{9} (\frac{1}{4})^2 - \frac{1}{13} (\frac{1}{4})^3 + \frac{1}{17} (\frac{1}{4})^4 - \text{etc.}]$$

$$\mathfrak{A} = \frac{1}{8} [1 - \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{5} (\frac{1}{4})^2 - \frac{1}{7} (\frac{1}{4})^3 + \frac{1}{9} (\frac{1}{4})^4 - \text{etc.}]$$

$$\mathfrak{O} = \frac{1}{4} [\frac{1}{3} - \frac{1}{7} \cdot \frac{1}{4} + \frac{1}{11} (\frac{1}{4})^2 - \frac{1}{15} (\frac{1}{4})^3 + \frac{1}{19} (\frac{1}{4})^4 - \text{etc.}]$$

§. 6. Cum igitur sit $\Theta = 2\mathfrak{h} + 2\mathfrak{A} + \mathfrak{O} = \frac{\pi}{4}$, per 4 multiplicando valor ipsius π per sequentes tres series ex-primitur

$$\pi = \begin{cases} 2(1 - \frac{1}{5} \cdot \frac{1}{4} + \frac{1}{9} \cdot \frac{1}{4^2} - \frac{1}{13} \cdot \frac{1}{4^3} + \frac{1}{17} \cdot \frac{1}{4^4} - \text{etc.}) \\ 1(1 - \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{1}{4^2} - \frac{1}{7} \cdot \frac{1}{4^3} + \frac{1}{9} \cdot \frac{1}{4^4} - \text{etc.}) \\ 1(\frac{1}{3} - \frac{1}{7} \cdot \frac{1}{4} + \frac{1}{11} \cdot \frac{1}{4^2} - \frac{1}{15} \cdot \frac{1}{4^3} + \frac{1}{19} \cdot \frac{1}{4^4} - \text{etc.}) \end{cases}$$

§. 7. Ex his certe ternis seriebus ratio peripheriae ad diametrum multo minore labore computari potuisset quam ex serie *Leibnitiana*, qua Autores illi meritissimi Sharp, Machin & Lagny sunt usi, quorum primus valorem ipsius π in fractione decimali usque ad 72 figuram, secundus ad 100, ac postremus adeo usque ad 128 determinavit. At vero sequentes casus multo magis istum labore sublevare possent.

Evo-

Evolutio casus secundi,

quo $x = \frac{1}{2}$.

§. 8. Hoc igitur casu erit $\frac{x^4}{4} = \frac{1}{64}$, unde tres illae series sequenti modo referentur:

$$\hat{b} = \frac{1}{3} \left(1 - \frac{1}{5} \cdot \frac{1}{64} + \frac{1}{9} \cdot \frac{1}{64^2} - \frac{1}{13} \cdot \frac{1}{64^3} + \text{etc.} \right)$$

$$2 = \frac{1}{32} \left(1 - \frac{1}{3} \cdot \frac{1}{64} + \frac{1}{5} \cdot \frac{1}{64^2} - \frac{1}{7} \cdot \frac{1}{64^3} + \text{etc.} \right)$$

$$\sigma = \frac{1}{32} \left(\frac{1}{3} - \frac{1}{7} \cdot \frac{1}{64} + \frac{1}{11} \cdot \frac{1}{64^2} - \frac{1}{15} \cdot \frac{1}{64^3} + \text{etc.} \right)$$

§. 9. Cum igitur $2\hat{b} + 2\hat{2} + \sigma = A \tan g. \frac{1}{3}$, erit

$$A \tan g. \frac{1}{3} = \left\{ \begin{array}{l} \frac{1}{4} \left(1 - \frac{1}{5} \cdot \frac{1}{64} + \frac{1}{9} \cdot \frac{1}{64^2} - \frac{1}{13} \cdot \frac{1}{64^3} + \text{etc.} \right) \\ \frac{1}{15} \left(1 - \frac{1}{3} \cdot \frac{1}{64} + \frac{1}{5} \cdot \frac{1}{64^2} - \frac{1}{7} \cdot \frac{1}{64^3} + \text{etc.} \right) \\ \frac{1}{32} \left(\frac{1}{3} - \frac{1}{7} \cdot \frac{1}{64} + \frac{1}{11} \cdot \frac{1}{64^2} - \frac{1}{15} \cdot \frac{1}{64^3} + \text{etc.} \right) \end{array} \right\}$$

Etsi autem hic tres computandae sunt series, tamen, quia singulae secundum eandem rationem $1 : 64$ decrescunt, laborem mirum in modum contrahere licebit.

Evolutio casus tertii,

quo $x = \frac{1}{4}$.

§. 10. Cum igitur hic sit $\frac{x^4}{4} = \frac{1}{1024}$, series nostrae tres principales ita se habebunt:

$$\hat{b} = \frac{1}{5} \left(1 - \frac{1}{5} \cdot \frac{1}{1024} + \frac{1}{9} \cdot \frac{1}{1024^2} - \frac{1}{13} \cdot \frac{1}{1024^3} + \text{etc.} \right)$$

$$2 = \frac{1}{1024} \left(1 - \frac{1}{3} \cdot \frac{1}{1024} + \frac{1}{5} \cdot \frac{1}{1024^2} - \frac{1}{7} \cdot \frac{1}{1024^3} + \text{etc.} \right)$$

$$\sigma = \frac{1}{256} \left(\frac{1}{3} - \frac{1}{7} \cdot \frac{1}{1024} + \frac{1}{11} \cdot \frac{1}{1024^2} - \frac{1}{15} \cdot \frac{1}{1024^3} + \text{etc.} \right)$$

§. 11. Cum igitur $2\frac{1}{3} + 2\frac{2}{7} + \sigma = A \tan\frac{\pi}{7}$, erit
his seriebus debite iunctis:

$$A \tan\frac{\pi}{7} = \left\{ \begin{array}{l} \frac{1}{8} \left(1 - \frac{1}{5} \cdot \frac{1}{1024} + \frac{1}{9} \cdot \frac{1}{1024^2} - \frac{1}{13} \cdot \frac{1}{1024^3} + \text{etc.} \right) \\ \frac{1}{64} \left(1 - \frac{1}{3} \cdot \frac{1}{1024} + \frac{1}{5} \cdot \frac{1}{1024^2} - \frac{1}{7} \cdot \frac{1}{1024^3} + \text{etc.} \right) \\ \frac{1}{256} \left(\frac{1}{3} - \frac{1}{7} \cdot \frac{1}{1024} + \frac{1}{11} \cdot \frac{1}{1024^2} - \frac{1}{15} \cdot \frac{1}{1024^3} + \text{etc.} \right) \end{array} \right\}$$

Applicatio binorum casuum posteriorum ad peripheriam circuli per series maxime convergentes exprimendam.

§. 12. Cum sit, uti iam observavimus,
 $\frac{\pi}{4} = 2A \tan\frac{1}{3} + A \tan\frac{1}{7}$ erit $\pi = 8A \tan\frac{1}{3} + 4A \tan\frac{1}{7}$
seriebus supra inventis substitutis valor ipsius π per sex sequentes series coniunctim exprimetur:

$$\pi = \left\{ \begin{array}{l} 2 \left(1 - \frac{1}{5} \cdot \frac{1}{64} + \frac{1}{9} \cdot \frac{1}{64^2} - \frac{1}{13} \cdot \frac{1}{64^3} + \text{etc.} \right) \\ \frac{1}{12} \left(1 - \frac{1}{3} \cdot \frac{1}{64} + \frac{1}{5} \cdot \frac{1}{64^2} - \frac{1}{7} \cdot \frac{1}{64^3} + \text{etc.} \right) \\ \frac{1}{4} \left(\frac{1}{3} - \frac{1}{7} \cdot \frac{1}{64} + \frac{1}{11} \cdot \frac{1}{64^2} - \frac{1}{15} \cdot \frac{1}{64^3} + \text{etc.} \right) \\ \frac{1}{2} \left(1 - \frac{1}{5} \cdot \frac{1}{1024} + \frac{1}{9} \cdot \frac{1}{1024^2} - \frac{1}{13} \cdot \frac{1}{1024^3} + \text{etc.} \right) \\ \frac{1}{16} \left(1 - \frac{1}{3} \cdot \frac{1}{1024} + \frac{1}{5} \cdot \frac{1}{1024^2} - \frac{1}{7} \cdot \frac{1}{1024^3} + \text{etc.} \right) \\ \frac{1}{64} \left(\frac{1}{3} - \frac{1}{7} \cdot \frac{1}{1024} + \frac{1}{11} \cdot \frac{1}{1024^2} - \frac{1}{15} \cdot \frac{1}{1024^3} + \text{etc.} \right) \end{array} \right\}$$

Hic ergo maxime notatu dignum occurrit, quod omnes istae series per solas potestates binarii procedant.