DE INSIGNI VSV
CALCVLI IMAGINARIORVM
IN CALCVLO INTEGRALI.

Auctore

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§. 1.

Cum nuper integrale formulae differentialis

$$\frac{\partial x(1+xx)^2}{(1-xx)\sqrt[4]{(1-6xx+x^4)}}$$

eruissem, quod, posito brevitatis gratia $\sqrt[4]{(1-6xx+x^4)}=v$, inveneram

$$=\frac{1}{2}l\frac{1+xx+vv-2vx}{1+xx+vv+2vx}-\text{Arc. tang.}\frac{2vx}{1+xx-vv},$$

affirmare non dubitavi hoc ipsum integrale non nisi ope Calculi Imaginariorum obtineri posse. Tractaveram enim ante istam formulam differentialem:

$$\frac{\partial y(1-yy)^2}{(1+yy)\sqrt[4]{(1+6yy+y^4)^3}}$$

ex qua illa oritur, si statuatur $y = x\sqrt{-1}$. Nunc ergo quoque, postquam in integrali posterioris loco y scripsissem $x\sqrt{-1}$, integrale superioris prodire debebat. Ad hoc autem requirebatur, ut tam logarithmi, quam arcus quantitatum imaginariarum ita evolverentur, ut ad formam generalem $A + B\sqrt{-1}$ reducerentur.

§. 2. Hoc autem phaenomenon in innumeris alii casibus occurrere potest, qui ex hac consideratione originem trahunt. Sit Z eiusmodi functio ipsius z, ut formulae diffe rentialis $Z\partial z$ integrale utcunque, sive algebraice, sive pe logarithmos, sive arcus circulares exprimi queat, quod in tegrale per litteram V designemus, ut sit $\int Z\partial z = V$. Ia loco z substituamus quantitatem imaginariam quamcunque quam uti constat semper tali forma repraesentare lice $z = y(\cos.\theta + \sqrt{-1}\sin.\theta)$, ubi angulum θ ut constante spectabimus, ita ut sola y sit variabilis; hoc modo e $\partial z = \partial y(\cos.\theta + \sqrt{-1}\sin.\theta)$; functio autem Z recipi similem formam $Z = M + N\sqrt{-1}$, ita ut iam formula tegranda sit

$$\int Z\partial z = \int \partial y(M\cos.\theta - N\sin.\theta) + \sqrt{-1}\int \partial y(M\sin.\theta + N\cos.\theta)$$

cuius prior pars est realis, posterior vero imaginaria.

§. 3. Fiat nunc eadem substitutio. nempe Z = (cos. $\theta + \sqrt{-1}$ sin. θ) in integrali invento V, unde pari forma imaginaria $P + Q\sqrt{-1}$ prodeat necesse est; et q niam partes reales et imaginariae seorsim inter se compa debent, hinc orientur duae sequentes aequalitates:

$$P = \cos.\theta\int M\partial y - \sin.\theta\int N\partial y;$$

$$Q = \sin.\theta\int M\partial y + \cos.\theta\int N\partial y;$$

unde colligimus

$$\int M \partial y = P \cos. \theta + Q \sin. \theta \quad \text{et}$$
$$\int N \partial y = Q \cos. \theta - P \sin. \theta$$

hocque modo si inventae fuerint binae quantitates P et Q, ambo integralia tam $\int M \partial y$ quam $\int N \partial y$ exhiberi poterunt.

§. 4. Nisi autem functio proposita Z fuerit admodum simplex, plerumque litterae M et N hinc proveniunt functiones tam complicatae novae variabilis y, ut vix alia via pateat, harum formularum $\int M \partial y$ et $\int N \partial y$ integralia investigandi, praeter hanc ipsam, quam modo indicavimus, et quae per imaginaria procedit; totum ergo negotium huc redit, ut ex invento integrali V ambae quantitates P et Q inde oriundae definiantur. Quatenus igitur istud integrale V partes continet algebraicas, ista operatio nulla laborat difficultate; quando autem logarithmos et arcus circulares involvit, haud exigua sagacitate opus est, ut eius valor in formam $P + Q \sqrt{-1}$ transmutetur, quam ob rem subsidia hic sum traditurus, quibus omnes huiusmodi transformationes perfici queant.

§. 5. Cuncta autem haec subsidia commodissime repeti possunt ex sola formula Arc. tang. $t \sqrt{-1}$: Cum enim eius differentiale sit $= \frac{\partial t \sqrt{-1}}{1 - tt}$, huius integrale viciffim erit $= \frac{\sqrt{-1}}{2} l \frac{1+t}{1-t}$, siquidem ita definiatur, ut evanescat posito $t = 0$, quandoquidem hoc casu etiam arcus evanescit. Hinc igitur iam nacti sumus hanc primam reductionem:

$$\text{Arc. tang. } t \sqrt{-1} = \frac{\sqrt{-1}}{2} l \frac{1+t}{1-t}$$

ubi in generali forma $A + B \sqrt{-1}$ est $A = 0$.

§. 6.

§. 6. Ponamus nunc $t = u\sqrt{-1}$, eritque $t\sqrt{-1} = -u$ et Arc. tang. $t\sqrt{-1} = -$ Arc. tang. u, ex quo habebimus

$$-\text{Arc. tang. } u = \frac{\sqrt{-1}}{2}\, l\, \frac{1+u\sqrt{-1}}{1-u\sqrt{-1}},$$

unde viciſſim colligitur

$$l\, \frac{1+u\sqrt{-1}}{1-u\sqrt{-1}} = -\frac{2}{\sqrt{-1}} \text{ Arc. tang. } u = +2\sqrt{-1} \text{ Arc. tang. } u.$$

Cum porro ſit

$$\frac{1+u\sqrt{-1}}{1-u\sqrt{-1}} = \frac{(1+u\sqrt{-1})^2}{1+uu}, \text{ erit}$$

$$l\, \frac{1+u\sqrt{-1}}{1-u\sqrt{-1}} = 2\, l\,(1+u\sqrt{-1}) - 2\, l\sqrt{(1+uu)}$$
$$= 2\sqrt{-1} \text{ Arc. tang. } u,$$

unde colligitur haec nova reductio:

$$l(1+u\sqrt{-1}) = l\sqrt{(1+uu)} + \sqrt{-1} \text{ Arc. tang. } u.$$

§. 7. Cum igitur omnes formulae imaginariae ad formam $p(\text{coſ. } \alpha + \sqrt{-1} \text{ ſin. } \alpha)$ reduci queant, erit

$$l\,p(\text{coſ. } \alpha + \sqrt{-1} \text{ ſin. } \alpha) = l\,p \text{ coſ. } \alpha + l(1+\sqrt{-1} \text{ tang. } \alpha)$$

et poſito $u = \text{tang. } \alpha$, fiet

$$l(1 + \text{tang. } \alpha\sqrt{-1}) = -l \text{ coſ. } \alpha + \alpha\sqrt{-1}.$$

Hinc deducimus iſtam reductionem non minus memorabilem:

$$l\,p(\text{coſ. } \alpha + \sqrt{-1} \text{ ſin. } \alpha) = l\,p + \alpha\sqrt{-1}$$

ideoque

$$l(\text{coſ. } \alpha + \sqrt{-1} \text{ ſin. } \alpha) = \alpha\sqrt{-1}.$$

§. 8. Hinc igitur iam facilem modum impetravimus omnium quantitatum imaginariarum logarithmos ad formam $A + B\sqrt{-1}$ revocandi. At vero pro arcubus imaginariis hanc

hanc ſolam reductionem adhuc ſumus nacti, qua erat Arc. tang. $t\sqrt{-1} = \frac{\sqrt{-1}}{2} l \frac{1+t}{1-t}$. Deſideratur ergo adhuc regula huiusmodi arcum imaginarium Arc. tang. $(p + q\sqrt{-1})$ ad formam $A + B\sqrt{-1}$ reducendi. Talis quidem regula iam paſſim reperitur, quia autem plerumque nimis operoſe eſt eruta, ſequenti modo eam immediate ex ſolo principio hic ſtabilito deducemus.

§. 9. Quaeramus ſcilicet primo ſummam huiusmodi binorum arcuum, quae ſit

$$\text{Arc. tang.}\,(p + q\sqrt{-1}) + \text{Arc. tang.}\,(p - q\sqrt{-1})$$

quam deſignemus littera R, et cum in genere ſit

$$\text{A tang.}\,a + \text{A tang.}\,b = \text{Arc. tang.}\,\frac{a+b}{1-ab},$$

ob $a = p + q\sqrt{-1}$ et $b = p - q\sqrt{-1}$, erit

$$R = \text{Arc. tang.}\,\frac{2p}{1-pp-qq}.$$

Simili modo ponatur eorundem arcuum differentia

$$\text{Arc. tang.}\,(p + q\sqrt{-1}) - \text{Arc. tang.}\,(p - q\sqrt{-1}) = S,$$

et quia

$$\text{Arc. tang.}\,a - \text{Arc. tang.}\,b = \text{Arc. tang.}\,\frac{a-b}{1+ab}, \text{ erit}$$

$$S = \text{Arc. tang.}\,\frac{2q\sqrt{-1}}{1+pp+qq}.$$

Initio autem vidimus eſſe

$$\text{Arc. tang.}\,t\sqrt{-1} = \frac{\sqrt{-1}}{2} l \frac{1+t}{1-t},$$

unde ſumto $t = \frac{2q}{1+pp+qq}$; erit

$$S = \frac{\sqrt{-1}}{2} l \frac{(1+q)^2 + pp}{(1-q)^2 + pp}.$$

§. 10. Inventis igitur binarum illarum formularum tam ſumma R quam differentia S, utramque ſeorſim exhibere

li-

licet; erit enim

$$\text{Arc. tang.}\,(p + q\sqrt{-1}) = \frac{R+S}{2}, \text{ ideoque}$$

$$\text{Arc. tang.}\,(p+q\sqrt{-1}) = \tfrac{1}{2}\,\text{Arc. tang.}\,\frac{2p}{1-pp-qq} + \frac{\sqrt{-1}}{4}\, l\, \frac{(1+q)^2+p}{(1-q)^2+p}$$

similique modo erit

$$\text{Arc. tang.}\,(p-q\sqrt{-1}) = \tfrac{1}{2}\,\text{Arc. tang.}\,\frac{2p}{1-pp-qq} - \frac{\sqrt{-1}}{4}\, l\, \frac{(1+q)^2+p}{(1-q)^2+p}$$

quae quidem ex priore sponte deducitur, loco q scriber $-q$. Hic commode Arc. tang. $\frac{2p}{1-pp-qq}$ in duos resolvere lic quo facto erit

$$\text{Arc. tang.}\,(p+q\sqrt{-1}) = \tfrac{1}{2}\,\text{Arc. tang.}\,\frac{p}{1-q} + \tfrac{1}{2}\,\text{Arc. tang.}\,\frac{p}{1+q} + \frac{\sqrt{-1}}{4}\, l\, \frac{(1+q)^2+pp}{(1-q)^2+pp}.$$

§. 11. Nunc igitur loco $p + q\sqrt{-1}$ substitua formam $r(\text{cos}.\,\alpha + \sqrt{-1}\,\text{sin}.\,\alpha)$, ut fit $p = r\,\text{cos}.\,\alpha$ et $q = r\,\text{si}$ ac reperietur

$$\text{Arc. tang.}\,r(\text{cos}.\,\alpha + \sqrt{-1}\,\text{sin}.\,\alpha) = \tfrac{1}{2}\,\text{Arc. tang.}\,\frac{2r\,\text{cos}.\,\alpha}{1-rr} + \frac{\sqrt{-1}}{4}\, l\, \frac{1+2r\,\text{sin}.\,\alpha+rr}{1-2r\,\text{sin}.\,\alpha+rr}.$$

Per posteriorem autem formam erit quoque

$$\text{Arc. tang.}\,r(\text{cos}.\,\alpha + \sqrt{-1}\,\text{sin}.\,\alpha) = \tfrac{1}{2}\,\text{Arc. tang.}\,\frac{r\,\text{cos}.\,\alpha}{1-r\,\text{sin}.\,\alpha} + \tfrac{1}{2}\,\text{Arc. tang.}\,\frac{r\,\text{cos}.\,\alpha}{1+r\,\text{sin}.\,\alpha} + \frac{\sqrt{-1}}{4}\, l\, \frac{1+2r\,\text{sin}.\,\alpha+rr}{1-2r\,\text{sin}.\,\alpha+rr},$$

§. 12. Hae iam formulae hactenus inventae c subsidia complectuntur, quibus indigebimus ad omnes rithmos et arcus circulares imaginarios resolvendos. F las autem inventas hic simul aspectui exponamus:

$$\text{I. } l(a+b\sqrt{-1}) = la + l\left(1+\frac{b\sqrt{-1}}{a}\right) = l\sqrt{(aa+bb)} + \sqrt{-1}\,\text{Arc. tang.}\,\frac{b}{a},$$

unde deducitur ista saepissime occurrens:

$$l\frac{a+b\sqrt{-1}}{a-b\sqrt{-1}} = 2\sqrt{-1}\ \text{Arc. tang.}\ \frac{b}{a}.$$

Porro etiam notetur haec formula:

$$l\,a(\text{cos.}\,\alpha+\sqrt{-1}\ \text{sin.}\,\alpha) = l\,a + \alpha\sqrt{-1}.$$

Pro arcubus autem has adepti sumus formulas:

$$\text{Arc. tang.}\,(a+b\sqrt{-1}) = \tfrac{1}{2}\ \text{Arc. tang.}\ \frac{2a}{1-aa-bb} + \frac{\sqrt{-1}}{4}\,l\frac{(1+b)^2+aa}{(1-b)^2+aa},$$

vel etiam

$$\text{Arc. tang.}\ a(\text{cos.}\,\alpha+\sqrt{-1}\ \text{sin.}\,\alpha) = \tfrac{1}{2}\ \text{Arc. tang.}\ \frac{2a\ \text{cos.}\,\alpha}{1-aa} + \frac{\sqrt{-1}}{4}\,l\frac{1+2a\ \text{sin.}\,\alpha+aa}{1-2a\ \text{sin.}\,\alpha+aa}.$$

§. 13. His fundamentis constitutis consideremus casus, quibus integrale $\int Z\,\partial z$ per logarithmos et arcus circulares exprimi potest, id quod semper evenit, quando Z est functio rationalis ipsius z, tum autem integrale componitur ex huiusmodi partibus:

I. $l(1 \pm z)$;

II. $l(1 - 2z\ \text{cos.}\,\alpha + zz)$;

III. Arc. tang. $\frac{z\ \text{sin.}\,\alpha}{1-z\ \text{cos.}\,\alpha}$;

vel saltem integralia, quae reperiuntur, facile ad tales formas redigi possunt. Harum ergo resolutionem, quando statuitur $z = y(\text{cos.}\,\theta + \sqrt{-1}\ \text{sin.}\,\theta)$, nonnullas in sequentibus problematibus expediemus.

Problema 1.

§. 14. *Hanc formulam logarithmicam $l(1 \pm z)$, posito*

$$z = y\ \text{cos.}\,\theta + \sqrt{-1}\ \text{sin.}\,\theta)$$

ad formam generalem $A + B\sqrt{-1}$ reducere.

 Solu-

Solutio.

Evolvamus primo formulam

$$l(1+z) = l(1 + y\cos.\theta + y\sqrt{-1}\sin.\theta)$$

et comparatione cum superiore forma generali facta erit

$$a = 1 + y\cos.\theta \text{ et } b = y\sin.\theta,$$

unde colligitur

$$l(1+z) = l\sqrt{(1 + 2y\cos.\theta + yy)} + \sqrt{-1}\,\text{Arc. tang.}\,\frac{y\sin.\theta}{1 + y\cos.\theta}.$$

Hinc autem alter casus $l(1-z)$ sponte derivatur, sumendo y negative, eritque ergo $l(1-z) =$

$$l\sqrt{(1 - 2y\cos.\theta + yy)} - \sqrt{-1}\,\text{Arc. tang.}\,\frac{y\sin.\theta}{1 - y\cos.\theta}.$$

Saepenumero autem in integralibus occurrere solet formula $l\frac{1+z}{1-z}$, cuius ergo valor, posito

$$z = y(\cos.\theta + \sqrt{-1}\sin.\theta),$$

sequenti modo exprimetur:

$$l\frac{1+z}{1-z} = \frac{1}{2}l\frac{1 + 2y\cos.\theta + yy}{1 - 2y\cos.\theta + yy} + \sqrt{-1}\,\text{Arc. tang.}\,\frac{y\sin.\theta}{1 + y\cos.\theta} + \sqrt{-1}\,\text{Arc. tang.}\,\frac{y\sin.\theta}{1 - y\cos.\theta},$$

quare si ambo arcus in unum contrahantur, prodibit

$$l\frac{1+z}{1-z} = \frac{1}{2}l\frac{1 + 2y\cos.\theta + yy}{1 - 2y\cos.\theta + yy} + \sqrt{-1}\,\text{Arc. tang.}\,\frac{2y\sin.\theta}{1 - yy}.$$

Problema 2.

§. 15. *Proposita formula logarithmica*

$$l(1 - 2z\cos.\alpha + zz),$$

si in ea ponatur $z = y(\cos.\theta + \sqrt{-1}\sin.\theta)$, *eius valorem ad formulam postulatam* $A + B\sqrt{-1}$ *reducere.*

Solu

Solutio.

Si hic immediate fubftitutionem facere vellemus, in calculos fatis moleftos delaberemur, quos ut evitemus, obfervaffe iuvabit, formulam $1 - 2z \cos.\alpha + zz$ effe productum ex his factoribus:

$$[1 - z(\cos.\alpha + \sqrt{-1}\,\sin.\alpha)\,(1 - z(\cos.\alpha - \sqrt{-1}\,\sin.\alpha)],$$

quorum ergo logarithmos invicem addi oportet.

Tractemus ergo primo formulam

$$l\,[1 - z\,(\cos.\alpha + \sqrt{-1}\,\sin.\alpha)],$$

et cum fit

$$y\,(\cos.\theta + \sqrt{-1}\,\sin.\theta)\,(\cos.\alpha + \sqrt{-1}\,\sin.\alpha) = z\,(\cos.\alpha + \sqrt{-1}\,\sin.\alpha)$$

quoniam in genere eft

$$(\cos.\beta + \sqrt{-1}\,\sin.\beta)\,(\cos.\gamma + \sqrt{-1}\,\sin.\gamma) = \cos.(\beta + \gamma) + \sqrt{-1}\,\sin.(\beta + \gamma),$$ erit

$$l\,[1 - z\,(\cos.\alpha + \sqrt{-1}\,\sin.\alpha)] = l\,[1 - y\cos.(\alpha + \theta) + \sqrt{-1}\,\sin.(\alpha + \theta)].$$

Hic ergo facta comparatione erit

$$a = 1 - y\cos.(\alpha + \theta) \text{ et } b = -y\sin.(\alpha + \theta);$$

unde eius valor refolutus erit

$$l\,[1 - z\,(\cos.\alpha + \sqrt{-1}\,\sin.\alpha)] = \tfrac{1}{2}\,l\,[1 - 2y\cos.(\alpha + \theta) + yy] - \sqrt{-1}\,\text{Arc.tang.}\frac{y\sin.(\alpha + \theta)}{1 - y\cos.(\alpha + \theta)}.$$

Hinc altera formula facile deducitur, fumendo angulum negative, eritque

$$l\,[1 - z\,(\cos.\alpha - \sqrt{-1}\,\sin.\alpha)] = \tfrac{1}{2}\,l\,[1 - 2y\cos.(\theta - \alpha) + yy] - \sqrt{-1}\,\text{Arc.tang.}\frac{y\sin.(\theta - \alpha)}{1 - y\cos.(\theta - \alpha)}.$$

 Nunc

Nunc igitur tantum opus eſt, ambos valores, quos modo invenimus, invicem addere, ſicque prodibit haec reductio:

$$l(1 - 2z \cos. \alpha + zz)$$
$$= \tfrac{1}{2} l[1 - 2y \cos.(\alpha+\theta) + yy] - \sqrt{-1}\, \text{Arc. tg.} \frac{y \sin. \alpha + \theta}{1 - y \cos.(\alpha+\theta)}$$
$$+ \tfrac{1}{2} l[1 - 2y \cos.(\theta-\alpha) + yy] - \sqrt{-1}\, \text{Arc. tg.} \frac{y \sin.(\theta-\alpha)}{1 - y \cos.(\theta-\alpha)}$$

Problema 3.

§. 16. *Propoſita formula pro arcu circulari*

$$T = \text{Arc. tang.} \frac{z \sin. \alpha}{1 - z \cos. \alpha},$$

ſi in ea ponatur $z = y(\cos. \theta + \sqrt{-1} \sin. \theta)$, *eius valorem inde reſultantem ad formam* $A + B\sqrt{-1}$ *revocare.*

Solutio.

Quia hic in numeratore et denominatore imaginari occurrunt, ad ſimpliciorem formam perveniemus, ſi utrin que addamus Arc. α, ſive Arc. tang. $\frac{\sin. \alpha}{\cos. \alpha}$; ſic enim erit

$$T + \alpha = \text{Arc. tang.} \frac{\sin. \alpha}{\cos. \alpha - z} = 90° - \text{Arc. tang.} \frac{\cos. (\alpha - z}{\sin. \alpha}$$

ideoque

$$T = 90° - \alpha - \text{Arc. tang.} \frac{\cos. (\alpha - z)}{\sin. \alpha}.$$

Iam in hac poſtrema formula ponamus

$$z = y(\cos. \theta + \sqrt{-1} \sin. \theta), \text{ fietque}$$

$$\text{Arc. tang.} \frac{\cos. (\alpha - z)}{\sin. \alpha} = \text{Arc. tang.} \frac{\cos. \alpha - y \cos. \theta - y\sqrt{-1} \sin}{\sin. \alpha}$$

quae expreſſio comparata cum formula generali

Ar

Arc. tang. $(a + b\sqrt{-1})$ dat

$$a = \frac{\cos. \quad - y\cos.\theta}{\sin.\alpha} \text{ et } b = -\frac{y\sin.\theta}{\sin.\alpha}.$$

Hinc igitur erit

$$1 - aa - bb = -\frac{\cos.2\alpha + 2y\cos.\alpha\cos.\theta - yy}{\sin.\alpha^2}$$

ideoque

$$\frac{2a}{1 - aa - bb} = \frac{\sin.2\alpha - 2y\sin.\alpha\cos.\theta}{-\cos.2\alpha + 2y\cos.\alpha\cos.\theta - yy}, \text{ ergo}$$

$$\text{Arc. tang.} \frac{2a}{1 - aa - bb} = -\text{Arc. tang.} \frac{\sin.2\alpha - 2y\sin.\alpha\cos.\theta}{\cos.2\alpha - 2y\cos.\alpha\cos.\theta + yy}.$$

Iam pro parte imaginaria erit

$$1 + aa + bb = \frac{1 - 2y\cos.\alpha\cos.\theta + yy}{\sin.\alpha^2},$$

unde colligitur numerator

$$(1 + b)^2 + aa = \frac{1 - 2y\cos.(\theta - \alpha) + yy}{\sin.\alpha^2}$$

et denominator

$$(1 - b)^2 + aa = \frac{1 - 2y\cos.(\alpha + \theta) + yy}{\sin.\alpha^2},$$

sicque pars imaginaria erit

$$\frac{\sqrt{-1}}{4} l \frac{(1 + b)^2 + aa}{(1 - b)^2 + aa} = \frac{\sqrt{-1}}{4} l \frac{1 - 2y\cos.(\theta - \alpha) + yy}{1 - 2y\cos.(\alpha + \theta) + yy},$$

quamobrem hinc colligimus

$$\text{Arc. tang.} \frac{\cos.(\alpha - z)}{\sin.\alpha} = -\tfrac{1}{2}\text{Arc. tang.} \frac{\sin.2\alpha - 2y\sin.\alpha\cos.\theta}{\cos.2\alpha - 2y\cos.\alpha\cos.\theta + yy}$$
$$+ \frac{\sqrt{-1}}{4} l \frac{1 - 2y\cos.(\theta - \alpha) + yy}{1 - 2y\cos.(\alpha + \theta) + yy}.$$

His iam formulis inventis reductio ipsius formulae propositae ita se habebit:

$$\text{Arc. tang.} \frac{z\sin.\alpha}{1 - z\cos.\alpha} = 90^0 - \alpha + \tfrac{1}{2}\text{Arc. tg.} \frac{\sin.2\alpha - 2y\sin.\alpha\cos.\theta}{\cos.2\alpha - 2y\cos.\alpha\cos.\theta + yy}$$
$$- \frac{\sqrt{-1}}{4} l \frac{1 - 2y\cos.(\theta - \alpha) + yy}{1 - 2y\cos.(\alpha + \theta) + yy}.$$

Hae

Hac iam reductiones haud difficulter ad omnes formulas accommodari poterunt, quod quo clarius appareat, ſequens exemplum adiungamus.

Integratio
Formulae differentialis

$$\frac{\partial x}{(3 - xx)\sqrt[3]{(1 - 3xx)}} = \partial V.$$

§. 17. Quoniam nondum apparet, quomodo han ipſam formulam tractari conveniat, eam ad ſequentem fo mam imaginariam, ponendo $x = z\sqrt{-1}$, reducamus, ut ſi

$$\partial V = \frac{\partial z \sqrt{-1}}{(3 + zz)\sqrt[3]{(1 + 3zz)}},$$

quae forma iam ita comparata deprehenditur, ut per prae cepta non ita pridem tradita ad integrationem perduci po ſit, eius ergo reſolutionem ſequenti modo expedire pot rimus.

§. 18. Ponamus igitur

$$\frac{\partial z}{(3 + zz)\sqrt[3]{(1 + 3zz)}} = \partial T,$$

ut ſit $V = T\sqrt{-1}$. Hanc autem formam ſequenti mod repraeſentemus: $\partial T = \frac{z\,\partial z}{(3z + z^3)\sqrt[3]{(1 + 3zz)}}$, ubi brevit tatis gratia ſtatuamus $\sqrt[3]{(1 + 3zz)} = v$, ut ſit

∂

$$\partial T = \frac{z \partial z}{v(3z + z^3)},$$

hicque fecundum noftra praecepta ftatuamus $p = \frac{1+z}{v}$ et $q = \frac{1-z}{v}$, unde fit $p + q = \frac{2}{v}$ et $p - q = \frac{2z}{v}$, hincque $z = \frac{p-q}{p+q}$, ideoque differentiando

$$\partial z = \frac{2q\partial p - 2p\partial q}{(p+q)^2} = \frac{1}{2} v v (q \partial p - p \partial q),$$

quo valore fubftituto impetramus

$$\partial T = \frac{v z (q \partial p - p \partial q)}{2(3z + z^3)}.$$

§. 19. Cum iam fit $1 + z = pv$ et $1 - z = qv$, erit primo $2z = v(p-q)$, tum vero fumma cuborum dabit

$$(1+z)^3 + (1-z)^3 = v^3(p^3 + q^3) = 2 + 6zz.$$

Quoniam igitur pofuimus $\sqrt[3]{(1+3zz)} = v$, erit $v^3 = 1 + 3zz$; quam ob rem habebimus $p^3 + q^3(1 + 3zz) = 2 + 6zz$, confequenter $p^3 + q^3 = 2$. Denique vero differentia cuborum praebet $(p^3 - q^3)v^3 = 6z + 2z^3$; unde patet effe $3z + z^3 = \frac{1}{2}(p^3 - q^3)v^3$; at vero differentia quadratorum dat $(pp - qq)vv = 4z$, unde fit $z = \frac{1}{4}vv(pp - qq)$.

§. 20. Subftituantur nunc ifti valores loco z et $3z + z^3$, atque noftra formula evadet

$$\partial T = \frac{(pp - qq)(q\partial p - p\partial q)}{4(p^3 - q^3)},$$

ubi ergo tantum binae litterae p et q occurrunt, quae ita a fe invicem pendent, vt fit $p^3 + q^3 = 2$, ideoque differentiando $pp\partial p + qq\partial q = 0$, confequenter five $\partial p = \frac{-qq\partial q}{pp}$, five $\partial q = \frac{-pp\partial p}{qq}$.

§. 21.

§. 21. Dividatur nunc haec forma in duas partes, ponendo

$$\frac{pp(q\partial p - p\partial q)}{p^3 - q^3} = \partial P \text{ et } \frac{qq(q\partial p - p\partial q)}{p^3 - q^3} = \partial Q,$$

vt fit $\partial T = \frac{1}{4}\partial P - \frac{1}{4}\partial Q$, hicque ſtatim patet, ſi in priore formula loco $pp\partial p$ ſcribatur $-qq\partial q$, tum prodire

$$\partial P = -\frac{\partial q(p^3 + q^3)}{p^3 - q^3}.$$

Quia vero eſt $p^3 + q^3 = 2$, ideoque $p^3 = 2 - q^3$, elementum ∂P per ſolam litteram q ita exprimetur, ut fit

$$\partial P = \frac{\partial q}{1 - q^3}.$$

§. 22. Simili modo ſi in altera formula ∂Q loco $qq\partial q$ ſcribatur $-pp\partial p$, prodibit $\partial Q = \frac{\partial p(p^3 + q^3)}{p^3 - q^3}$, quae ergo ob relationem inter p et q ſuppeditat hanc formulam:

$$\partial Q = \frac{\partial p}{p^3 - 1} = \frac{-\partial p}{1 - p}.$$

His igitur coniunctis erit

$$4\partial T = \frac{\partial p}{1 - p^3} - \frac{\partial q}{1 - q^3},$$

ſicque totum negotium perductum eſt ad duas formulas differentiales rationales, quas ergo per logarithmos et arcus circulares integrare licet.

§. 23. Ad haec integralia invenienda ſtatuatur

$$\frac{1}{1 - p^3} = \frac{F}{1 - p} + \frac{G}{1 + p + pp},$$

ubi notetur fore

$$F = \frac{1 - p}{1 - p^3} = \frac{1}{1 - p + pp},$$

poſito $1 - p = 0$, ſive $p = 1$, unde fit $F = \frac{1}{3}$, tum vero erit

$$G = \frac{1 + p - pp}{1 - p^3} = \frac{1}{1 - p},$$

po-

posito $1 + p + pp = 0$. Hic iam ut littera p ex denominatore tolli queat, multiplicetur supra et infra per $2 + p$, fiet $G = \frac{2 + p}{2 - p - pp}$. Quia igitur est $p + pp = -1$, erit $G = \frac{2+p}{3}$; quam ob rem habebimus $\frac{3 \partial p}{1 - p^3} = \frac{\partial p}{1 - p} + \frac{(2 + p) \partial p}{1 + p + pp}$. Constat autem esse $\int \frac{\partial p}{1 - p} = - l (1 - p)$ et

$$\int \frac{p \partial p + 2 \partial p}{pp + p + 1} = \frac{1}{2} l (1 + p + pp) + \frac{3}{2} \int \frac{\partial p}{1 + p + pp}.$$

Novimus autem in genere esse

$$\int \frac{\partial p}{1 - 2 p \cos. \alpha + pp} = \frac{1}{\sin. \alpha} \text{ Arc. tang. } \frac{p \sin. \alpha}{1 - p \cos. \alpha},$$

unde patet sumi debere $\alpha = 120^\circ$, et ob $\sin. \alpha = \frac{\sqrt{3}}{2}$, erit

$$\int \frac{\partial p}{1 + p + pp} = \frac{2}{\sqrt{3}} \text{ Arc. tang. } \frac{p \sqrt{3}}{2 + p},$$

sicque totum integrale erit

$$3 \int \frac{\partial p}{1 - p^3} = - l (1 - p) + \frac{1}{2} l (1 + p + pp)$$
$$+ \sqrt{3} \text{ Arc. tang. } \frac{p \sqrt{3}}{2 + p},$$

similique modo erit

$$3 \int \frac{\partial q}{1 - q^3} = - l (1 - q) + \frac{1}{2} l (1 + q + qq)$$
$$+ \sqrt{3} \text{ Arc. tang. } \frac{q \sqrt{3}}{3 + q}.$$

§. 24. His igitur inventis erit

$$12 T = l \frac{1 - q}{1 - p} + \frac{1}{2} l \frac{1 + p + pp}{1 + q + qq}$$
$$+ \sqrt{3} \text{ Arc. tang. } \frac{p \sqrt{3}}{2 + p} - \sqrt{3} \text{ Arc. tang. } \frac{q \sqrt{3}}{2 + q}.$$

Quare cum sit $p = \frac{1 + z}{v}$ et $q = \frac{1 - z}{v}$, habebimus

$$T = \frac{1}{12} l \frac{v - 1 + z}{v - 1 - z} + \frac{1}{24} l \frac{vv + v (1 + z) + (1 + z)^2}{vv + v (1 - z) + (1 - z)^2}$$
$$+ \frac{1}{4 \sqrt{3}} \text{ Arc. tang. } \frac{(1 + z) \sqrt{3}}{2 v + 1 + z} - \frac{1}{4 \sqrt{3}} \text{ Arc. tang. } \frac{(1 - z) \sqrt{3}}{2 v + 1 - z}.$$

§. 25. Nunc secundum praecepta supra exposita, ubi sumsimus $z = y(\text{cos.}\,\theta + \sqrt{-1}\,\text{sin.}\,\theta)$, quia est $x = z\sqrt{-1}$, erit

$$z = -x\sqrt{-1} = y(\text{cos.}\,\theta + \sqrt{-1}\,\text{sin.}\,\theta);$$

unde patet statui debere $\theta = 90°$ et $y = -x$, hocque notato, ut superiores reductiones ad nostrum casum propius accommodemus, ibi ubique loco z et y scribamus $\frac{z}{s}$ et $\frac{y}{s}$, quo facto reductiones erunt

I. $l(s+z) = +\frac{1}{2}l(ss + 2sy\,\text{cos.}\,\theta + yy) + \sqrt{-1}\,\text{Arc. tg.}\,\frac{y\,\text{sin.}\,\theta}{s + y\,\text{cos.}\,\theta}$.

II. $l(ss - 2sz\,\text{cos.}\,\alpha + zz) = \frac{1}{2}l[ss - 2sy\,\text{cos.}(\alpha+\theta) + yy]$
$$-\sqrt{-1}\,\text{Arc. tang.}\,\frac{y\,\text{sin.}(\alpha+\theta)}{s - y\,\text{cos.}(\alpha+\theta)}$$
$$+\frac{1}{2}l(ss - 2sy\,\text{cos.}(\theta-\alpha) + yy) - \sqrt{-1}\,\text{Arc. tang.}\,\frac{y\,\text{sin.}(\theta-\alpha)}{s - y\,\text{cos.}(\theta-\alpha)}.$$

III. $\text{Arc. tang.}\,\frac{z\,\text{sin.}\,\alpha}{s - z\,\text{cos.}\,\alpha} = 90° - \alpha + \frac{1}{9}\,\text{Arc. tg.}\,\frac{ss\,\text{sin.}\,2\alpha - 2sy\,\text{sin.}\,\alpha\,\text{cos.}\,\theta}{ss\,\text{cos.}\,2\alpha - 2sy\,\text{cos.}\,\alpha\,\text{cos.}\,\theta + yy}$
$$-\frac{\sqrt{-1}}{4}\,l\,\frac{ss - 2sy\,\text{cos.}(\theta-\alpha) + yy}{ss - 2sy\,\text{cos.}(\alpha+\theta) + yy}.$$

§. 26. Iam haec praecepta ad singulas partes integralis inventi applicemus, ac primo quidem pro formula $l(v - 1 + z)$ erit $s = v - 1$, et ob $y = -x$ et $\theta = 90°$ colligitur

I. $l(v - 1 + z) = \frac{1}{2}l[(v-1)^2 + xx] - \sqrt{-1}\,\text{Arc. tang.}\,\frac{x}{v-1}$.

II. $l(v - 1 - z) = \frac{1}{2}l[(v-1)^2 + xx] + \sqrt{-1}\,\text{Arc. tang.}\,\frac{x}{v-1}$.

III. Pro formula $l[vv + v(1+z) + (1+z)^2]$ patet fore
$$ss = vv + v + 1, \text{ seu } s = \sqrt{(vv + v + 1)};$$
$$\text{cos.}\,\alpha = -\frac{v-2}{2\sqrt{(vv+v+1)}} \text{ et sin.}\,\alpha = \frac{v\sqrt{3}}{2\sqrt{(vv+v+1)}},$$

unde ob $\theta = 90°$ erit
$$\text{cos.}(\alpha+\theta) = -\text{sin.}\,\alpha;\ \text{cos.}(\theta-\alpha) = \text{sin.}\,\alpha;$$
$$\text{sin.}(\alpha+\theta) = \text{cos.}\,\alpha \text{ et sin.}(\theta-\alpha) = \text{cos.}\,\alpha;$$

quo

quo obſervato erit:

$$l[vv+v(1+z)+(1+z)^2]=\tfrac{1}{2}l[vv+v+1-vx\sqrt{3}+xx]$$
$$+\tfrac{1}{2}l[vv+v+1+vx\sqrt{3}+xx]$$
$$-\sqrt{-1}\,\text{Arc. tang.}\,\frac{x(v+2)}{2(vv+v+1)-vx\sqrt{3}}$$
$$-\sqrt{-1}\,\text{Arc. tang.}\,\frac{x(v+2)}{2(vv+v+1)+vx\sqrt{3}},$$

hinc ſimul mutato ſigno litterarum z et x erit

$$l[vv+v(1-z)+(1-z)^2]=\tfrac{1}{2}l[vv+v+1+vx\sqrt{3}+xx]$$
$$+\tfrac{1}{2}l[vv+v+1-vx\sqrt{3}+xx]$$
$$+\sqrt{-1}\,\text{Arc. tang.}\,\frac{x(v+2)}{2(vv+v+1)+vx\sqrt{3}}$$
$$+\sqrt{-1}\,\text{Arc. tang.}\,\frac{x(v+2)}{2(vv+v+1)-vx\sqrt{3}}.$$

§. 27. Nunc porro pro Arc. tang. $\frac{(1+z)\sqrt{3}}{2v+1+z}$, quae forma in regula noſtra non continetur, notetur eſſe

$$\text{Arc. tg.}\,\frac{(1+z)\sqrt{3}}{2v+1+z}=\text{Arc. tg.}\,\frac{\sqrt{3}}{2v+1}+\text{Arc. tg.}\,\frac{vz\sqrt{3}}{2(vv+v+1)+(v+2)z},$$

qui poſtremus valor comparatus cum Arc. tang. $\frac{z\,\text{ſin.}\,\alpha}{s-z\,\text{coſ.}\,\alpha}$ iterum praebet ſin. $\alpha=\frac{v\sqrt{3}}{2\sqrt{(vv+v+1)}}$, ſicque anguli $\alpha+\theta$ et $\theta-\alpha$ manent iidem, ut ante; unde reductio praebet

$$\text{Arc. tang.}\,\frac{vz\sqrt{3}}{2(vv+v+1)+(v+2)z}=-\tfrac{1}{2}\text{Arc. tg.}\,\frac{v(v+2)\sqrt{3}}{-vv+2v+2+2xx}$$
$$-\frac{\sqrt{-1}}{4}\,l\,\frac{vv+v+1+vx\sqrt{3}+xx}{vv+v+1-vx\sqrt{3}+xx}.$$

Nacti ergo ſumus has reductiones:

$$\text{Arc. tg.}\,\frac{(1+z)\sqrt{3}}{2v+1+z}=\text{Arc. tg.}\,\frac{\sqrt{3}}{2v+1}-\tfrac{1}{2}\text{Arc. tg.}\,\frac{v(v+2)\sqrt{3}}{-vv+2v+2+2xx}$$
$$-\frac{\sqrt{-1}}{4}\,l\,\frac{vv+v+1+vx\sqrt{3}+xx}{vv+v+1-vx\sqrt{3}+xx}\ \text{et}$$

$$\text{Arc. tg.}\,\frac{(1-z)\sqrt{3}}{2v+1-z}=\text{Arc. tg}\,\frac{\sqrt{3}}{2v+1}-\tfrac{1}{2}\text{Arc. tg.}\,\frac{v(v+2)\sqrt{3}}{-vv+2v+2+2xx}$$
$$-\frac{\sqrt{-1}}{4}\,l\,\frac{vv+v+1-vx\sqrt{3}+xx}{vv+v+1+vx\sqrt{3}+xx}.$$

§. 28. Quodſi iam omnes has partes rite colligamus, reperiemus

$$T = -\frac{\sqrt{-1}}{6}\,\text{Arc. tang.}\,\frac{x}{v-1} - \frac{\sqrt{-1}}{12}\,\text{Arc. tg.}\,\frac{x(v+2)}{2(vv+v+1)-vx\sqrt{3}}$$
$$-\frac{\sqrt{-1}}{12}\,\text{Arc. tg.}\,\frac{x(v+2)}{2(vv+v+1)+vx\sqrt{3}} - \frac{\sqrt{-1}}{8\sqrt{3}}\,l\,\frac{vv+v+1+vx\sqrt{3}+xx}{vv+v+1-vx\sqrt{3}+xx}.$$

§. 29. Hic igitur commode uſu venit, ut omnes partes reales ſe mutuo deſtruxerint, imaginariae vero duplicatae prodierint, quemadmodum natura rei manifeſto poſtulat. Cum igitur integrale quaeſitum ſit $V = T\sqrt{-1}$, nunc eius valor pulcherrime prodit realis, quocirca perducti ſumus ad hanc integrationem:

$$\int \frac{\partial x}{(3-xx)\sqrt[3]{(1-3xx)}} = +\frac{1}{6}\,\text{Arc. tang.}\,\frac{x}{v-1}$$
$$+\frac{1}{12}\,\text{Arc. tang.}\,\frac{x(v+2)}{2(vv+v+1)-vx\sqrt{3}}$$
$$+\frac{1}{12}\,\text{Arc. tang.}\,\frac{x(v+2)}{2(vv+v+1)+vx\sqrt{3}}$$
$$+\frac{1}{8\sqrt{3}}\,l\,\frac{(vv+v+1+vx\sqrt{3}+xx)}{vv+v+1-vx\sqrt{3}+xx},$$

ubi eſt $v = \sqrt[3]{(1-3xx)}$. Hae formulae aliquanto fir pliciores reddi poſſunt, conſiderando quod ſit $1 - v^3 = 3x$ ideoque $1 + v + vv = \frac{3xx}{1-v}$, unde cum plures ſubſtitution adhiberi queant, iis hic non immorandum cenſemus, ſe contenti eſſe poſſumus, iſtius formulae differentialis integra eruiſſe, ad quod per nullam aliam methodum aditus p tere videtur.

§. 3

§. 30. Caeterum calculus facilior evadet, si in integrali primum invento ambo arcus per $\frac{1}{4\sqrt{3}}$ multiplicati in unum colligantur: inde enim prodit $\frac{1}{4\sqrt{3}}$ Arc. tg. $\frac{vz\sqrt{3}}{vv+v+1-zz}$. Hic iam statim ponatur $z = -x\sqrt{-1}$, ut formula prodeat $-\frac{1}{4\sqrt{3}}$ Arc. tang. $\frac{vx\sqrt{3}\sqrt{-1}}{vv+v+1+xx}$, quae comparata cum canonica Arc. tang. $t\sqrt{-1} = \frac{\sqrt{-1}}{2} l \frac{1+t}{1-t}$, ob $t = \frac{vx\sqrt{3}}{vv+v+1+xx}$, statim perducit ad hanc formulam reductam:

$$\frac{-\sqrt{-1}}{8\sqrt{3}} l \frac{vv+v+1+vx\sqrt{3}+xx}{vv+v+1-vx\sqrt{3}+xx}.$$

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