E711

METHODVS NOVA AC FACILIS OMNIVM AEQVATIONVM ALGEBRAICARVM RADICES NON SOLVM IPSAS SED ETIAM QVASCVNQVE EARVM POTESTATES PER SERIES CONCINNAS EXPRIMENDI.

Auctore

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§. 1.

Primus, qui hoc argumentum satis felici successu tractavit, erat ingeniosissimus *Lambert* non ita pridem beate defunctus, qui huiusmodi series pro aequationibus trinomialibus methodo prorsus singulari per approximationes procedente elicuit. Verum ista methodus calculos maxime operosos ac taediosos requirebat, ita ut inventis aliquot terminis initialibus eundem calculum ulterius prosequi non potuerit, sed tantum ex egregio ordine, qui in prioribus terminis observabatur, per inductionem sequentes concludere fuerit coactus. Quamobrem iam ex illo tempore plurimum studii collocavi in methodum directam et planiorem inquirendi, quae ad easdem series perduceret.

§. 2.

§. 2. In huiusmodi autem methodum non multo post incidi, quam in *Commentar. Novor. Tomo XV.* fusius exposui; ubi in genere aequationem sub hac forma contentam:

$$1 = \frac{A}{x} - \frac{B}{x^2} + \frac{C}{x^3} - \frac{D}{x^4} + \frac{E}{x^5} - \text{etc.}$$

sum contemplatus, cuius radices si fuerint α, β, γ, etc. notum est earum summam esse $= A$, summam quadratorum $= A^2 - 2B$, summam cuborum $= A^3 - 3AB + 3C$, et ita porro. Hinc igitur pro summa potestatum quarumcunque $\alpha^n + \beta^n + \gamma^n + \delta^n +$ etc. similem expressionem investigavi, cuius legem progressionis in infinitum extendi observavi; cum tamen pro quovis casu eos tantum terminos accipi oporteat, qui a fractionibus sint liberati; unde mihi in mentem venit in valores istarum serierum, si in infinitum continuentur, inquirere. Mox autem facili ratiocinio intellexi, earum summam eandem potestatem solius maximae radicis, puta α^n, definire.

§. 3. Cum igitur seriem infinitam essem adeptus, quae potestatem exponentis n maximae radicis, scilicet α, exprimeret, mox perspexi istam seriem egregie cum Lambertina convenire; tum vero haud amplius difficile erat similes series pro omnibus aequationibus sub hac forma multo generaliori contentas:

$$1 = \frac{A}{x^\alpha} + \frac{B}{x^\beta} + \frac{C}{x^\gamma} + \frac{D}{x^\delta} + \text{etc.}$$

exhibere, ubi exponentibus α, β, γ, δ, etc. adeo omnes plane valores sive positivos, sive negativos, sive integros, sive fractos tribuere licet. Nihilo vero minus singuli huius seriei termini egregio ordine procedunt, hosque adeo, quousque libue-

libuerit, facile continuare licet, ita ut hic nihil plane inductioni vel coniecturae concedere necesse sit.

§. 4. Cum autem haec methodus ex principio prorsus alieno, et per ambages non parum molestas, sit deducta, plurimum laboravi, ut methodum magis directam, et faciliori negotio ad scopum perducentem, perscrutarer, quin etiam labores meos in aliquot dissertationibus cum Academia communicatis accuratius exposui. Nunc autem idem argumentum retractans in methodum longe faciliorem, ac per nullas ambages procedentem, incidi, quam hoc loco clarius explicare constitui.

§. 5. Hic igitur consideraturus sum aequationem algebraicam sub hac forma generalissima contentam:

$$1 = \frac{A}{x^\alpha} + \frac{B}{x^\beta} + \frac{C}{x^\gamma} + \frac{D}{x^\delta} + \text{etc.}$$

ubi ante omnia facile patet sine ulla restrictione loco litterae A unitatem scribi posse, ita ut aequatio, quam hic tractare suscipio, sit:

$$1 = \frac{1}{x^\alpha} + \frac{B}{x^\beta} + \frac{C}{x^\gamma} + \frac{D}{x^\delta} + \text{etc.}$$

ex qua statim patet, si litterae B, C, D, etc. evanescerent, fore $x = 1$; unde sequitur in genere radicem x certe seriei infinitae aequalem statui posse, cuius primus terminus sit unitas, sequentes vero litteras B, C, D, utcunque inter se compositas, complectantur, quandoquidem in eam praeter ipsas has litteras singulas tam omnia producta ex binis quam ex ternis et pluribus ingredi debent.

 §. 6.

§. 6. Quoniam autem hic mihi propositum est non solum in ipsam radicem x, sed in genere in eius potestatem quamcunque x^n inquirere, ipsam aequationem hac forma repraesentabo:

$$x^n - x^{n-\alpha} = B x^{n-\beta} + C x^{n-\gamma} + D x^{n-\delta} + \text{etc.}$$

ubi primum terminum a dextra in sinistram transtuli, ut ad dextram tantum litterae B, C, D, etc. cum suis potestatibus coniunctae occurrant, ex quarum permistione verum valorem potestatis x^n investigari oportet; ubi facile perspicitur eiusmodi seriem pro x^n prodire debere, in qua post terminum primum 1 non solum singulae litterae B, C, D, etc. sed etiam omnia producta tam ex binis quam ternis pluribusque occurrere debeant, ita ut totum negotium iam huc redeat, ut singulis his productis debiti coëfficientes assignentur, qui utique potissimum pendebunt ab exponente n, praeter reliquos exponentes datos α, β, γ, δ, ε, etc.

§. 7. Hic autem plurimum iuvabit istos coëfficientes per idoneos characteres repraesentare, quibus scilicet omnis confusio ex infinita multitudine terminorum orionda evitari queat. Ita coëfficientes ipsarum litterarum B, C, etc. quatenus ad potestatem exponentis n referuntur, his signis denotabo: $(\overset{n}{B})$, $(\overset{n}{C})$, $(\overset{n}{D})$, etc. Vnde si alius quicunque exponens, puta m, proponeretur, hi coëfficientes ita forent designandi: $(\overset{m}{B})$, $(\overset{m}{C})$, $(\overset{m}{D})$, etc., quod idem tenendum est de omnibus productis ex binis pluribusve harum litterarum compositis. Veluti si in genere occurrat hoc productum $B^b. C^c. D^d.$ etc., eius coëfficientem pro potestate exponentis m ita sum repraesentaturus: $(\overset{m}{B^b. C^c. D^d. \text{etc.}})$.

§. 8.

§. 8. Hoc igitur signandi modo constituto valor potestatis quaesitae x^n per huiusmodi seriem exprimetur:

$$\begin{aligned} x^n = 1 &+ (\overset{n}{B})B + (\overset{n}{C})C + (\overset{n}{D})D + (\overset{n}{E})E + \text{etc.} \\ &+ (\overset{n}{B^2})B^2 + (\overset{n}{C^2})C^2 + (\overset{n}{D^2})D^2 + (\overset{n}{E^2})E^2 + \text{etc.} \\ &+ (\overset{n}{BC})BC + (\overset{n}{BD})BD + (\overset{n}{BE})BE + \text{etc.} \end{aligned}$$

cuius ergo seriei terminus generalis omnes plane in se complectens erit

$$(\overset{n}{B^b.\,C^c.\,D^d.\,E^e.\,\text{etc.}})\; B^b.\,C^c.\,D^d.\,E^e.\,\text{etc.}$$

§. 9. Ne autem hic opus sit calculum ad plures horum terminorum simul applicare, praecipuum momentum huc redit, ut singulos coëfficientes ex paucioribus terminis iam cognitis investigare doceamus. Ac primo quidem si quaeratur coëfficiens $(\overset{n}{B})$, sive terminus $(\overset{n}{B})\,B$, pro potestate x^n, evidens est pro $x^{n-\alpha}$ hunc terminum fore $(\overset{n-\alpha}{B})\,B$, unde ex ipsa aequatione, quatenus hic tantum de terminis formae B agitur, erit

$$(\overset{n}{B})\,B - (\overset{n-\alpha}{B})\,B = B\,x^{n-\beta} = B;$$

propterea quod potestas $x^{n-\beta}$ nullas harum litterarum involvere debet, ideoque pro $x^{n-\beta}$ scribi debet unitas, utpote prima pars valoris veri. Quoniam nunc hic per B dividi potest, habebimus pro coëfficiente quaesito $(\overset{n}{B})$ hanc aequationem: $(\overset{n}{B}) - (\overset{n-\alpha}{B}) = 1$; simili modo pro reliquis habebimus has aequationes:

$$(\overset{n}{C}) - (\overset{n-\gamma}{C}) = 1;\; (\overset{n}{D}) - (\overset{n-\alpha}{D}) = 1;\; (\overset{n}{E}) - (\overset{n-\alpha}{E}) = 1;\; \text{etc.}$$

§. 10. Sin autem quaeramus coëfficientem $(\overset{n}{B^2})$, evidens eſt ex parte ſiniſtra noſtrae aequationis ſolum terminum $B\,x^{n-\beta}$ in computum venire, quia nullae aliae litterae hic occurrunt. Erit igitur

$(\overset{n}{B^2})\,B^2 + (\overset{n-\alpha}{B^2})\,B^2 = (\overset{n-\beta}{B})\,B^2$; ergo per B^2 dividendo

$(\overset{n}{B^2}) - (\overset{n-\alpha}{B^2}) = (\overset{n-\beta}{B})$. Simili modo erit

$(\overset{n}{C^2}) - (\overset{n-\alpha}{C^2}) = (\overset{n-\gamma}{C})$, tum vero

$(\overset{n}{D^2}) - (\overset{n-\alpha}{D^2}) = (\overset{n-\delta}{D})$,

etc.

§. 11. Sin autem porro quaeratur coëfficiens $(\overset{n}{BC})$ ſive terminus $(\overset{n}{BC})\,B\,C$, manifeſtum eſt ex parte aequationis dextra binos terminos $B\,x^{n-\beta}$ et $C\,x^{n-\gamma}$ hic in ſubſidium vocari debere. Vt enim forma $B\,C$ reſultet, pro priore parte pro $x^{n-\beta}$ ſumi debet $(\overset{n-\beta}{C})\,C$; pro poſteriore autem loco $x^{n-\gamma}$ ſcribi debet $(\overset{n-\gamma}{B})\,B$, ſicque noſtra aequatio per $B\,C$ diviſa erit:

$(\overset{n}{BC}) - (\overset{n-\alpha}{BC}) = (\overset{n-\beta}{C}) + (\overset{n-\gamma}{B})$. Eodem modo erit

$(\overset{n}{BD}) - (\overset{n-\alpha}{BD}) = (\overset{n-\beta}{D}) + (\overset{n-\delta}{B})$

$(\overset{n}{CD}) - (\overset{n-\alpha}{CD}) = (\overset{n-\gamma}{D}) + (\overset{n-\delta}{C})$

et ita porro.

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ſimilique modo evidens eſt fore

$$(B\overset{n}{C}D) - (B\overset{n-\alpha}{C}D) = (\overset{n-\beta}{CD}) + (\overset{n-\gamma}{BC}) + (\overset{n-\delta}{BC})$$

atque porro

$$(BC\overset{n}{D}E) - (BC\overset{n-\alpha}{D}E) = (\overset{n-\varepsilon}{BCD}) + (\overset{n-\beta}{CDE}) + (\overset{n-\gamma}{BDE}) + (\overset{n-\delta}{BCE}).$$

§. 12. Quod ſi eadem littera ſaepius occurrat, ipſa quidem quadrata iam evolvimus, pro cubis vero habebimus:

$$(\overset{n}{B^3}) - (\overset{n-\alpha}{B^3}) = (\overset{n-\beta}{B^2})$$
$$(\overset{n}{C^3}) - (\overset{n-\alpha}{C^3}) = (\overset{n-\gamma}{C^2})$$
$$(\overset{n}{D^3}) - (\overset{n-\alpha}{D^3}) = (\overset{n-\delta}{D^2})$$

etc.

Eodemque modo erit pro ſuperioribus poteſtatibus

$$(\overset{n}{B^4}) - (\overset{n-\alpha}{B^4}) = (\overset{n-\beta}{B^3})$$
$$(\overset{n}{B^5}) - (\overset{n-\alpha}{B^5}) = (\overset{n-\beta}{B^4})$$
$$(\overset{n}{B^6}) - (\overset{n-\alpha}{B^6}) = (\overset{n-\beta}{B^5})$$

etc.

§. 13. Sin autem plures litterae ingrediantur, ex parte aequationis dextra etiam plures termini in ſubſidium vocari debent, veluti ex ſequentibus formulis pateſcit:

$$(B^2\overset{n}{C}) - (B^2\overset{n-\alpha}{C}) = (\overset{n-\gamma}{B^2}) + (\overset{n-\beta}{BC})$$
$$(B^2\overset{n}{C^2}) - (\overset{n-\alpha}{B^2C^2}) = (\overset{n-\gamma}{B^2C}) + (\overset{n-\beta}{BC^2})$$

(B^3

$$(\overset{n}{B^3 C}) - (\overset{n-\alpha}{B^3 C}) = (\overset{n-\gamma}{B^3}) + (\overset{n-\beta}{B^2 C})$$

$$(\overset{n}{B^3 C^2}) - (\overset{n-\alpha}{B^3 C^2}) = (\overset{n-\gamma}{B^3 C}) + (\overset{n-\beta}{B^2 C^2})$$

$$(\overset{n}{B^3 C^3}) - (\overset{n-\alpha}{B^3 C^3}) = (\overset{n-\gamma}{B^3 C^2}) + (\overset{n-\beta}{B^2 C^3})$$

Simili modo perspicuum est fore

$$(\overset{n}{B^3 C^2 D}) - (\overset{n-\alpha}{B^3 C^2 D}) = (\overset{n-\beta}{B^2 C^2 D}) + (\overset{n-\gamma}{B^3 C D}) + (\overset{n-\delta}{B^3 C^2}).$$

Haecque exempla abunde sufficiunt ad coëfficientes omnium plane productorum per huiusmodi aequationes designandos.

§. 14. Per tales autem aequationes investigatio coëfficientium utcunque complexorum ad coëfficientes simpliciorum productorum reducitur, quos tanquam iam cognitos spectare licet, quandoquidem a determinatione simpliciorum operationes inchoamus. Scilicet si coëfficiens quaesitus quicunque designetur per $\Phi : n$, siquidem tanquam functio ipsius n spectari potest: resolutio omnium harum aequationum revocatur ad hanc formam:

$$\Phi : n - \Phi : (n - \alpha) = \Pi,$$

ubi Π est functio iam cognita litterae n. At vero mox videbimus, huius aequationis resolutionem pro nostro instituto satis commode expediri posse.

§. 15. Resolutio huius aequationis ad calculum differentiarum finitarum est referenda, & perinde ac differentialium quantitatem constantem arbitrariam recipiet. Quare ne hinc ulla incertitudo relinquatur, ante omnia probe est notandum, omnes coëfficientes, quos quaerimus, ita comparaparatos esse debere, ut evanescant posito $n = 0$. Cum enim hoc casu fiat $x^n = 1$, ideoque ipsi primo termino nostrae seriei aequalis, sequentes termini omnes litteras B, C, D, etc. involventes hoc casu evanescere debebunt; unde necesse est ut eorum coëfficientes factorem n involvant.

§. 16. Resolutio autem generalis huius aequationis $\Phi : n - \Phi : (n - \alpha) = \Pi$, parum adiumenti in hoc negotio esset allatura. At vero datur solutio particularis ad nostrum institutum imprimis accommodata, quam hic evolvi conveniet. Denotante scilicet n' ipsam quantitatem variabilem n, constante quapiam c sive auctam sive minutam, ita ut sit $n' = n \pm c$, si fuerit

$$\Phi : n = \Delta\, n\, (n' + \alpha)\, (n' + 2\,\alpha) \ldots\ldots (n' + i\,\alpha)$$

ubi Δ itidem significat quantitatem constantem, erit

$$\Phi : (n - \alpha) = \Delta\, (n - \alpha)\, n' (n' + \alpha)\, (n' + 2\,\alpha) \ldots\ldots [n' + (i - 1)\,\alpha],$$

unde ob factores communes

$$(n' + \alpha)\, (n' + 2\,\alpha) \ldots\ldots [n' + (i - 1)\,\alpha], \text{ erit}$$

$$\Phi : n - \Phi : (n - \alpha) = \Delta\, (n' n + i\, n\, \alpha - n' n + \alpha\, n') \ldots\ldots$$

$$= \Delta\, \alpha\, (n' + i\, n)\, (n' + \alpha)\, (n' + 2\,\alpha) \ldots\ldots n' + (i - 1)\,\alpha$$

quae expressio ergo aequabitur quantitati illi Π, unde sequens Lemma fundamenti loco hic constituamus.

Lemma.

§. 17. *Proposita aequatione resolvenda*

$$\Phi : n - \Phi : (n - \alpha) = \Pi,$$

qualis ista quantitas Π *in hac forma continebitur:*

$\Pi =$

$\Pi = \Delta\alpha(n' + in)[(n' + \alpha)(n' + 2\alpha)(n' + 3\alpha)\ldots[n' + (i - 1)]]$ tum semper erit $\Phi : n = \Delta n(n' + \alpha)(n' + 2\alpha)\ldots(n' + i\alpha)$ existente $n' = n \pm c$. Haec forma iam ita est comparata, ut evanescat posito $n = o$, sicque ad coëfficientes quaesitos definiendos apprime est accommodata.

§. 18. Iam huius Lemmatis beneficio omnes nostros coëfficientes satis expedite determinare licebit; et quia magis compositos perpetuo ex simplicioribus derivari oportet, omnes terminos seriei generalis, quam quaerimus, pro potestate indefinita x^ν in certos ordines distinguamus, quorum primus comprehendat terminos ipsas litteras B, C, D, etc. simpliciter continentes; ad ordinem secundum referamus producta ex binis harum litterarum, cuiusmodi sunt B^2, BC, C^2, etc.; tertius ordo contineat productum ex ternis, cuiusmodi sunt B^3, B^2C, BCD, etc. quartus ordo producta ex quaternis, et ita porro. Pro singulis ergo his ordinibus coëfficientes investigabimus.

Investigatio

Terminorum primi ordinis.

§. 19. Omnium horum terminorum unica est forma B, pro cuius coëfficiente supra habuimus hanc aequationem: $(\overset{n}{B})B - (\overset{n-\alpha}{B}) = 1$; unde posito $(\overset{n}{B}) = \Phi : n$, erit hic $\Pi = 1$. Sumatur ergo in Lemmate praemisso $\Phi : n = \Delta n$, ita ut hic sit $i = o$, et quoniam hinc fit $\Phi : (n - \alpha) = \Delta(n - \alpha)$, erit $\Pi = \Delta\alpha = 1$, unde fit $\Delta = \frac{1}{\alpha}$; quamobrem coëfficiens no-

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fter erit $(\overset{n}{B}) = \frac{n}{\alpha}$, fimilique modo pro caeteris huius ordinis erit $(\overset{n}{C}) = \frac{n}{\alpha}$, $(\overset{n}{D}) = \frac{n}{\alpha}$, etc. ita ut ipfi termini huius ordinis futuri fint $\frac{n}{\alpha}B + \frac{n}{\alpha}C + \frac{n}{\alpha}D + \frac{n}{\alpha}E +$ etc.

Inveftigatio
Terminorum fecundi ordinis.

§. 20. Horum igitur terminorum dabitur duplex forma, vel B^2, vel BC, quorum ergo coëfficientes funt $(\overset{n}{B^2})$, vel $(\overset{n}{BC})$, quos indagari oportet. Pro priore autem fupra iam dedimus hanc aequationem: $(\overset{n}{B^2}) - (\overset{n'}{B^2}) = (\overset{n\ \beta}{B})$; ita ut pofito $(\overset{n}{B^2}) = \Phi : n$ fit $\Pi = (\overset{n\ \beta}{B})$. Cum igitur modo invenerimus effe $(\overset{n}{B}) = \frac{n}{\alpha}$, erit $\Pi = \frac{n-\beta}{\alpha}$. In Lemmate igitur fiat $i = 1$, ita ut fit $\Phi : n = \Delta n (n' + \alpha)$, unde oritur $\Pi = \Delta \alpha (n' + n) = \frac{n-\beta}{\alpha}$. Hinc ergo reftituto $n' = n + c$, erit $\frac{n-\beta}{\alpha} = \Delta \alpha (2n + c)$, unde fequitur fore $2 \Delta \alpha n = \frac{n}{\alpha}$ et $\Delta \alpha c = -\frac{\beta}{\alpha}$, unde fit $\Delta = \frac{1}{2\alpha\alpha}$ et $c = -\frac{\beta}{\Delta\alpha\alpha} = -2\beta$, ita ut fit $n' = n - 2\beta$.

§. 21. Cum igitur fit $\Delta = \frac{1}{2\alpha\alpha}$ et $n' = n - 2\beta$, erit coëfficiens ipfius B^2 quaefitus, fcilicet

$$(\overset{n}{B^2}) = \frac{n(n+\alpha-2\beta)}{2\alpha\alpha} = \frac{n}{\alpha} \cdot \frac{n+\alpha-2\beta}{2\alpha}.$$

Quare termini fecundi ordinis formae B^2 erunt fequentes:

$$\frac{n}{\alpha} \cdot \frac{n+\alpha-2\beta}{2\alpha} \cdot B^2 + \frac{n}{\alpha} \cdot \frac{n+\alpha-2\gamma}{2\alpha} C^2 + \frac{n}{\alpha} \cdot \frac{n+\alpha-2\delta}{2\alpha} \cdot D + \text{etc.}$$

§. 22. Pro altera forma B C supra attulimus hanc aequationem:

$$(\overset{n}{B}C) - (\overset{n-\gamma}{B}C) = (\overset{n-\beta}{C}) + (\overset{n-\gamma}{B}).$$

Cum igitur sit $(\overset{n}{B}) = (\overset{n}{C}) = \frac{n}{\alpha}$, si ponamus $(\overset{n}{B}C) = \Phi : n$, erit

$$\Pi = \frac{n-\beta}{\alpha} + \frac{n-\gamma}{\alpha} = \frac{\alpha n - \beta - \gamma}{\alpha}.$$

In Lemmate igitur nostro sumamus $i = 1$, ut sit

$\Phi : n = \Delta n (n' + \alpha)$, fierique debet

$$\Delta \alpha (n' + n) = \frac{2n - \beta - \gamma}{\alpha}.$$

Sumatur ergo primo $n' = n - \beta - \gamma$, ut fiat $\Delta \alpha = \frac{1}{\alpha}$, ideoque $\Delta = \frac{1}{\alpha\alpha}$, sicque erit coëfficiens quaesitus

$$(\overset{n}{B}C) = \frac{n(2n - \beta - \gamma)}{\alpha\alpha} = \frac{2n}{\alpha} \cdot \frac{n + \alpha - \beta - \gamma}{2\alpha}.$$

§. 23. Hinc ergo pro secundo ordine termini formae BC erunt $(\overset{n}{B}C) = \frac{2n}{\alpha} \cdot \frac{n + \alpha - \beta - \gamma}{2\alpha}$, hincque

$$\frac{n}{\alpha} \cdot \frac{n+\alpha-\beta-\gamma}{2\alpha} \cdot 2\,BC + \frac{n}{\alpha} \cdot \frac{n+\alpha-\beta-\delta}{2\alpha} \cdot 2\,BD$$
$$+ \frac{n}{\alpha} \cdot \frac{n+\alpha-\gamma-\delta}{2\alpha} \cdot 2\,CD + \text{etc.}$$

quibus si adiungantur termini formae B^2 modo ante inventi, totus ordo secundus iam est absolutus.

Investigatio

Terminorum tertii ordinis.

§. 24. Prima forma in hoc ordine occurrens est B^3, pro cuius coëfficiente supra nacti sumus hanc aequationem:

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$$(\overset{n}{B^3}) - (\overset{n-\alpha}{B^3}) = (\overset{n-\beta}{B^3}).$$

Cum igitur modo invenerimus

$$(\overset{n}{B^2}) = \frac{n(n+\alpha-2\beta)}{\alpha . 2\alpha}, \text{ erit hic}$$

$$(\overset{n-\beta}{B^2}) = \Pi = \frac{n-\beta}{\alpha} . \frac{n+\alpha-3\beta}{2\alpha}.$$

Hinc in Lemmate praemiſſo ſumamus $i = 2$, ut inde fiat

$$\Pi = \Delta\,\alpha\,(n' + 2n)(n' + \alpha).$$

Vt igitur poſteriores factores evadant aequales, ſtatui debet $n' = n - 3\beta$, quo factore communi ſublato relinquetur haec aequatio:

$$\Delta\,\alpha\,(n' + 2n) = \Delta\,\alpha\,(3n - 3\beta) = \frac{n-\beta}{2\alpha^2}.$$

Hic igitur commode diviſio per $n - \beta$ ſuccedit, ita ut hinc fiat $\Delta = \frac{1}{6\alpha^3}$, ſicque coëfficiens quaeſitus erit

$$(\overset{n}{B^3}) = \frac{n(n+\alpha-3\beta)(n+2\alpha-3\beta)}{\alpha . 2\alpha . 3\alpha};$$

unde per ſe patet termini C^3 coëfficientem fore

$$(\overset{n}{C^3}) = \frac{n(n+\alpha-3\gamma)(n+2\alpha-3\gamma)}{\alpha . 2\alpha . 3\alpha}.$$

§. 25. Secunda forma huius ordinis erit B^2C, pro cuius coëfficiente ſupra reperimus hanc aequationem:

$$(\overset{n}{B^2C}) - (\overset{n-\alpha}{B^2C}) = (\overset{n-\gamma}{B^2}) + (\overset{n-\beta}{BC}) = \Pi,$$

ergo ex valoribus iam inventis derivamus iſtas duas partes:

$$(\overset{n-\gamma}{B^2}) = \frac{(n-\gamma)(n+\alpha-2\beta-\gamma)}{\alpha . 2\alpha};$$

deinde

$$(\overset{n-\beta}{B\,C}) = \frac{2(n-\beta)(n+\alpha-2\beta-\gamma)}{\alpha.\,2\alpha},$$

ubi evidens eſt poſteriores factores aequales inter ſe prodire debuiſſe; unde ex additione orietur

$$\Pi = \frac{(3n-2\beta-\gamma)(n+\alpha-2\beta-\gamma)}{\alpha.\,2\alpha}.$$

In Lemmate igitur noſtro ſumi debet $i = 2$, indeque fiet

$$\Pi = \Delta\,\alpha\,(n' + 2\,n)\,(n' + \alpha);$$

quare, ut poſteriores factores congruant, ſumi debet $n' = n - 2\beta - \gamma$, quibus ſublatis remanebit haec aequatio:

$$\frac{3n-2\beta-\gamma}{2.2} = \Delta\,\alpha\,(3\,n - 2\,\beta - \gamma),$$

ubi iterum diviſio per $3\,n - 2\,\beta - \gamma$ ſuccedit, ita ut hinc $\Delta = \frac{1}{2\,\alpha^3}$, conſequenter coëfficiens quaeſitus erit

$$(\overset{n}{B^2\,C}) = \frac{n(n+\alpha-2\beta-\gamma)(n+2\alpha-2\beta-\gamma)}{\alpha.\,2\alpha.\,\alpha},$$

ſive hoc modo

$$\tfrac{1}{3}(\overset{n}{B^2\,C}) = \frac{n(n+\alpha-2\beta-\gamma)(n+2\alpha-2\beta-\gamma)}{\alpha.\,2\alpha.\,3\alpha}.$$

§. 26. Tertia denique forma huius ordinis eſt BCD pro cuius coëfficiente ſupra data eſt haec aequatio:

$$(\overset{n}{B\,C\,D}) - (\overset{n-\alpha}{B\,C\,D}) = (\overset{n-\beta}{C\,D}) + (\overset{n-\gamma}{B\,D}) + (\overset{n-\delta}{B\,C}) = \Pi,$$

ita ut hic Π componatur ex tribus partibus, quae ad praeſentes indices reductae erunt

$$1^\circ.\ (\overset{n-\beta}{C\,D}) = \frac{2(n-\beta)(n+\alpha-\beta-\gamma-\delta)}{\alpha.\,2\alpha}$$

$$2^\circ.\ (\overset{n-\gamma}{B\,D}) = \frac{2(n-\gamma)(n+\alpha-\beta-\gamma-\delta)}{\alpha.\,2\alpha}$$

$$3^\circ.\ (\overset{n-\delta}{B\,C}) = \frac{2(n-\delta)(n+\alpha-\beta-\gamma-\delta)}{\alpha.\,2\alpha}$$

ub

ubi evidens eſt poſteriores factores neceſſario inter ſe aequales prodire debuiſſe, ſicque his iunctis erit

$$\Pi = \frac{2(3n - \beta - \gamma - \delta)(n + \alpha - \beta - \gamma - \delta)}{\alpha \cdot 2\alpha}.$$

In noſtro igitur Lemmate ſumi oportet $i = 2$, ut inde prodeat $\Pi = \Delta\,\alpha\,(n' + 2n)(n' + \alpha)$; ubi manifeſto ſumi debet $n' = n - \beta - \gamma - \delta$; ſicque etiam priores factores tolli poterunt, hincque concludetur fore $\Delta = \frac{1}{\alpha^3}$, conſequenter coëfficiens quaeſitus producti B C D erit

$$(\overset{n}{\text{B C D}}) = \frac{6n(n - \alpha - \beta - \gamma - \delta)(n + 2\alpha - \beta - \gamma - \delta)}{\alpha \cdot 2\alpha \cdot 3\alpha}.$$

Inveſtigatio
Terminorum quarti ordinis.

§. 27. Prima forma in hoc ordine occurrens, quando ſcilicet omnes quatuor factores ſunt inter ſe aequales, eſt B^4, pro cuius coëfficiente ſupra haec aequatio eſt data:

$$(\overset{n}{B^4}) - (\overset{n-\alpha}{B^4}) = (\overset{n-\beta}{B^3}) = \Pi.$$

Cum igitur modo invenerimus

$$(\overset{n}{B^3}) = \frac{n}{\alpha} \cdot \frac{n + \alpha - 3\beta}{2\alpha} \cdot \frac{n + 2\alpha - 3\beta}{3\alpha}, \text{ erit}$$

$$(\overset{n-\beta}{B^3}) = \frac{n - \beta}{\alpha} \cdot \frac{n + \alpha - 4\beta}{2\alpha} \cdot \frac{n + 2\alpha - 4\beta}{3\alpha} = \Pi.$$

Quia hic habentur tres factores, in Lemmate praemiſſo ſumi debet $i = 3$, indeque orietur

$$\Pi = \Delta\,\alpha\,(n' + 3n)(n' + \alpha)(n' + 2\alpha);$$

ubi bini poſteriores factores ſponte ſe tollunt, ponendo $n' = n - 4\beta$; tum autem relinquetur haec aequatio: $\frac{n - \beta}{6\alpha^3} =$

$4\Delta\alpha(n-\beta)$, unde fit $\Delta = \frac{1}{24\alpha^4}$, ficque coëfficiens qua-
tus pro forma B^4 erit

$$\overset{n}{(B^4)} = \frac{n}{\alpha} \cdot \frac{n+\alpha-4\beta}{2\alpha} \cdot \frac{n+2\alpha-4\beta}{3\alpha} \cdot \frac{n+3\alpha-4\beta}{4\alpha}.$$

§. 28. Secunda forma hic occurrens eft B^3C, p
cuius coëfficiente fupra dedimus hanc aequationem:

$$\overset{n}{(B^3C)} - \overset{n-\alpha}{(B^3C)} = \overset{n-\gamma}{(B^3)} + \overset{n-\beta}{(B^2C)} = \Pi.$$

Colligantur ergo ex formis fupra inventis hae duae part-
ac reperietur

$$\overset{n-\gamma}{(B^3)} = \frac{n-\gamma}{\alpha} \cdot \frac{n+\alpha-3\beta-\gamma}{2\alpha} \cdot \frac{n+2\alpha-3\beta-\gamma}{3\alpha}$$

$$\overset{n-\beta}{(B^2C)} = \frac{3(n-\beta)}{\alpha} \frac{n+\alpha-3\beta-\gamma}{2\alpha} \cdot \frac{n+2\alpha-3\beta-\gamma}{3\alpha};$$

vnde oritur

$$\Pi = \frac{4n-3\beta-\gamma}{\alpha} \cdot \frac{n+\alpha-3\beta-\gamma}{2\alpha} \cdot \frac{n+2\alpha-3\beta-\gamma}{3\alpha}.$$

In Lemmate ergo pro hoc cafu fumi debet $i = 3$, ut pr
deat inde

$$\Pi = \Delta\alpha(n'+3n)(n'+\alpha)(n'+2\alpha),$$

ubi ftatim patet fumi debere $n' = n-3\beta-\gamma$, hocque m
do omnes factores litteram n involventes fe tolli patiuntu
quo facto reperietur $\Delta = \frac{1}{6\alpha^4}$, confequenter formae B^3C co-
ciens quaefitus erit

$$\overset{n}{(B^3C)} = \frac{4n}{\alpha} \cdot \frac{n+\alpha-3\beta-\gamma}{2\alpha} \cdot \frac{n+2\alpha-3\beta-\gamma}{3\alpha} \cdot \frac{n+3\alpha-3\beta-\gamma}{4\alpha}.$$

§. 29. Tertia forma in hoc ordine eft B^2C^2, pro c
ius coëfficiente fupra data eft haec aequatio:

(I

$$(\overset{n}{B^2 C^2}) - (\overset{n-\alpha}{B^2 C^2}) = (\overset{n-\gamma}{B^2 C}) + (\overset{n-\beta}{B\,C^2}) = \Pi.$$

Hic vero erit primo

$$(\overset{n-\gamma}{B^2 C}) = \frac{3(n-\gamma)}{\alpha} \cdot \frac{n+\alpha-2\beta-2\gamma}{2\alpha} \cdot \frac{n+2\alpha-2\beta-2\gamma}{3\alpha}$$

$$(\overset{n-\beta}{B\,C^2}) = \frac{3(n-\beta)}{\alpha} \cdot \frac{n+\alpha-2\beta-2\gamma}{2\alpha} \cdot \frac{n+2\alpha-2\beta-2\gamma}{3\alpha},$$

unde fit

$$\Pi = \frac{2n-\beta-\gamma}{2\alpha\alpha} \cdot (n+\alpha-2\beta-2\gamma)(n+2\alpha-2\beta-2\gamma).$$

Quare fi in Lemmate fumamus $i = 3$, quoque ut ante effe debet

$$\Pi = \Delta\,\alpha\,(n' + 3\,n)(n' + \alpha)(n' + 2\,\alpha),$$

ubi manifefto fumi debet $n' = n - 2\beta - 2\gamma$, quo facto reperitur $\Delta = \frac{1}{4\alpha^4}$, ficque huius formae B^2C^2 coëfficiens quaefitus erit

$$(\overset{n}{B^2 C^2}) = \frac{6n}{\alpha} \cdot \frac{n+\alpha-2\beta-2\gamma}{2\alpha} \cdot \frac{n+2\alpha-2\beta-2\gamma}{3\alpha} \cdot \frac{n+3\alpha-2\beta-2\gamma}{4\alpha}.$$

§. 30. Quarta forma ad hunc ordinem referenda eft $B^2 C D$, pro cuius coëfficiente ex principiis fupra ftabilitis haec aequatio ftatui debet:

$$(\overset{n}{B^2 C D}) - (\overset{n-\alpha}{B^2 C D}) = (\overset{n-\delta}{B^2 C}) + (\overset{n-\gamma}{B^2 D}) + (\overset{n-\beta}{B\,C\,D}) = \Pi,$$

ficque Π conftat ex tribus partibus

$$(\overset{n-\delta}{B^2 C}) = \frac{3(n-\delta)}{\alpha} \cdot \frac{n+\alpha-2\beta-\gamma-\delta}{2\alpha} \cdot \frac{n+2\alpha-2\beta-\gamma-\delta}{3\alpha}$$

$$(\overset{n-\gamma}{B^2 D}) = \frac{3(n-\gamma)}{\alpha} \cdot \frac{n+\alpha-2\beta-\gamma-\delta}{2\alpha} \cdot \frac{n+2\alpha-2\beta-\gamma-\delta}{3\alpha}$$

$$(\overset{n-\beta}{B\,C\,D}) = \frac{6(n-\beta)}{\alpha} \cdot \frac{n+\alpha-2\beta-\gamma-\delta}{2\alpha} \cdot \frac{n+2\alpha-2\beta-\gamma-\delta}{3\alpha}$$

qui-

quibus ergo collectis fit

$$\Pi = \frac{4n - 2\beta - \gamma - \delta}{2 \cdot 3} (n + \alpha - 2\beta - \gamma - \delta)(n + 2\alpha - 2\beta - \gamma - \delta).$$

cui ex Lemmate haec expressio:

$$\Delta\, \alpha\, (n' + 3n)(n' + \alpha)(n' + 2\alpha)$$

reddi debet aequalis, quod egregie succedet sumendo $n' = n - 2\beta - \gamma - \delta$; hinc enim colligetur $\Delta = \frac{1}{2\alpha^4}$, consequenter huius formae B^2CD coëfficiens debitus ita exprimetur:

$$(B^2 \overset{n}{C} D) = \frac{12n}{\alpha} \cdot \frac{n + \alpha - 2\beta - \gamma - \delta}{2\alpha} \cdot \frac{n + 2\alpha - 2\beta - \gamma - \delta}{3\alpha} \cdot \frac{n + 3\alpha - 2\beta - \gamma - \delta}{4\alpha}.$$

§. 31. Postrema forma huius ordinis est BCDE, pro cuius coëfficiente habetur haec aequatio:

$$(BC\overset{n}{D}E) - (B\overset{n-\alpha}{C}DE) = (B\overset{n-\varepsilon}{C}D) + (BC\overset{n-\delta}{E}) + (B\overset{n-\gamma}{D}E) + (C\overset{n-\beta}{D}E).$$

Has igitur quatuor partes hic evolvamus

$$(B\overset{n-\varepsilon}{C}D) = \frac{6(n-\varepsilon)}{\alpha} \cdot \frac{n + \alpha - \beta - \gamma - \delta - \varepsilon}{2\alpha} \cdot \frac{n + 2\alpha - \beta - \gamma - \delta - \varepsilon}{3\alpha}$$

$$(B\overset{n-\delta}{C}E) = \frac{6(n-\delta)}{\alpha} \cdot \frac{n + \alpha - \beta - \gamma - \delta - \varepsilon}{2\alpha} \cdot \frac{n + 2\alpha - \beta - \gamma - \delta - \varepsilon}{3\alpha}$$

$$(B\overset{n-\gamma}{D}E) = \frac{6(n-\gamma)}{\alpha} \cdot \frac{n + \alpha - \beta - \gamma - \delta - \varepsilon}{2\alpha} \cdot \frac{n + 2\alpha - \beta - \gamma - \delta - \varepsilon}{3\alpha}$$

$$(C\overset{n-\beta}{D}E) = \frac{6(n-\beta)}{\alpha} \cdot \frac{n + \alpha - \beta - \gamma - \delta - \varepsilon}{2\alpha} \cdot \frac{n + 2\alpha - \beta - \gamma - \delta - \varepsilon}{3\alpha}$$

His igitur in unam summam collectis erit

$$\Pi = \frac{4n - \beta - \gamma - \delta - \varepsilon}{3} (n + \alpha - \beta - \gamma - \delta - \varepsilon)(n + 2\alpha - \beta - \gamma - \delta - \varepsilon).$$

Quare ut Lemmatis forma pro Π data huic evadat aequalis, manifesto sumi debet $n = n - \beta - \gamma - \delta - \varepsilon$, unde fit $\Delta = \frac{1}{\alpha^4}$, sicque istius formae B C D E coëfficiens erit

$$(BC\overset{n}{D}E) = \frac{24n}{\alpha} \cdot \frac{n + \alpha - \beta - \gamma - \delta - \varepsilon}{2\alpha} \cdot \frac{n + 2\alpha - \beta - \gamma - \delta - \varepsilon}{3\alpha} \cdot \frac{n + 3\alpha - \beta - \gamma - \delta - \varepsilon}{4\alpha}$$

Con-

Conclusio generalis.

§. 32. Lex qua istae expressiones ulterius progrediuntur, iam ita est manifesta, ut superfluum foret has operationes ulterius continuare, id quod unico exemplo illustrasse sufficiet. Sit igitur proposita haec forma ordinis noni $B^4 . C^3 . D^2$, cuius coëfficiens naturalis ex lege combinationis ortus est, ut constat,

$$N = \frac{1 . 2 . 3 \ldots\ldots 9}{1 . 2 . 3 . 4 \times 1 . 2 . 3 \times 1 . 2}.$$

Iam si brevitatis gratia ponamus $4\beta + 3\gamma + 2\delta = \lambda$, coëfficiens huius formae pro nostro instituto erit

$$(B^4 C^3 D^2)^{n} = N . \frac{n}{\alpha} . \frac{n+\alpha-\lambda}{2\alpha} . \frac{n+2\alpha-\lambda}{3\alpha} . \frac{n+3\alpha-\lambda}{4\alpha} . \frac{n+4\alpha-\lambda}{5\alpha} \ldots\ldots \frac{n+8\alpha-\lambda}{9\alpha}$$

ubi si loco N valorem modo datum substituamus, nanciscemur

$$(B^4 C^3 D^2)^{n} = \frac{1}{\alpha^9} \frac{n(n+\alpha-\lambda)(n+2\alpha-\lambda)\ldots\ldots(n+8\alpha-\lambda)}{1 . 2 . 3 . 4 \times 1 . 2 . 3 \times 1 . 2}.$$

§. 33. Hinc iam in genere pro producto $B^b . C^c . D^d . E^e$. etc. eundem coëfficientem reperimus, quem iam olim in Tomo Comentar. XV. ex longe aliis principiis elicueram; scilicet, si summa omnium exponentium $b + c + d + e +$ etc. $= i$, quo numero ordo, ad quem hoc productum est referendum, indicatur; tum vero statuatur

$$b\beta + c\gamma + d\delta + e\varepsilon + \text{etc.} = \lambda,$$

coëfficiens istius producti ita exprimetur:

$$\frac{1}{\alpha^i} . \frac{n(n+\alpha-\lambda)(n+2\alpha-\lambda)(n+3\alpha-\lambda)\ldots\ldots[n+(i-1)\alpha-\lambda]}{1 . 2 . 3 \ldots b \times 1 . 2 . 3 \ldots c \times 1 . 2 . 3 \ldots d \times 1 . 2 . 3 \ldots e \times \text{etc.}}.$$

§. 34. Quod si ergo omnia possibilia producta litterarum B, C, D, E, etc. cum his coëfficientibus coniuncta

 in

in unam fummam colligantur, eique unitas praefigatur, habetur valor poteftatis indefinitae x^n, qui convenit huic aequationi algebraicae:

$$1 = \frac{1}{x^\alpha} + \frac{B}{x^\beta} + \frac{C}{x^\gamma} + \frac{D}{x^\delta} + \frac{E}{x^\varepsilon} + \text{etc.}$$

§. 35. In hac aequatione omnes numeratores litteris diverfis B, C, D, E, etc. defignavimus, propterea quod ad diverfas poteftates ipfius x referuntur. Vnde intelligitur, etiamfi forte effet $C = D$, tamen producti quod effet B^2 coefficientem neutiquam ex forma $(\overset{n}{B}B)$, fed perpetuo ex forma $(\overset{n}{B}C)$ repeti debere.

§. 36. Quoniam ope huius methodi valorem cuiuscunque poteftatis indefinitae x^n formavimus, nil certe facilius eft, quam hinc ipfam aequationis propofitae radicem x determinare, ponendo fcilicet $n = 1$. Vnde hoc infigne Paradoxon fe offert, quod eadem methodus nullum plane ufum praeftatura fuiffet, fi eius ope ipfam radicem x elicere voluiffemus, propterea quod vis iftius methodi in eo ipfo eft conftituenda, quod poteftatem plane indefinitam x^n iam ftatim ab initio fimus contemplati, unde omnium aliarum poteftatum $x^{n-\alpha}$, $x^{n-\beta}$, $x^{n-\gamma}$, etc. valores exprimere licuit. Ceterum alia infignia Phaenomena, quae feries hoc modo formatae offerunt, hic non commemoranda cenfeo, cum hoc argumentum iam alibi fufius fim profecutus, hoc vero loco mihi potiffimum fuerit propofitum methodum directam in medium afferre, tales feries fatis expedite inveniendi.

D. I