
E X E M P L A
QUARUNDAM
MEMORABILIUM AEQUATIONUM
DIFFERENTIALIUM,

quas adeo algebraice integrare licet,
etiamsi nulla via pateat variables a se invicem
separandi.

Auctore *L. EVLERO.*

Conventui exhibita die 19 Ian. 1778.

§. 1.

Facile quidem est hujusmodi aequationes, quotquot lubuerit, exhibere, quarum integralia assignari queant. Si enim pro V accipiatur quaecunque functio binarum variarum x et y , ita ut sit $\partial V = M\partial x + N\partial y$, evidens est huic aequationi differentiali $\partial x (P.V + MS) + \partial y (QV + NS) = 0$ semper satisfacere aequationem finitam $V = 0$. Verum hoc integrale tantum est particulare. Praeterea vero si ejusmodi aequatio proponatur, plerumque haud difficulter ista functio V vel divinando inveniri potest, ita ut hujusmodi aequationes parum in recessu habere sunt censendae. Hic autem tales aequationes in medium sum allaturus, quarum integratio omnes methodos adhuc cognitae respicere videatur, cum tamen nihilominus earum integralia completa, atque adeo algebraica, exhiberi queant.

§. 2. Hujusmodi scilicet aequationes differentiales deducere licet ex hac aequatione differentiali hactenus primum tractata: $\frac{\partial x}{\sqrt{X}} = \frac{\partial y}{\sqrt{Y}}$, in qua est $X = a + 2\beta x + \gamma xx + \delta x^3 + \varepsilon x^4$ et $Y = a + 2\beta y + \gamma yy + 2\delta y^3 + \varepsilon y^4$ cujus integrale completum hac aequatione finita exprimitur.

I. $\frac{\sqrt{X} + \sqrt{Y}}{x - y} = \sqrt{2\lambda + \gamma + 2\delta(x+y) + \varepsilon(x+y)^2}$, vbi λ denotat constantem arbitrariam integratione ingressam quod ergo integrale etiam hoc modo exhiberi potest.

II. $\sqrt{XY} = \lambda(x-y)^2 - a - \beta(x+y) - \gamma xy - \delta xy(x+y) + \varepsilon xxyy$. Quin etiam irrationalitatem penitus tollendo hoc integrale sequentem induet formam:

III. $0 = \lambda\lambda(x-y)^2 - 2\lambda(a + \beta(x+y) + \gamma xy + \delta xy(x+y) + \varepsilon xxyy) + (\beta\beta - a\gamma) - 2a\delta(x+y) - a\varepsilon(x+y)^2 - 2\beta\delta xy - 2\beta\varepsilon xy(x+y) + (\delta\delta - \gamma\varepsilon)xxyy$.

Hinc jam sequentia exempla evolvamus.

Exemplum I.

§. 3. Cum ex aequatione $\frac{\partial x}{\sqrt{X}} = \frac{\partial y}{\sqrt{Y}}$ fit $\frac{\partial x}{\partial y} = \sqrt{\frac{X}{Y}}$ habebimus $\frac{\partial x}{\partial y} = \frac{\sqrt{XY}}{Y}$, vbi, si valores pro Y et \sqrt{XY} in forma integralis secunda substitutamur, prodibit

$$\frac{\partial x}{\partial y} = \frac{\lambda(x-y)^2 - a - \beta(x+y) - \gamma xy - \delta xy(x+y) - \varepsilon xxyy}{a + 2\beta y + \gamma yy + 2\delta y^3 + \varepsilon y^4}$$

quae more solito in ordinem redacta hanc induet formam

$$\partial x(a + 2\beta y + \gamma yy + 2\delta y^3 + \varepsilon y^4) + \partial y(a + \beta(x+y) + \gamma xy + \delta xy(x+y) + \varepsilon xxyy) = \lambda \partial y(x-y)^2$$

cujus aequationis ergo integrale est aequatio finita, quae sub triplici forma exhibuimus, Quoniam autem in hoc integrali nulla nova constans occurrit, quae in differentiali non inest, hoc integrale tantum pro particulari est habendum.

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Interim tamen haec aequatio differentialis jam ita est comparata, ut nemo certe ejus integrale divinando elicere potuerit, cum sex quantitates diversae ibi occurrant. Quin etiam si quatuor adeo litterae evanescant, tamen integrale adhuc satis absconditumprehenditur. Veluti si sumamus $\beta = \gamma = \delta = \varepsilon = 0$, oritur haec aequatio differentialis $\alpha dx + \alpha dy = \lambda dy (x - y)^2$, cujus ergo integrale ex prima forma erit $\frac{2\sqrt{\alpha}}{x-y} = \sqrt{2\lambda}$, sive $x - y = \frac{\sqrt{2\alpha}}{\lambda}$, sive $x = y + \frac{\sqrt{2\alpha}}{\lambda}$, qui valor utique satisfacit, sed tantum particulariter. Pro integrali autem completo inveniendofaturatur $x - y = v$, sive $x = y + v$, unde aequatio differentialis evadet $\partial y = \frac{\alpha \partial v}{\lambda v^2 - \alpha}$, cujus ergo integrale completum sive a logarithmis, sive ab arcibus circularibus pendet.

§. 5. Ponamus nunc esse $\alpha = \gamma = \delta = \varepsilon = 0$, et aequatio nostra differentialis erit

$$2\beta y \partial x + \beta(x+y) \partial y = \lambda \partial y (x-y)^2,$$

cui ergo satisfacit hoc integrale ex I. forma

$$\frac{\sqrt{2\beta x + \sqrt{2\beta} y}}{x-y} = \sqrt{2\lambda}, \text{ vel ex II. forma}$$

$$2\beta \sqrt{xy} = \lambda(x-y)^2 - \beta(x+y).$$

Illa autem forma praebet $\sqrt{x} + \sqrt{y} = (x-y) \sqrt{\frac{\lambda}{\beta}}$, quae divisa per $\sqrt{x} + \sqrt{y}$ dat $1 = (\sqrt{x} - \sqrt{y}) \sqrt{\frac{\lambda}{\beta}}$, sive $\sqrt{x} = \sqrt{y} + \sqrt{\frac{\beta}{\lambda}}$, hincque $x = y + 2\sqrt{\frac{\beta}{\lambda}}\sqrt{y} + \frac{\beta}{\lambda}$, ideoque $\partial x = \partial y + \frac{\partial y \sqrt{\beta}}{\sqrt{\lambda y}}$, qui valores substituti aequationem identicam producant.

§. 6. Cum igitur isti casus simplicissimi jam profundorem indagacionem requirant, hinc evidentissime elucet, si omnes sex litterae in calculo relinquuntur, tum neminem certe

certe vnquam ejus integrale faltem particulare esse erunturum; unde haec ipfa aequatio generalis:

$$\partial x (\alpha + 2\beta y + \gamma y^2 + 2\delta y^3 + \varepsilon y^4) + \partial y (\alpha + \beta(x+y) + \gamma xy + \delta xy(x+y) + \varepsilon xxyy) = \lambda \partial y (x-y)$$

omni attentione maxime digna videtur, cum ejus integrale licet particulare, fit ipfa aequatio supra §. 2. assignata fit triplici forma. In fequentibus autem exemplis hujusmodi aequationes differentiales proferemus, quarum adeo integra lia completa algebraice exhiberi queant.

Exemplum II.

§. 7. Cum fit $\partial x : \partial y = \sqrt{X} : \sqrt{Y}$, erit $\frac{\partial x + \partial y}{\partial x - \partial y} = \frac{\sqrt{X} + \sqrt{Y}}{\sqrt{X} - \sqrt{Y}}$. Jam haec fractio supra et infra multiplicetur per $\sqrt{X} + \sqrt{Y}$ fietque $\frac{\partial x + \partial y}{\partial x - \partial y} = \frac{(\sqrt{X} + \sqrt{Y})^2}{X - Y}$, cujus numerator ex prima forma integralis est $(x - y)^2 (2\lambda + \gamma + 2\delta(x + yy) + \varepsilon(x + y)^2)$ denominator vero erit

$$2\beta(x - y) + \gamma(xx - yy) + 2\delta(x^3 - y^3) + \varepsilon(x^4 - y^4)$$

ficque haec fractio per $x - y$ deprimi potest, ita vt habeamus

$$\frac{\partial x + \partial y}{\partial x - \partial y} = \frac{(x - y)(2\lambda + \gamma + 2\delta(x + y) + \varepsilon(x + y)^2)}{2\beta + \gamma(x + y) + 2\delta(xx + xy + yy) + \varepsilon(x + y)(xx + y^2)}$$

cujus ergo integrale pariter erit ipfa aequatio finita supra assignata, quae cum praeter quantitates constantes, in ipfa aequationem differentialem ingredientibus, quae sunt $\beta, \delta, \varepsilon$ et λ , infuper litteram α contineat, vtique pro integrale completo est habenda.

§. 8. Quo hanc aequationem in ordinem redigamus primo eam in hanc formam convertamus:

$$\frac{\partial x}{\partial y} = \frac{\beta + \lambda(x - y) + \gamma x + \delta x(2x + y) + \varepsilon x x(x + y)}{\lambda(x - y) - \beta - \gamma y - \delta y(2y + x) - \varepsilon y y(x + y)}$$

Nunc igitur
 $\frac{\lambda \partial x (x - y) - \beta \partial y (x - y) - \gamma \partial y (x - y) - \delta \partial y (2y + x) - \varepsilon \partial y y (x + y)}{\lambda \partial y (x - y) - \beta \partial y (x - y) - \gamma \partial y (x - y) - \delta \partial y (2y + x) - \varepsilon \partial y y (x + y)}$
 Hujus aequationis, in solutionem poterit
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igitur fractionibus sublatis prodibit haec aequatio:

$$\frac{\lambda \partial x (x-y) - \beta \partial x - \gamma \gamma \partial x - \delta \gamma \partial x (2\gamma+x) - \epsilon \gamma \gamma \partial x (x+y)}{\lambda \partial y (x-y) - \beta \partial y - \gamma \gamma \partial y - \delta \gamma \partial y (2x+\gamma) - \epsilon \gamma \gamma \partial y (x+y)} = 0.$$

Huius ergo aequationis integrale completum est ipsa illa aequatio finita, quam supra sub triplici forma repraesentavimus, in qua littera α est constans arbitraria per integrationem ingressa, unde ex tertia forma integrale ita referri poterit

$$\alpha (\lambda (x-y)^2 + 2\delta (x+y) + \epsilon (x+y)^2) = \lambda \lambda (x-y)^2 - 2\lambda \beta (x+y) - 2\lambda \gamma \gamma x - 2\lambda \delta \gamma (x+y) - 2\lambda \epsilon x \gamma \gamma + \beta \beta - 2\beta \delta x \gamma - 2\beta \epsilon x \gamma (x+y) + (\delta \delta - \gamma \epsilon) x \gamma \gamma.$$

sive

$$\alpha = \frac{\lambda \lambda (x-y)^2 - 2\lambda \beta (x+y) - 2\lambda \gamma \gamma x - 2\lambda \delta \gamma (x+y) - 2\lambda \epsilon x \gamma \gamma + \beta \beta - 2\beta \delta x \gamma - 2\beta \epsilon x \gamma (x+y) + (\delta \delta - \gamma \epsilon) x \gamma \gamma}{2\lambda + \gamma + 2\delta (x+y) + \epsilon (x+y)^2}$$

§. 9. Quia in hac aequatione plures occurrunt litterae, scilicet $\lambda, \beta, \gamma, \delta, \epsilon$, contemplemur primo casus speciales, quibus duae tantum litterae occurrunt, reliquis ad nihilum redactis.

Casus I.

quo $\gamma = \delta = \epsilon = 0$.

§. 10. Aequatio ergo differentialis erit

$$\lambda \partial x (x-y) - \lambda \partial y (x-y) - \beta \partial x - \beta \partial y = 0$$

sive $\lambda (x+y) (\partial x - \partial y) - \beta (\partial x + \partial y) = 0$

cujus integrale sponte se prodit

$$\lambda (x-y)^2 - 2\beta (x+y) = \text{const.}$$

Generalis vero integralis forma hoc casu praebet

$$\alpha = \frac{\lambda \lambda (x-y)^2 - 2\lambda \beta (x+y) + \beta \beta}{2\lambda}$$

Casus

Casus II.

quo $\beta = \delta = \varepsilon = 0.$

Hoc casu aequatio differentialis erit
 $\lambda \partial x (x - y) - \lambda \partial y (x - y) - \gamma (y \partial x + x \partial y) =$
 cujus integrale pariter sponte se offert, quandoquidem
 $\lambda (x - y)^2 - 2 \gamma xy = \text{const.}$

Ex forma generali integrale fit $\alpha = \frac{\lambda \lambda (x - y)^2 - 2 \lambda \gamma xy}{2 \lambda + \gamma}$
 Quin etiam si fuerit tantum $\delta = \varepsilon = 0$, qui fit

Casus III.

Aequatio differentialis erit
 $\lambda (x - y) (\partial x - \partial y) - \beta (\partial x + \partial y) - \gamma (y \partial x + x \partial y) =$
 cujus integrale est manifesto
 $\lambda (x - y)^2 - 2 \beta (x + y) - 2 \gamma xy = \text{const.}$

Forma generalis autem praebet
 $\alpha = \frac{\lambda \lambda (x - y)^2 - 2 \lambda \beta (x + y) - 2 \lambda \gamma xy + \beta \beta}{2 \lambda + \gamma}$

vbi consensus est manifestus, sicque quoties ambae litterae
 δ et ε evanescunt, res nihil plane habet in recessu; vel
 si litterarum δ et ε , vel altera tantum, vel ambae affuerint
 ejusmodi oriuntur aequationes differentiales, quarum
 ratio per methodos usitatas non parum difficultatis in-
 vit; hujusmodi igitur casus hic data opera evolvamus

Casus IV.

quo $\beta = \gamma = \varepsilon = 0.$

§. II. Hoc ergo casu aequatio differentialis erit
 $\lambda (x - y) (\partial x - \partial y) - \delta \gamma \partial x (2y + x) - \delta x \partial y (2x + y)$
 cujus integrale ex forma generali resultat
 $\alpha = \frac{\lambda \lambda (x - y)^2 - 2 \lambda \delta \gamma (x + y) + \delta \delta x x y y}{2 \lambda + 2 \delta (x + y)}$

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veritas neququam tam clare perspicitur, quam can-
 precedentibus; namque posito brevitatis gratia $\lambda = n\delta$,
 habeatur haec aequatio:

$(x - y)(\partial x - \partial y) = y^{\lambda} x (2y + x) + x \partial y (2x + y)$,
 prius membrum sponte est integrabile, hincque etiam si
 multiplicetur per functionem quamcunque $x - y$. Verum
 nulla hujusmodi functio datur, qua etiam posterius mem-
 brum integrabile reddatur. Ut autem more solito in ejus
 integrale inquiramus, ponamus $x + y = p$ et $x - y = q$,
 ut sit $x = \frac{p+q}{2}$ et $y = \frac{p-q}{2}$, atque aequatio nostra in-
 duet hanc formam: $nq\partial q = \frac{1}{4}\partial p (3pp + qq) - pq\partial q$. Po-
 namus hic $qq = v$, ut sit $2q\partial q = \partial v$, et aequatio nostra
 erit: $2n\partial v + 2p\partial v - v\partial p = 3pp\partial p$. In qua aequatione
 quia v unquam tantum habet dimensionem, ea methodo con-
 sueta resolvi poterit: divisa enim per $2n + 2p$, praebet

$$\frac{\partial v}{2n + 2p} = \frac{v\partial p}{2n + 2p} = \frac{3pp\partial p}{2n + 2p}$$

§. 12. Constat autem hanc aequationem generalem:
 $P\partial v = Q\partial p$, ubi P et Q sint functiones quaecunque
 ipsius p integrabilem reddi, si ducatur in $e^{\int P\partial p}$; tum enim
 integrale fit $e^{\int P\partial p} v = \int e^{\int P\partial p} Q\partial p$. Hinc autem pro nostro
 casu habebimus $P = \frac{-1}{2n + 2p}$ et $Q = \frac{3pp}{2n + 2p}$; quamobrem
 $\frac{\partial v}{2n + 2p} = -\frac{1}{2}l(2n + 2p) + \frac{1}{2}l 2 = -\frac{1}{2}l(n + p)$, ideoque $e^{\int P\partial p}$
 $= \frac{1}{\sqrt{(n+p)}}$, ergo aequatio integralis erit $\frac{v}{\sqrt{(n+p)}} = \frac{3}{2} \int \frac{pp\partial p}{(n+p)^{\frac{3}{2}}}$.
 Pro postremo membro ponatur $n + p = zz$, five $p = zz - n$
 eritque $(n + p)^{\frac{3}{2}} = z^3$, tum vero fiet $\frac{pp\partial p}{(n+p)^{\frac{3}{2}}} = \frac{2\partial z(z^4 - 2nzz + nn)}{z^3}$
 $= 2zz\partial z - 4n\partial z + \frac{2nn\partial z}{zz}$, cujus integrale est
 $z^3 - 4nz - \frac{2nn}{z}$, consequenter nostra aequatio integralis
 erit: $\frac{v}{\sqrt{(n+p)}} = z^3 - 6nz - \frac{3nn}{z} + \text{const.}$

five $\frac{v}{\sqrt{(n+p)}} = (n+p)^{\frac{3}{2}} - 6n\sqrt{(n+p)} - \frac{3n^2}{\sqrt{(n+p)}} + C$, quae aequatio reducitur ad hanc formam:

$$v = (n+p)^2 - 6n(n+p) - 3nn + C\sqrt{(n+p)},$$

five $v = pp - 4np - 8nn + C\sqrt{(n+p)}$.

§. 13. Erat autem $v = qq$, sicque integrale nostrum erit $qq = pp - 4np - 8nn + C\sqrt{(n+p)}$. At vero integrale supra datum, si pariter ad quantitates p et q reducatur, in hanc formam transmutatur:

$$\frac{2\alpha}{\delta} = \frac{nnqq - \frac{np(pp-qq)}{2} + \frac{(pp-qq)^2}{16}}{n+p}$$

$$= \frac{16nnqq - 8np(pp-qq) + (pp-qq)^2}{16(n+p)}$$

Ex forma autem inventa constans arbitraria C hoc modo definitur: $C = \frac{pp - qq - 4np - 8nn}{\sqrt{(n+p)}}$,

cujus quadratum praebet

$$CC = \frac{(pp-qq)^2 - 8np(pp-qq) - 16nn(pp-qq) + 16nnpp + 64n^3p + 64n^4}{(n+p)^2}$$

hincque jam elicitur $\frac{32\alpha}{\delta} - CC = 64n^3$. Unde patet ambae haec integralia perfecte inter se convenire, siquidem tantum quantitate constante a se invicem discrepant.

§. 14. Ob tantas ergo ambages, quibus vti sumus ad integrale eliciendum, iste casus tanto majore attentione dignus est censendus. Interim tamen, quoniam integrale denominatorem habet $n+p$, atque ipsa fractio differentialis nostram aequationem differentialem reproducere debet, necesse est ut ipsa nostra aequatio differentialis

$$4nq\partial q + 4pq\partial q - 3pp\partial p - qq\partial p = 0$$

integrabilis reddatur, si per certam fractionem, quae reperitur $\frac{pp - qq + 4np + 8nn}{(n+p)^2}$, multiplicetur, id quod calculum inveni-

tuen-

per plures demum ambages patebit, si formulam pro $\frac{32 \epsilon}{5}$ laborat exhibitam differentiari voluerit, quem laborem autem hic suscipere non vacat, praesertim postquam consensum amborum integralium jam ostenderit; quam ob causam iste casus maximam attentionem meretur.

Casus V.

quo $\beta = \gamma = \delta = 0$.

§. 15. Hoc ergo casu aequatio differentialis erit $\lambda(x-y)(\partial x - \partial y) - \epsilon(x+y)(yy \partial x + xx \partial y) = 0$, cuius ergo integrale completum erit

$$a = \frac{\lambda \lambda (x-y)^2 - 2 \lambda \epsilon x x y y}{2 \lambda + \epsilon (x+y)^2}$$

Fiat nunc iterum $x+y = p$ et $x-y = q$, ponaturque $\lambda = n\epsilon$, et aequatio differentialis prodibit

$$nq \partial q - \frac{1}{4} p \partial p (pp + qq) + \frac{1}{2} ppq \partial q = 0.$$

Integrale vero erit $\frac{a}{\epsilon} = \frac{nnqq - \frac{1}{8} n (pp - qq)^2}{2n + pp}$

Ista autem aequatio pariter nulla laborat difficultate; posito enim $qq = v$, ut sit $2q \partial q = \partial v$, prodibit haec forma:

$$2n \partial v - p v \partial p + pp \partial v = p^3 \partial p,$$

haecque divisa per $2n + pp$, erit $\partial v - \frac{v p \partial p}{2n + pp} = \frac{p^3 \partial p}{2n + pp}$,

quae cum aequatione generali §. 12. comparata dat

$P = \frac{-p}{2n + pp}$ et $Q = \frac{p^3}{2n + pp}$. Fiet ergo $\int P \partial p = -\frac{1}{2} l(2n + pp)$,

ideoque $e^{\int P \partial p} = \frac{1}{\sqrt{(2n + pp)}}$, ergo aequatio integralis erit

$$\frac{v}{\sqrt{(2n + pp)}} = \int \frac{p^3 \partial p}{(2n + pp)^2} = \frac{4n + pp}{\sqrt{(2n + pp)}} + \text{Const.}$$

Sicque integrale completum erit $qq = 4n + pp + C \sqrt{(2n + pp)}$,

sive habebimus $C = \frac{qq - pp - 4n}{\sqrt{(2n + pp)}}$, quae forma, ut cum fu-

ra assignata comparari possit, quadretur, fietque

$$CC = \frac{q^4 - 2ppqq - 8nqq + p^4 + 8npp + 16n^2}{2n + pp}. \text{ Erat autem } \\ = + \frac{(pp - qq)^2 - 8nqq}{2n + pp}, \text{ quarum expressionum differentia} \\ CC + \frac{8\alpha}{n\epsilon} = 8n; \text{ unde patet constantem C ita definiri,} \\ \text{fit } CC = 8n - \frac{8\alpha}{n\epsilon}.$$

Casus generalis,

vbi omnes litterae admittuntur.

§. 16. Posito nunc in genere $x + y =$ et $x - y =$ aequatio nostra differentialis erit

$$\lambda q \partial q - \beta \partial p - \frac{1}{2} \gamma (p \partial p - q \partial q) - \frac{1}{4} \delta \partial p (3pp + q^2) \\ + \delta p q \partial q - \frac{1}{4} \epsilon p \partial p (pp + qq) + \frac{1}{2} \epsilon p p q \partial q = 0$$

cujus ergo integrale completum erit

$$\alpha = \frac{\left\{ \begin{array}{l} + \lambda \lambda q q - 2 \lambda \beta p - \frac{1}{2} \lambda \gamma (pp - qq) - \frac{1}{2} \lambda \delta p (pp - qq) \\ - \frac{1}{8} \lambda \epsilon (pp - qq)^2 + \beta \beta - \frac{1}{2} \beta \epsilon p (pp - qq) + \frac{1}{16} (\delta \delta - \gamma \epsilon) (pp - qq) \end{array} \right\}}{2 \lambda + \gamma + 2 \delta p + \epsilon p p}.$$

§. 17. Postquam autem nostra aequatio ad hanc formam est reducia; ejus resolutio nulla amplius difficultate laborat; posito enim $qq = v$, et terminis sive v , sive continentibus in vnam partem translatis, ista forma provenit

$$(2\lambda + \gamma + 2\delta p + \epsilon p p) \partial v - v(\delta + \epsilon p) \partial p \\ = (4\beta + 2\gamma p + 3\delta p p + \epsilon p^3) \partial p.$$

$$\partial v = \frac{v \partial p (\delta + \epsilon p)}{2\lambda + \gamma + 2\delta p + \epsilon p p} = \frac{\partial p (4\beta + 2\gamma p + 3\delta p p + \epsilon p^3)}{2\lambda + \gamma + 2\delta p + \epsilon p p}$$

haec forma cum generali §. 12. comparata dat

$$P = \frac{-\delta - \epsilon p}{2\lambda + \gamma + 2\delta p + \epsilon p p} \text{ et } Q = \frac{4\beta + 2\gamma p + 3\delta p p + \epsilon p^3}{2\lambda + \gamma + 2\delta p + \epsilon p p}$$

fiet ergo $\int P \partial p = -\frac{1}{2} l(2\lambda + \gamma + 2\delta p + \epsilon p p)$

$$\text{ideoque } e^{\int P \partial p} = \frac{1}{\sqrt{(2\lambda + \gamma + 2\delta p + \epsilon p p)^2}}$$

quocirca

Integratio dabit

$$\frac{A + Bp + Cpp}{\sqrt{(2\lambda + \gamma + 2\delta p + \epsilon pp)^3}} = \int \frac{\partial p (4\beta + 2\gamma p + 3\delta pp + \epsilon p^3)}{(2\lambda + \gamma + 2\delta p + \epsilon pp)^{\frac{3}{2}}}$$

§. 18. Ut nunc postremam formulam integram facile
 hinc evolvamus, ponamus ejus integrale esse $\frac{A + Bp + Cpp}{\sqrt{(2\lambda + \gamma + 2\delta p + \epsilon pp)^3}}$,
 cujus formae differentiale debitum habebit denominatorem,
 at vero numerator ad hanc formam reducitur:

$$\partial p (2\lambda + \gamma) B - A\delta + p \partial p (B\delta + 2C(2\lambda + \gamma) - A\epsilon) - p p \partial p \cdot 3\delta C + p^3 \partial p \cdot \epsilon C;$$

hinc ergo obtinemus quatuor sequentes aequationes:

1. $4\beta = (2\lambda + \gamma) B - A\delta,$
2. $2\gamma = B\delta + 2C(2\lambda + \gamma) - A\epsilon.$
3. $3\delta = 3\delta C,$
4. $\epsilon = \epsilon C.$

ubi hae postremae manifesto praebent $C = 1$, tum vero
 secunda fit $B\delta + 4\lambda - A\epsilon = 0$, ex qua cum prima con-
 juncta elicitur $B = \frac{4\beta\epsilon + 4\lambda\delta}{(2\lambda + \gamma)\epsilon - \delta\delta}$,
 ac denique $A = \frac{4\beta\delta + 4\lambda(2\lambda + \gamma)}{(2\lambda + \gamma)\epsilon - \delta\delta}$, quibus valoribus inven-
 tia aequatio nostra integralis erit

$$\frac{A + Bp - Cpp}{\sqrt{(2\lambda + \gamma + 2\delta p + \epsilon pp)^3}} + \Delta,$$

sive $\Delta = \frac{q\delta - A - Bp - Cpp}{\sqrt{(2\lambda + \gamma + 2\delta p + \epsilon pp)^3}}$, sive $-\Delta = \frac{C\delta p - q\delta + A + Bp}{\sqrt{(2\lambda + \gamma + 2\delta p + \epsilon pp)^3}}$

cujus quadratum a valore ipsius $\frac{16\alpha}{\delta\delta - \gamma\epsilon - 2\lambda\epsilon}$ subtractum re-
 linquit quantitatem constantem.