DEMONSTRATIO
INSIGNIS THEOREMATIS NUMERICI CIRCA UNCIAS POTESTATUM BINOMIALIUM.

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§. 1. Si iste character $(\frac{p}{q})$ designet coëfficientem potestatis x^q, qui ex evolutione Binomii $(1-x)^p$ oritur, ita ut sit

$$(\tfrac{p}{q}) = \frac{p}{1} \cdot \frac{p-1}{2} \cdot \frac{p-2}{3} \cdot \ldots \ldots \frac{p-q+1}{q},$$

non ita pridem ostendi, summam huiusmodi productorum: $(\frac{m}{0})(\frac{n}{c}) + (\frac{m}{1})(\frac{n}{c+1}) + (\frac{m}{2})(\frac{n}{c+2}) +$ etc. semper hac formula exprimi $(\frac{m+n}{m+c}) = (\frac{m+n}{n-c})$, quandoquidem hi duo characteres sunt inter se aequales, quia in genere est $(\frac{p}{q}) = (\frac{p}{p-q})$.

§. 2. Hoc elegans theorema tum temporis deduxi ex casibus specialibus, quibus erat primo $m = 1$, unde fit

$$1(\tfrac{n}{c}) + 1(\tfrac{n}{c+1}) = (\tfrac{1+n}{n-c}) = (\tfrac{1+n}{1+c}).$$

Deinde sumpto $m = 2$ etiam haud difficulter perspicitur esse

$$1(\tfrac{n}{c}) + 2(\tfrac{n}{c+1}) + 1(\tfrac{n}{c+2}) = (\tfrac{2+n}{2+c}).$$

Casu autem $m = 3$ habebitur

$$1(\tfrac{n}{c}) + 3(\tfrac{n}{c+1}) + 3(\tfrac{n}{c+2}) + 1(\tfrac{n}{c+3}) = (\tfrac{3+n}{3+c})$$

Ex quibus casibus conclusio generalis satis tuto est deducta, ita ut demonstrationi rigidae aequivalens sit censenda.

§. 3. Interim tamen istud ratiocinium non nisi ad casus, quibus m est numerus integer positiuus, extendi potest, etiamsi veritas multo latius patere atque adeo ad omnes plane valores litterae m extendi deprehendatur; unde etiamnunc pro hoc theoremate demonstratio completa desideratur, qua ejus veritas pro omnibus casibus, sive litterae m et n denotent numeros integros, sive positivos, sive negativos, sive integros, sive fractos, ostendatur. Talem igitur demonstrationem hic sum traditurus.

Lemma.

§. 4. Si formula $\frac{x^p}{(1-x)^{q+1}}$ in seriem evoluatur secundum potestates ipsius x procedentem, tum in hac serie potestatis x^n coefficiens erit $(\frac{n-p+q}{q})$. Cum enim sit
$(1-x)^{-q-1} = 1 + (\frac{q+1}{1}) x + (\frac{q+2}{2}) xx + (\frac{q+3}{3}) x^3 + (\frac{q+4}{4}) x^4$ etc.
in genere potestatis x^λ coëfficiens erit $(\frac{q+\lambda}{\lambda})$, qui ergo etiam erit coëfficiens potestatis $x^{p+\lambda}$ ex evolutione formulae $\frac{x^p}{(1-x)^{q+1}}$ resultantis. Fiat nunc $p+\lambda = n$, sive $\lambda = n-p$, atque coëfficiens potestatis x^p erit $= (\frac{n-p+q}{n-p}) = (\frac{n-p+q}{q})$.

§. 5. Hoc lemmate praemisso consideremus hanc expressionem: $\frac{z^c}{(1-z)^{c+1}} (1 + \frac{z}{1-z})^m = V$, pro qua cum more solito fiat
$(1 + \frac{z}{1-z})^m = 1 + (\frac{m}{1}) \frac{z}{1-z} + (\frac{m}{2}) \frac{zz}{(1-z)^2} + (\frac{m}{3}) \frac{z^3}{(1-z)^3} +$ etc.
erit per seriem
$V = \frac{z^c}{(1-z)^{c+1}} + (\frac{m}{1}) \frac{z^{c+1}}{(1-z)^{c+2}} + (\frac{m}{2}) \frac{z^{c+2}}{(1-z)^{c+3}} + (\frac{m}{3}) \frac{z^{c+3}}{(1-z)^{c+4}}$ etc.
ubi primo termino praefigi potest character $(\frac{m}{0})$. Concipiantur

nunc

nunc singula membra hujus seriei more solito in series evoluta, et ex singulis colligantur termini potestate z^n affecti, atque per lemma praemissum ex primo membro, ob $p = c$ et $q = c$, coefficiens hujus potestatis z^n erit $= \left(\frac{m}{0}\right)\left(\frac{n}{c}\right)$. Deinde ex secundo membro, ob $p = c + 1$ et $q = c + 1$, erit ipsius z^n coefficiens $\left(\frac{m}{1}\right)\left(\frac{n}{c+1}\right)$. Simili modo ex tertio membro nascitur potestatis z^n coefficiens: $\left(\frac{m}{2}\right)\left(\frac{n}{c+2}\right)$; sicque porro. Hinc manifestum est ex tota forma V hujus potestatis z^n coëfficientem esse prodíturum $= \left(\frac{m}{0}\right)\left(\frac{n}{c}\right) + \left(\frac{m}{1}\right)\left(\frac{n}{c+1}\right) + \left(\frac{m}{2}\right)\left(\frac{n}{c+2}\right) +$ etc. quem brevitatis gratia littera C indicemus, haecque est ea ipsa progressio, cujus summa demonstranda est aequari huic characteri $\left(\frac{m+n}{m+c}\right)$.

§. 6. Hoc autem facile ostendetur, si modo observemus esse $\frac{1-z}{1-z} = \frac{1}{1-z}$. Sic igitur forma nostra erit $V = \frac{z^c}{(1-z)^{m+c+1}}$, ex cujus evolutione potestatis z^n coëfficiens, ob $p = c$ et $q = m + c$, elicitur $= \left(\frac{m+n}{m+c}\right) = \left(\frac{m+n}{n-c}\right)$. Quare cum hi duo coefficientes ipsius x^n, ex eadem expressione V oriundi, inter se necessario debeant esse aequales, erit utique

$$\left(\frac{m}{0}\right)\left(\frac{n}{c}\right) + \left(\frac{m}{1}\right)\left(\frac{n}{c+1}\right) + \left(\frac{m}{2}\right)\left(\frac{n}{c+2}\right) + \text{etc.} = \left(\frac{m+n}{n-c}\right)$$

quae est demonstratio maxime rigorosa nostri theorematis, cujus ergo veritas semper subsistit, quicunque numeri litteris m et n tribuantur.

§. 7. Casus hic singularis, quo $m = 0$ et potestas $\left(\frac{1+z}{1-z}\right)^m$ abire censenda est in $l\left(1 + \frac{z}{1-z}\right)$, peculiarem evolutionem postulat. Cum igitur hic sit $V = \frac{z^c}{(1-z)^{c+1}}\, l\left(1 + \frac{z}{1-z}\right)$, ob

$$l\left(1 + \frac{z}{1-z}\right) = \frac{z}{1-z} - \frac{1}{2} \cdot \frac{zz}{(1-z)^2} + \frac{1}{3} \cdot \frac{z^3}{(1-z)^3} - \frac{1}{4} \cdot \frac{z^4}{(1-z)^4} \text{ etc.}$$

 erit

erit

$$V = \frac{z^{c+1}}{(1-z)^{c+2}} - \frac{1}{2}\cdot\frac{z^{c+2}}{(1-z)^{c+3}} + \frac{1}{3}\cdot\frac{z^{c+3}}{(1-z)^{c+4}} - \frac{1}{4}\cdot\frac{z^{c+4}}{(1-z)^{c+5}} \text{ etc.}$$

§. 8. Hinc jam, ut supra fecimus, investigemus coëfficientem potestatis z^n, atque ex primo membro is prodit $= (\frac{n}{c+1})$; ex secundo membro oritur $- \frac{1}{2}\cdot(\frac{n}{c+2})$; ex tertio membro $\frac{1}{3}\cdot(\frac{n}{c+3})$; ex quarto $-\frac{1}{4}\cdot(\frac{n}{c+4})$, et ita porro; sicque totus coëfficiens potestatis z^n, ex evolutione expressionis V ortus, erit

$$(\frac{n}{c+1}) - \frac{1}{2}\cdot(\frac{n}{c+2}) + \frac{1}{3}\cdot(\frac{n}{c+3}) - \frac{1}{4}\cdot(\frac{n}{c+4}) + \frac{1}{5}\cdot(\frac{n}{c+5}) \text{ etc.} = C.$$

§. 9. Cum vero per transformationem sit
$l(1+\frac{z}{1-z}) = l\frac{1}{1-z} = -l(1-z)$, erit quoque
$V = -\frac{z^c l\, 1-z}{(1-z)^{c+1}}$. Quare cum sit
$-l(1-z) = z + \frac{1}{2}zz + \frac{1}{3}z^3 + \frac{1}{4}z^4 + \frac{1}{5}z^5 + \text{etc.}$
erit $V = \frac{z^{c+1}}{(1-z)^{c+1}} + \frac{1}{2}\cdot\frac{z^{c+2}}{(1-z)^{c+1}} + \frac{1}{3}\cdot\frac{z^{c+3}}{(1-z)^{c+1}} + \text{etc.}$
ex cujus evolutione propterea si quaeratur coëfficiens potestatis z^n, is illi, quem modo ante invenimus, aequalis esse debet.

§. 10. Nunc vero per lemma praemissum primum membrum pro hoc coëfficiente praebet $(\frac{n-1}{c})$; secundum membrum autem dat $\frac{1}{2}\cdot(\frac{n-2}{c})$; tertium $= \frac{1}{3}\cdot(\frac{n-3}{c})$, et ita porro; ita ut hinc totus coëfficiens potestatis z^n sit:

$$C = (\frac{n-1}{c}) + \frac{1}{2}\cdot(\frac{n-2}{c}) + \frac{1}{3}\cdot(\frac{n-3}{c}) + \frac{1}{4}\cdot(\frac{n-4}{c}) + \text{etc.}$$

§. 11. Hinc igitur adepti sumus sequentem aequationem inter binas progressiones inventas, quandoquidem semper erit:

$(\frac{n}{c+1})$

$$\left(\frac{n}{c+1}\right) - \frac{1}{2}\cdot\left(\frac{n}{c+2}\right) + \frac{1}{3}\cdot\left(\frac{n}{c+3}\right) - \frac{1}{4}\cdot\left(\frac{n}{c+4}\right) \text{ etc.} =$$
$$= \left(\frac{n-1}{c}\right) + \frac{1}{2}\cdot\left(\frac{n-2}{c}\right) + \frac{1}{3}\cdot\left(\frac{n-3}{c}\right) + \frac{1}{4}\cdot\left(\frac{n-4}{c}\right) + \text{ etc.}$$

quae duae progressiones debent esse inter se aequales, quicunque valores litteris n et c tribuantur, cujus veritatis nonnullos casus perpendisse juvabit.

Casus I.

quo $c = 0$

§. 12. Hoc ergo casu casu prior series evadet

$$\left(\frac{n}{1}\right) - \frac{1}{2}\cdot\left(\frac{n}{2}\right) + \frac{1}{3}\cdot\left(\frac{n}{3}\right) - \frac{1}{4}\cdot\left(\frac{n}{4}\right) + \frac{1}{5}\cdot\left(\frac{n}{5}\right) \text{ etc.}$$

cujus progressionis postremus terminus erit $\pm \frac{1}{n}\left(\frac{n}{n}\right)$, quia statim atque in his characteribus numerus inferior superiorem excedit, eorem valores evanescunt, siquidem numeri integri adhibeantur. Posterior vero series evadet:

$$\left(\frac{n-1}{0}\right) + \frac{1}{2}\cdot\left(\frac{n-2}{0}\right)\ \frac{1}{3}\cdot\left(\frac{n-3}{0}\right) + \frac{1}{4}\cdot\left(\frac{n-4}{0}\right) + \frac{1}{5}\cdot\left(\frac{n-5}{0}\right) + \text{ etc.}$$

Ubi notandum est, omnium harum formularum $\left(\frac{n-\lambda}{0}\right)$ valorem esse $= 1$, quamdiu λ non excedit n, hancque adeo seriem tantum usque ad terminum $\left(\frac{n-n}{0}\right)$ esse continuandam, hocque modo posterior series ita est repraesentanda: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots\ldots + \frac{1}{n}$.

§. 13. Hinc ergo nacti sumus sequentem aequationem maxime memorabilem:

$$\left(\frac{n}{1}\right) - \frac{1}{2}\left(\frac{n}{2}\right) + \frac{1}{3}\left(\frac{n}{3}\right) - \frac{1}{4}\cdot\left(\frac{n}{4}\right) + \frac{1}{5}\cdot\left(\frac{n}{5}\right) \text{ etc.} \ldots\ldots \pm \frac{1}{n}\left(\frac{n}{n}\right) =$$
$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \text{ etc.} \ldots\ldots \frac{1}{n}.$$

Cujus veritatem aliquot exemplis ostendamus.

§. 14. Sit 1°. $n = 1$, fiet prior series $\left(\frac{1}{1}\right) = 1$, altera vero pariter dat 1.

2°.

2°. Sit $n = 2$, et ob $(\frac{n}{1}) = 2$ et $(\frac{n}{2}) = 1$, erit prior series $= 2 - \frac{1}{2} = \frac{3}{2}$; posterior vero series dat $1 + \frac{1}{2} = \frac{3}{2}$.

3°. Sit $n = 3$, ob $(\frac{n}{1}) = 3$; $(\frac{n}{2}) = 3$ et $(\frac{n}{3}) = 1$, prior series dat $3 - \frac{3}{2} + \frac{1}{3} = \frac{11}{6}$; posterior vero series praebet: $1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$.

4°. Si $n = 4$, ob $(\frac{n}{1}) = 4$; $(\frac{n}{2}) = 6$; $(\frac{n}{3}) = 4$ et $(\frac{n}{4}) = 1$, prior series dabit $4 - \frac{6}{2} + \frac{4}{3} - \frac{1}{4} = 2 + \frac{1}{3} - \frac{1}{4}$; altera vero series dat $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$, cui ille valor est aequalis, ob $1 - \frac{1}{4} = \frac{1}{2} + \frac{1}{4}$.

5°. Si $n = 5$, ob $(\frac{n}{1}) = 5$; $(\frac{n}{2}) = 10$; $(\frac{n}{3}) = 10$; $(\frac{n}{4}) = 5$ et $(\frac{n}{5}) = 1$, erit prior series: $5 - \frac{10}{2} + \frac{10}{3} - \frac{5}{4} + \frac{1}{5}$; posterior vero dat $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$, qui valores calculo instituto accurate evadunt aequales.

Simili modo erit quoque:

$$6 - \frac{15}{2} + \frac{20}{3} - \frac{15}{4} + \frac{6}{5} - \frac{1}{6} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}.$$

Item erit

$$7 - \frac{21}{2} + \frac{35}{3} - \frac{35}{4} + \frac{21}{5} - \frac{7}{6} + \frac{1}{7} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}.$$

Singulis enim terminis subtractis remanet:

$$6 - 11 + 11\tfrac{1}{3} - 9 + 4 - 1\tfrac{1}{3} = 0.$$

Casus II.

quo $c = 1$.

§. 15. Hoc casu erit prior series

$(\frac{n}{2}) - \frac{1}{2}(\frac{n}{3}) + \frac{1}{3}(\frac{n}{4}) - \frac{1}{4}(\frac{n}{5}) + \frac{1}{5}(\frac{n}{6})$ etc. altera vero fit:

$(\frac{n-1}{1}) + \frac{1}{2}(\frac{n-2}{1}) + \frac{1}{3}(\frac{n-3}{1}) + \frac{1}{4}(\frac{n-4}{1})$ etc. quae in has duas resoluitur: $\frac{n}{1} + \frac{n}{2} + \frac{n}{3} + \frac{n}{4} + \frac{n}{5} +$ etc.

$- 1 - 1 - 1 - 1 - 1 - 1 -$ etc. quae eo usque sunt continuandae, quoad superiores termini unitate fiant minores; huic ergo expressioni prior series semper erit aequalis.

§. 16.

§. 16. Sit 1°) $n = 1$, ac prior series tota evanescit, quod etiam in posteriore evenit.

2°) Sit $n = 2$, ac prior series dat 1; posterior vero dat $1 + 0$.

3°) Si $n = 3$, prior series dat $3 - \frac{1}{2} = 2\frac{1}{2}$; posterior vero series dat $2\frac{1}{2}$.

4°) Si $n = 4$, prior series praebet $6 - \frac{4}{2} + \frac{1}{3}$; posterior vero series dat $4\frac{1}{3}$.

5°) Si $n = 5$, prior series dat $11 - \frac{10}{2} + \frac{5}{3} - \frac{1}{4}$; posterior vero dat $4 + \frac{3}{2} + \frac{2}{3} + \frac{1}{4}$.

Casus III.

quo $c = 2$.

§. 17. Hoc ergo casu prior series erit: $\left(\frac{n}{3}\right) - \frac{1}{2}\left(\frac{n}{4}\right) + \frac{1}{3}\left(\frac{n}{5}\right) - \frac{1}{4}\left(\frac{n}{6}\right) + \frac{1}{5}\left(\frac{n}{7}\right)$ etc. posterior vero series praebet: $\left(\frac{n-1}{2}\right) + \frac{1}{2}\left(\frac{n-2}{2}\right) + \frac{1}{3}\left(\frac{n-3}{2}\right) + \frac{1}{4}\left(\frac{n-4}{2}\right) +$ etc. Hic jam, quamdiu $n < 3$, omnes termini prioris seriei abeunt in nihilum, quod etiam in altera usu venire deprehenditur. Tantum autem hic unicum casum, quo $n = 6$, evoluamus; quo casu prior series evadit: $20 - \frac{15}{2} + \frac{6}{3} - \frac{1}{4}$; altera vero series dat: $10 + \frac{6}{2} + \frac{3}{3} + \frac{1}{4}$.

Nota.

§. 18. In serie posteriore, quae erat: $\left(\frac{n-1}{c}\right) + \frac{1}{2}\left(\frac{n-2}{c}\right) + \frac{1}{3}\left(\frac{n-3}{c}\right) + \frac{1}{4}\left(\frac{n-4}{c}\right) +$ etc. dubium videri potest, quod ea tantum usque ad terminum $\frac{1}{n}\left(\frac{n-n}{c}\right)$ continuari debeat, cum tamen sequentes termini, in quibus superior numerus fit negativus, non evanescant. Verum hic observandum est,

in

in his characteribus numerum inferiorem, immediate ex analysi ortum, conversum esse in suum complementum, siquidem ex forma generali $\frac{z^p}{(1-z)^{q+1}}$ coefficiens ipsius z^n deductus est $\left(\frac{n-p+q}{n-p}\right)$, cujus loco scripsimus $\left(\frac{n-p+q}{q}\right)$, vi aequationis $\left(\frac{a}{b}\right) = \left(\frac{a}{a-b}\right)$. Ubi probe observandum est, talem conversionem non valere, nisi superior numerus fuerit positivus, quemadmodum hactenus assumsimus; unde si etiam ad numeros negativos nostras progressiones extendere velimus, in serie saltem posteriori in singulis characteribus complementa inferiorum numerorum scribi debebunt, hocque modo posterior progressio ita est repraesentanda:

$$\left(\frac{n-1}{n-1-c}\right) + \frac{1}{2}\left(\frac{n-2}{n-2-c}\right) + \frac{1}{3}\left(\frac{n-3}{n-3-c}\right) + \frac{1}{4}\left(\frac{n-4}{n-4-c}\right) + \text{etc.}$$

Hic probe notetur, omnes terminos, ubi inferiores numeri sunt negativi, pro nihilo esse habendos. Ita postremo casu, quo erat $n = 6$ et $c = 2$, haec progressio erit:

$$\left(\frac{5}{3}\right) + \frac{1}{2}\left(\frac{4}{2}\right) + \frac{1}{3}\left(\frac{3}{1}\right) + \frac{1}{4}\left(\frac{2}{0}\right) + \frac{1}{5}\left(\frac{1}{-1}\right).$$

Hic ergo omnes termini post $\left(\frac{2}{0}\right)$ sequentes evanescunt. Hoc autem observato etiam nostras expressiones ad valores negativos ipsius c extendere licebit.

Casus IV.

quo $c = -1$.

§. 19. Hoc ergo casu prior progressio erit:

$$\left(\frac{n}{0}\right) - \frac{1}{2}\left(\frac{n}{1}\right) + \frac{1}{3}\left(\frac{n}{2}\right) - \frac{1}{4}\left(\frac{n}{3}\right) + \frac{1}{5}\left(\frac{n}{4}\right) \text{ etc.}$$

altera vero progressio nunc ita se habebit:

$$\left(\frac{n-1}{n}\right) + \frac{1}{2}\left(\frac{n-2}{n-1}\right) + \frac{1}{3}\left(\frac{n-3}{n-2}\right) + \frac{1}{4}\left(\frac{n-4}{n-3}\right) + \frac{1}{5}\left(\frac{n-5}{n-4}\right) \text{ etc.}$$

cujus seriei priores termini omnes evanescunt, donec superiores numeri evadant negativi, tum vero sequentium terminorum ii tantum signi-

significatum habent, in quibus numerus inferior adhuc est positivus, vel 0; generatim enim omnes isti characteres, simul ac numeri inferiores evadunt negativi, semper evanescunt.

§. 20. Hinc ergo intelligitur, ex progressione posteriore unicum terminum relinqui, qui erit $\frac{1}{n+1}\left(\frac{-1}{0}\right)$, cujus valor est $+\frac{1}{n+1}$, cui ergo progressio prior semper est aequalis. Si enim ponamus $n=1$, prior progressio dat $1-\frac{1}{2}$; posterior vero dat etiam $\frac{1}{2}$.

2°. Si $n=2$, prior series dat $1-\frac{2}{2}+\frac{1}{3}=\frac{1}{3}$; posterior vero etiam dat $\frac{1}{3}$.

3°. Si $n=3$, erit $1-\frac{3}{2}+\frac{3}{3}-\frac{1}{4}=\frac{1}{4}$.
Similique modo porro habebitur:

$$1-\frac{4}{2}+\frac{6}{3}-\frac{4}{4}+\frac{1}{5}=\frac{1}{5}.$$
$$1-\frac{5}{2}+\frac{10}{3}-\frac{10}{4}+\frac{5}{5}-\frac{1}{6}=\frac{1}{6}.$$
etc. etc.

Casus V.

quo $c=-2$.

§. 21. Prior progressio erit:
$\left(\frac{n}{-1}\right)-\frac{1}{2}\left(\frac{n}{0}\right)+\frac{1}{3}\left(\frac{n}{1}\right)-\frac{1}{4}\left(\frac{n}{2}\right)+\frac{1}{5}\left(\frac{n}{3}\right)$ etc. ubi primus terminus evanescit; posterior vero series erit:
$\left(\frac{n-1}{n+1}\right)+\frac{1}{2}\left(\frac{n-2}{n}\right)+\frac{1}{3}\left(\frac{n-3}{n-1}\right)+\frac{1}{4}\left(\frac{n-4}{n-2}\right)$ etc. cujus terminus generalis est $\frac{1}{\lambda}\left(\frac{n-\lambda}{n-\lambda+2}\right)$. Hic igitur ab initio omnes termini evanescent, donec fiat $\lambda=n+1$, unde terminus fit $\frac{1}{n+1}\left(\frac{-1}{1}\right)=\frac{-1}{n+1}$, quem sequitur terminus $\frac{1}{n+2}\left(\frac{-2}{0}\right)$, qui adhuc valorem dat $\frac{1}{n+2}$; sequentes autem omnes iterum evanescunt, ita

significatum habent, in quibus numerus inferior adhuc est positivus, vel o; generatim enim omnes isti characteres, simul ac numeri inferiores evadunt negativi, semper evanescunt.

§. 20. Hinc ergo intelligitur, ex progressione posteriore unicum terminum relinqui, qui erit $\frac{1}{n+1}\left(\frac{-1}{0}\right)$, cujus valor est $+\frac{1}{n+1}$, cui ergo progressio prior semper est aequalis. Si enim ponamus $n=1$, prior progressio dat $1-\frac{1}{2}$; posterior vero dat etiam $\frac{1}{2}$.

2°. Si $n=2$, prior series dat $1-\frac{2}{2}+\frac{1}{3}=\frac{1}{3}$; posterior vero etiam dat $\frac{1}{3}$.

3°. Si $n=3$, erit $1-\frac{3}{2}+\frac{3}{3}-\frac{1}{4}=\frac{1}{4}$.
Similique modo porro habebitur:

$$1-\frac{4}{2}+\frac{6}{3}-\frac{4}{4}+\frac{1}{5}=\frac{1}{5}.$$
$$1-\frac{5}{2}+\frac{10}{3}-\frac{10}{4}+\frac{5}{5}-\frac{1}{6}=\frac{1}{6}.$$
etc. etc.

Casus V.

quo $c=-2$.

§. 21. Prior progressio erit:
$\left(\frac{n}{-1}\right)-\frac{1}{2}\left(\frac{n}{0}\right)+\frac{1}{3}\left(\frac{n}{1}\right)-\frac{1}{4}\left(\frac{n}{2}\right)+\frac{1}{5}\left(\frac{n}{3}\right)$ etc. ubi primus terminus evanescit; posterior vero series erit:
$\left(\frac{n-1}{n+1}\right)+\frac{1}{2}\left(\frac{n-2}{n}\right)+\frac{1}{3}\left(\frac{n-3}{n-1}\right)+\frac{1}{4}\left(\frac{n-4}{n-2}\right)$ etc. cujus terminus generalis est $\frac{1}{\lambda}\left(\frac{n-\lambda}{n-\lambda+2}\right)$. Hic igitur ab initio omnes termini evanescent, donec fiat $\lambda=n+1$, unde terminus fit $\frac{1}{n+1}\left(\frac{-1}{1}\right)=$ $=\frac{-1}{n+1}$, quem sequitur terminus $\frac{1}{n+2}\left(\frac{-2}{0}\right)$, qui adhuc valorem dat $\frac{1}{n+2}$; sequentes autem omnes iterum evanescunt, ita

ut

ut tota posterior series contrahatur in hos duos terminos: $-\frac{1}{n+1}+\frac{1}{n+2}=\frac{-1}{(n+1)(n+2)}$, qui ergo est valor seriei prioris.

§. 22. Ad hoc ostendendum sit primo $n=1$, et prior series erit $-\frac{1}{2}\left(\frac{1}{0}\right)+\frac{1}{3}\left(\frac{1}{1}\right)=-\frac{1}{2}+\frac{1}{3}=\frac{1}{6}$.

2°. Si $n=2$, habebitur $-\frac{1}{2}\left(\frac{2}{0}\right)+\frac{1}{3}\left(\frac{2}{1}\right)-\frac{1}{4}\left(\frac{2}{2}\right)$, sive $-\frac{1}{2}+\frac{2}{3}-\frac{1}{4}=-\frac{1}{12}=-\frac{1}{3\cdot 4}$.
Si $n=3$, erit $-\frac{1}{2}+\frac{3}{3}-\frac{3}{4}+\frac{1}{5}=-\frac{}{20}=-\frac{1}{4.5}$.

§. 23. Hic ergo prior progressio erit: $\left(\frac{n}{-2}\right)-\frac{1}{2}\left(\frac{n}{-1}\right)+\frac{1}{3}\left(\frac{n}{0}\right)-\frac{1}{4}\left(\frac{n}{1}\right)+\frac{1}{5}\left(\frac{n}{2}\right)-\frac{1}{6}\left(\frac{n}{3}\right)$ etc. ubi duo priores termini in nihilum abeunt. Pro posteriore vero serie, cujus terminus generalis est $\frac{1}{\lambda}\left(\frac{n-\lambda}{n-\lambda+3}\right)$, primus terminus significatum habens est $\frac{1}{n+1}\left(\frac{-1}{2}\right)=\frac{1}{n+1}$; sequens autem terminus erit $\frac{1}{n+2}\left(\frac{-2}{1}\right)=\frac{-2}{n+2}$; denuo sequens erit: $\frac{1}{n+3}\left(\frac{-3}{0}\right)=\frac{1}{n+3}$; reliqui vero omnes evanescunt, ita ut summa prioris semper futura sit $\frac{1}{n+1}-\frac{2}{n+2}+\frac{1}{n+3}=\frac{2}{(n+1)(n+2)(n+3)}$.

§. 24. Ut rem exemplis illustremus, sit 1°. $n=0$, quo casu summa dabit esse $\frac{2}{1\cdot 2\cdot 3}=\frac{1}{3}$, ipsa vero progressio dat $\frac{1}{3}\left(\frac{0}{0}\right)=\frac{1}{3}$.

2°. Casu $n=1$ fit summa $\frac{2}{2\cdot 3\cdot 4}=\frac{1}{12}$; ipsa vero progressio praebet $\frac{1}{3}\left(\frac{1}{0}\right)-\frac{1}{4}\left(\frac{1}{1}\right)=\frac{1}{12}$.

3°. Casu $n=2$ fit summa $\frac{2}{3\cdot 4\cdot 5}=\frac{1}{30}$; ipsa autem progressio erit $\frac{1}{3}-\frac{2}{4}+\frac{1}{5}=\frac{1}{30}$.

Eo-

Eodem modo habebimus:

$$\frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6} = \frac{2}{4.5.6}$$
$$\frac{1}{3} - \frac{4}{4} + \frac{6}{5} - \frac{4}{6} + \frac{1}{7} = \frac{2}{5.6.7.}$$
$$\frac{1}{3} - \frac{5}{4} + \frac{10}{5} - \frac{10}{6} + \frac{5}{7} - \frac{1}{8} = \frac{2}{6.7.8.}$$

§. 25. Superfluum foret haec ulterius prosequi. Hinc enim satis patet, si fuerit $c = -4$, posteriorem progressionem, atque adeo summam prioris, futuram esse:

$$\frac{-1}{n+1} + \frac{3}{n+2} - \frac{3}{n+3} + \frac{1}{n+4} = \frac{-1.2.3.}{(n+1)(n+2)(n+3)(n+4)}.$$

Prior vero series, omissis terminis nihilo aequalibus, erit:

$$-\frac{1}{4}\left(\frac{n}{0}\right) + \frac{1}{5}\left(\frac{n}{1}\right) - \frac{1}{6}\left(\frac{n}{2}\right) - \frac{1}{7}\left(\frac{n}{3}\right) - \frac{1}{8}\left(\frac{n}{4}\right) + \text{etc.}$$