ACCURATIOR EVOLUTIO PROBLEMATIS DE LINEA BREVISSIMA IN SUPERFICIE QUACUNQUE DUCENDA

AUCTORE

L. EULERO.

Conventui exhibita die 25. Ianuarii 1779.

§ 1.

Pro superficie, in qua lineam brevissimam duci oportet, data sit inter ternas coordinatas orthogonales x, y, z, haec aequatio differentialis: $\partial z = f\partial x + g\partial y$, ubi f et g sint functiones binarum x et y, ita ut sit $\partial f = \alpha\partial x + \beta\partial y$ et $\partial g = \beta\partial x + \gamma\partial y$. His positis, cum lineae cuiuscunque in hac superficie ductae elementum sit $\sqrt{\partial x^2 + \partial y^2 + \partial z^2}$, loco ∂z hoc valore posito erit elementum istius curvae $= \sqrt{\partial x^2 + \partial y^2 + (f\partial x + g\partial y)^2}$; unde si statuamus $\partial y = p\partial x$, hoc elementum erit $\partial x\sqrt{1 + pp + (f + gp)^2}$.

§ 2. Formula igitur integralis, quam ad minimum revocari oportet, erit $\int\partial x\sqrt{1 + pp + (f + gp)^2}$, quam in Tractatu meo: *Methodus inveniendi lineas curvas Maximi Minimive proprietate gaudentes*, in genere per $\int Z\partial x$ indicavi, ita ut pro hoc casu sit $Z = \sqrt{1 + pp + (f + gp)^2}$. Tum vero, posito $\partial Z = M\partial x + N\partial y + P\partial p$, ostendi naturam Minimi vel Maximi hac aequatione exprimi: $N\partial x = \partial P$, quam ergo patet ad differentialia secundi gradus assurgere.

§ 3.

§. 3. Cum igitur sit $Z^2 = 1 + pp + (f + gp)^2$, differentietur haec formula, ac distinguantur triplicis generis elementa, scilicet ∂x, ∂y, ∂p, hocque modo reperietur:

$$Z \partial Z = \partial x (\alpha + \beta p)(f + gp) + \partial y (\beta + \gamma p)(f + gp) + \partial p (p + g (f + gp)).$$

Cum igitur in genere posuerim $\partial Z = M \partial x + N \partial y + P \partial p$, hoc casu habebimus:

$$M = \frac{(\alpha + \beta p)(f + gp)}{Z}$$

$$N = \frac{(\beta + \gamma p)(f + gp)}{Z}$$

$$P = \frac{p + g(f + gp)}{Z}$$

Hinc ergo (ob $\beta \partial x + \gamma p \partial x = \partial g$) fiet $N \partial x = \frac{\partial g (f + gp)}{Z}$, unde aequatio pro curua nostra quaesita erit $\frac{\partial g (f + gp)}{Z} = \partial . \frac{p + g (f + gp)}{Z}$. Pro qua aequatione evoluenda ponatur brevitatis gratia $p + g (f + gp) = S$, atque habebimus:

$$\frac{\partial g (f + gp)}{Z} = \frac{\partial S}{Z} - \frac{S \partial Z}{ZZ}, \text{ siue}$$

$$\partial g (f + gp) = \partial S - \frac{S \partial Z}{Z}.$$

Quia igitur est $\partial S = \partial p + \partial g (f + gp) + g . \partial . (f + gp)$, erit nostra aequatio $o = \partial p + g \partial (f + gp) - \frac{S \partial Z}{Z}$. Porro vero est: $\frac{\partial Z}{Z} = \frac{p \partial p + (f + gp) \partial . (f + gp)}{1 + pp + (f + gp)^2}$, quod multiplicari debet per $S = p + g (f + gp)$. Hinc multiplicando per denominatorem $1 + pp + (f + gp)^2$, habebimus:

$$o = \partial p + (g - fp) \partial . (f + gp) - gp \partial f (f + gp) + \partial p (f + gp)^2$$

seu $o = \partial p + (g - fp) \partial . (f + gp) + f \partial p (f + gp)$, quae aequatio porro transmutatur in hanc formam:

$$o = \partial p (1 + ff + gg) + (g - fp)(\partial f + p \partial g).$$

§ 4.

§ 4. Quanquam haec aequatio satis est simplex, tamen non patet, quomodo eam ad differentialia primi gradus revocare liceat. Observavi autem sequenti substitutione negotium confici posse, scilicet: $v = \frac{g - fp}{f + gp}$; unde fit $p = \frac{g - fv}{gv + f}$, hinc iam differentiando deducitur $\partial p = - \frac{(ff + gg)\,\partial v + (1 + vv)\,(f\partial g - g\partial f)}{(f + gv)^2}$. Porro erit $g - fp = \frac{v\,(ff + gg)}{f + gv}$, denique $\partial f + p\partial g = \frac{f\partial f + g\partial g + v\,(g\partial f - f\partial g)}{f + gv}$, quibus substitutis aequatio prodit:

$$0 = - \partial v\,(ff + gg)\,(1 + ff + gg) + v\,(ff + gg)\,(f\partial f + g\partial g)$$
$$+ (1 + vv)\,(f\partial g - g\partial f) + (ff + gg)\,(f\partial g - g\partial f).$$

§ 5. Ad hanc aequationem simpliciorem reddendam statuamus $ff + gg = hh$, eritque $f\partial f + g\partial g = h\partial h$, deinde vero sit $\frac{g}{f} = k$, ut fiat $f\partial g - g\partial f = ff\partial k$, sicque aequatio nostra contrahetur in hanc formam:

$$0 = - hh\partial v\,(1 + hh) + h^3 v\partial h + (1 + hh + vv)\,ff\partial k.$$

Cum autem $g = fk$, erit $ff\,(1 + kk) = hh$, ideoque $ff = \frac{hh}{1 + kk}$, unde habebimus:

$$0 = - \partial v\,(1 + hh) + vh\partial h + (1 + hh + vv)\,\frac{\partial k}{1 + kk}$$

quae aequatio porro, ponendo $v = s\sqrt{1 + hh}$, reducitur ad hanc formam:

$$0 = - \partial s \sqrt{(1 + hh)} + \frac{\partial k\,(1 + ss)}{1 + kk}.$$

Nunc igitur quantitatem s a reliquis separatam exhibere licet, cum sit $\frac{\partial s}{1 + ss} = \frac{\partial k}{(1 + kk)\sqrt{1 + hh}}$, quae forma simplicissima esse videtur, ad quam in genere pertingere licet.

§ 6. Quoniam autem hic binas variabiles y et x per eandem z determinare sumus conati, cum tamen omnes tres aequali ratione in calculum ingrediantur, universam hanc quaestionem ita tractare mihi est visum, ut omnes formulae pari ratione tres coordinatas x, y, z involuant, quo pacto speculationi potius consulatur, quam usui, hancque ob rem investigationes sequentes subiungam.

Supplementum.

§ 7. Pro superficie data sit haec aequatio differentialis: $p\partial x + q\partial y + r\partial z = o$, ubi p, q, r sint functiones coordinatarum x, y, z; unde, ut aequatio sit possibilis, haec conditio inesse debet:

$$\frac{p\partial q - q\partial p}{\partial z} + \frac{q\partial r - r\partial q}{\partial x} + \frac{r\partial p - p\partial r}{\partial y} = o.$$

Hoc posito pro linea brevissima in hac superficie ducenda sequens habebitur aequatio, quam ternae coordinatae x, y, z pari ratione ingrediuntur:

$$\partial\partial x(q\partial z - r\partial y) + \partial\partial y(r\partial x - p\partial z) + \partial\partial z(p\partial y - q\partial x) = o.$$

Vel si brevitatis gratia ponamus:

$$\partial y\partial\partial z - \partial z\partial\partial y = f;$$
$$\partial z\partial\partial x - \partial x\partial\partial z = g;$$
$$\partial x\partial\partial y \ \partial y\partial\partial x = h;$$

erit $fp + gq + hr = o$; tum vero etiam $f\partial x + g\partial y + h\partial z = o$. Deinde si elementum curvae brevissimae ponatur $= \partial s$, erit $\partial s^2 = \partial x^2 + \partial y^2 + \partial z^2$; tum vero quoque

$$\frac{\partial\partial s}{\partial s} = \frac{q\partial\partial z - r\partial\partial y}{q\partial z - r\partial y} = \frac{r\partial\partial x - p\partial\partial z}{r\partial x - p\partial z} = \frac{p\partial\partial y - q\partial\partial x}{p\partial y - q\partial x}.$$

Appli-

Applicatio ad superficiem sphaericam.

§ 8. Sit aequatio pro hac superficie $x\partial x + y\partial y + z\partial z = 0$, ita ut hic habeamus $p = x$, $q = y$, $r = z$, et prima aequatio pro linea brevissima erit sequens:

$$\partial\partial x(y\partial z - z\partial y) + \partial\partial y(z\partial x - x\partial z) + \partial\partial z(x\partial y - y\partial x) = 0,$$

cuius ergo integrale completum est $\alpha x + \beta y + \gamma z = 0$, uti ex rei natura patet. Quaestio igitur huc redit, quomodo hoc integrale erui possit.

§. 9. Cum iam altera aequatio sit $fx + gy + hz = 0$, si pro hac aequatione ponamus $\Pi = \frac{z\partial x - x\partial z}{y\partial x - x\partial y}$, erit $\partial\Pi = \partial . \frac{z\partial x - x\partial z}{y\partial x - x\partial y}$, ideoque $\partial\Pi = \frac{z\partial\partial x - x\partial\partial z}{y\partial x - x\partial y} - \frac{(z\partial x - x\partial z)(y\partial\partial x - x\partial\partial y)}{(y\partial x - x\partial y)^2}$, sive evoluendo

$$\partial\Pi = \frac{x}{(y\partial x - x\partial y)^2}[(\partial y\partial\partial z - \partial z\partial\partial y)x + (\partial z\partial\partial x - \partial x\partial\partial z)y + (\partial x\partial\partial y - \partial y\partial\partial x)^2]$$

et introductis f, g, h, erit $\partial\Pi = x\frac{(fx + gy + hz)}{(y\partial x - x\partial y)^2}$. Cum autem sit $fx + gy + hz = 0$, erit $\partial\Pi = 0$, ideoque Π quantitas constans, quam si statuamus $= A$, erit aequatio differentialis primi gradus $\Pi = \frac{z\partial x - x\partial z}{y\partial x - x\partial y}$, ita expressa: $A(y\partial x - x\partial y) = z\partial x - x\partial z$, quae divisa per xx erit integrabilis; fiet enim $\frac{Ay}{x} = \frac{z}{x} + B$, sive $Ay - Bx - z = 0$, vel mutatis constantibus $\alpha x + \beta y + \gamma z = 0$, quae aequatio cum sit pro plano quocunque per centrum sphaerae ducto, in superficie sphaerica nascentur circuli maximi; unde sequitur omnes circulos maximos esse lineas brevissimas omnium, quae in superficie sphaerae duci possunt.

§ 10.

§ 10. Quoniam in huiusmodi calculis omnia ad unicam variabilem reduci solent, si pro hoc efficiendo ponamus $\partial y = t\partial x$ et $\partial z = u\partial x$, sumto ∂x pro constante, erit prima aequatio vt sequitur:

$$\partial t\,(r - pu) + \partial u\,(pt - q) = o.$$

At aequatio pro superficie erit $p + qt + ru = o$; unde cum hinc fiat $p = -qt - ru$, prior aequatio hanc induet formam:

$$\partial t\,(r + qtu + ruu) - \partial u\,(q + rtu + qtt) = o.$$

Porro erit

$$f = \partial x^2\,(t\partial u - u\partial t);\; g = -\partial x^2\,\partial u;\; h = \partial x^2\,\partial t;$$

tum vero $\partial s^2 = \partial x^2\,(1 + tt + uu)$, et denique

$$\frac{\partial\partial s}{\partial s} = \frac{t\partial t + u\partial u}{1 + tt + uu} = \frac{q\partial u - r\partial t}{qu - rt} = -\frac{-p\partial u}{r - pu} = \frac{p\partial t}{pt - q}.$$

§ 11. At si malimus quartam quandam variabilem, puta angulum Φ introducere, ponendo $\partial x = t\partial\Phi$; $\partial y = u\partial\Phi$; $\partial z = v\partial\Phi$; aequatio pro superficie erit $pt + qu + rv = o$. Porro pro litteris f, g, h, habebimus

$$f = \partial\Phi^2\,(u\partial v - v\partial u)$$
$$g = \partial\Phi^2\,(v\partial t - t\partial v)$$
$$h = \partial\Phi^2\,(t\partial u - u\partial t)$$

hinc ergo erit $ft + gu + hv = o$. Aequatio pro linea brevissima erit:

$$fp + gq + hr = p\,(u\partial v - v\partial u) + q\,(v\partial t - t\partial v) + r\,(t\partial u - u\partial t) = o,$$

denique fiet $\partial s^2 = \partial\Phi^2\,(tt + uu + vv)$, ideoque

$$\frac{\partial\partial s}{\partial s} = \frac{t\partial t + u\partial u + v\partial v}{tt + uu + vv} = \frac{q\partial v - r\partial u}{qv - ru} = \frac{r\partial t - p\partial v}{rt - pv} = \frac{p\partial u - q\partial t}{pu - qt}.$$

§ 12

quae ergo hanc induit formam:

$$-\partial\pi(1+pp+qq)+\partial p(\pi p-q)+\pi\partial q(\pi p-q)=0, \text{ seu}$$

$$\partial\pi(1+pp+qq)+(\partial p+\pi\partial q)(q-\pi p))=0.$$

§ 16. Quoniam in hac aequatione potissimum binae formulae $p+\pi q$ et $q-\pi p$ occurrunt, plurimum iuuabit rationem inter eas inducere. Statuatur hunc in finem $\frac{q-\pi p}{p+\pi q}=v$, unde iam fit $\pi=\frac{q-vp}{p+vq}$; tum vero vicissim $q-\pi p=\frac{v(pp+qq)}{p+vq}$, porro autem erit $\partial p+\pi\partial q=\frac{p\partial p+q\partial q+v(q\partial p-p\partial q)}{p+qv}$.

Si nunc ponatur $q=up$, erit $\pi=\frac{u-v}{1+uv}$, hincque $\partial\pi=\frac{\partial u(1+vv)-\partial v(1+uu)}{(1+uv)^2}$. Ponatur porro $pp+qq=tt$, et cum sit $q=up$, erit $pp=\frac{tt}{1+uu}$ et $\partial.\frac{q}{p}=\partial u=\frac{p\partial q-q\partial p}{pp}$, hincque

$$p\partial q-q\partial p=pp\partial u=\frac{tt\partial u}{1+uu},$$

quibus valoribus substitutis, ob $q=\pi p=\frac{vtt}{p(1+vu)}$ et $\partial p+\pi\partial q=\frac{t\partial t-(vtt\partial u):(1+uu)}{p(1+uv)}$, erit

$$0=-\frac{(\partial u(1+vv)-\partial v(1+uu))(1+tt)}{(1+uv)^2}-\frac{vtt(t(1+uu)\partial t-vtt\partial u)}{pp(1+uu)(1+vv)^2}$$

sive

$$(1+tt)(\partial u(1+vv)-\partial v(1+uu))$$
$$+vt((1+uu)\partial t-vt\partial u)=0,$$

quae aequatio porro reducitur ad hanc formam:

$$\partial u((1+vv)(1+tt)-vvtt)-\partial v(1+tt)(1+uu)$$
$$+vt\partial t(1+uu)=0,$$

sive ad hanc concinniorem:

$$\frac{\partial u}{1+uu}(1+vv+tt)-\partial v(1+tt)+vt\partial t=0.$$

Ponatur nunc $v=w\sqrt{1+tt}$, eritque $\partial w=\frac{\partial v(1+tt)-vt\partial t}{(1+tt)^{\frac{3}{2}}}$

seu

seu erit $\partial v\,(1+tt) - vt\partial t = (1+tt)^{\frac{3}{2}}\,\partial w$;
tum vero erit

$$1+tt+vv=(1+tt)(1+ww),$$

quibus substitutis aequatio nostra ita se habebit

$$\frac{\partial u}{1+uu}(1+tt)(1+ww)-(1+tt)^{\frac{3}{2}}\,\partial w=0,$$

hinc separando nanciscimur $\frac{\partial u}{1+uu}=\frac{\partial w\sqrt{1+tt}}{1+ww}$, consequenter

$$\frac{\partial w}{1+ww}=\frac{\partial u}{(1+uu)\sqrt{1+tt}},$$

quae ergo aequatio semper integrari potest, quoties t fuerit functio ipsius u, sive quoties $pp+qq$ fuerit functio ipsius $\frac{q}{p}$, sive q functio ipsius p.

§ 17. Evenit autem, ut q sit functio ipsius p, primo si z et y ita determinentur per x et aliam novam variabilem ω, ut sit $y=\mathrm{A}x$ et $z=\mathrm{B}x$, existentibus A et B functionibus quibuscunque ipsius ω. Cum ergo posuerimus $\partial z=p\partial x+q\partial y$, erit

$$\mathrm{B}\partial x+x\partial\mathrm{B}=p\partial x+q\mathrm{A}\partial x+qx\partial\mathrm{A},$$

ubi terminos differentiale ∂x involuentes seorsim inter se comparari oportet, unde fit $p=\mathrm{B}-\mathrm{A}q$; et comparatis seorsim terminis ipsam quantitatem x continentibus, erit $q=\frac{\partial\mathrm{B}}{\partial\mathrm{A}}$, ideoque $p=\frac{\mathrm{B}\partial\mathrm{A}-\mathrm{A}\partial\mathrm{B}}{\partial\mathrm{A}}$. Sicque p et q sunt functiones ipsius ω, ideoque et $tt=pp+qq$ et $u=\frac{q}{p}$ erunt functiones eiusdem quantitatis ω, et $\sqrt{1+tt}$ erit functio ipsius u. Quocirca aequatio supra inventa pro linea brevissima integrationem admittit. Hoc autem casu, quo scilicet $y=\mathrm{A}x$ et $z=\mathrm{B}x$, prodit superficies conica super basi quacunque constructa.

§ 18.

§ 18. Aequatio supra tradita porro fit integrabilis statuendo $y = Ax + C$ et $z = Bx + D$; tum enim erit

$$\partial z = p\partial x + q\partial y = B\partial x + x\partial B + \partial D.$$

et quia $\partial y = A\partial x + x\partial A + \partial C$, erit etiam

$$\partial z = p\partial x + q\partial y = p\partial x + Aq\partial x + xq\partial A + q\partial C$$

ideoque, comparatis inter se membris ipsam quantitatem x continentibus, tum vero iis quae differentiali ∂x affecta sunt, erit

$$B = p + Aq \quad \text{et} \quad \partial B = q\partial A$$

hinc $q = \frac{\partial B}{\partial A}$ et $p = \frac{B\partial A - A\partial B}{\partial A}$. Praeterea vero esse debet $\partial D = q\partial C = \frac{\partial B\partial C}{\partial A}$, sive functiones A, B, C, D, ita debent esse comparatae ut $\partial A\partial D = \partial B\partial C$, quod si contigerit, erunt iterum p et q functiones eiusdem variabilis ω, hincque erit etiam $\sqrt{1 + tt}$ functio ipsius u, quo ergo casu quoque lineam brevissimam definire licebit. Hic vero casus complecti videtur omnes plane superficies, quae in planum explicari possunt.