DE SUMMATIONE SERIERUM

IN HAC FORMA CONTENTARUM:

$$\frac{a}{1} + \frac{a^2}{4} + \frac{a^3}{9} + \frac{a^4}{10} + \frac{a^5}{25} + \frac{a^6}{36} + \text{ etc.}$$

AUCTORE

L. EULERO.

Conventui exhibita die 31 Maji 1779.

§. 1. Ex iis quae olim primus de summatione potestatum reciprocarum in medium attuli, duo tantum casus derivari possunt, quibus summam seriei hic propositae assignare licet: alter scilicet quo $a = 1$, ubi ostendi hujus seriei: $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} +$ etc. summam esse $= \frac{\pi\pi}{6}$, denotante π peripheriam circuli, cujus diameter $= 1$; alter vero casus est quo $a = -1$; tum enim, mutatis signis, hujus seriei: $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} -$ etc. summa est $= \frac{\pi\pi}{12}$. Praeterea vero methodo prorsus singulari inveni, casu $a = \frac{1}{2}$, hujus seriei: $\frac{1}{1 \cdot 2} + \frac{1}{4 \cdot 2^2} + \frac{1}{9 \cdot 2^3} + \frac{1}{16 \cdot 2^4} +$ etc. summam esse $\frac{\pi\pi}{12} - \frac{1}{2}(l\,2)^2$, denotante $l\,2$ logarithmum hyperbolicum binarii, qui est $0{,}693147180$. Neque vero, praeter hos casus, ullus alius adhuc constat, quo summam assignare liceat.

§. 2. Methodus autem, qua hunc postremum casum sum adeptus, ulterius extendi potest, ita ut inde plurimae insignes relationes inter binas pluresve series hujus formae reperiri queant. Innititur autem ista methodus hoc lemmate:

Lemma.

Si ponatur $p = \int \frac{\partial x}{x} l y$ *et* $q = \int \frac{\partial y}{y} l x$, *erit summa* $p + q = lx \,.\, ly + C$, *siquidem constans ita definiatur, ut unico casui satisfaciat.*

Hinc igitur sequentia problemata percurramus, pro varia scilicet relatione inter x et y.

Problema I.

Si fuerit $x + y = 1$, *binas illas formulas:* $p = \int \frac{dx}{x} l y$ *et* $q = \int \frac{dy}{y} l x$ *in series resolvere, ita ut hinc prodeat* $p + q = lx \,.\, ly + C$.

Solutio.

§. 3. Cum igitur sit $y = 1 - x$, erit $ly = -x - \frac{xx}{2} - \frac{x^3}{3} - \text{etc.}$ hincque $p = \int \frac{\partial x}{x} ly = -\frac{x}{1} - \frac{xx}{4} - \frac{x^3}{9} - \frac{x^4}{16} - \text{etc.}$ Similique modo, ob $x = 1 - y$ et $lx = -y - \frac{yy}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \text{etc.}$ erit $q = \int \frac{\partial y}{y} lx = -\frac{y}{1} - \frac{y^2}{4} - \frac{y^3}{9} - \frac{y^4}{16} - \text{etc.}$ quamobrem harum duarum serierum summa erit $lx \,.\, ly + C$. Pro constante C definienda consideremus casum quo $x = 0$ et $y = 1$, ideoque $lx \,.\, ly = 0$; tum igitur erit:

$$p+q=-1-\frac{1}{4}-\frac{1}{9}-\frac{1}{16}-\text{etc.}=-\frac{\pi\pi}{6}$$
unde elicitur $C=-\frac{\pi\pi}{6}$.

§. 4. Quoties ergo fuerit $x+y=1$, summa harum duarum serierum junctim sumtarum: $\frac{x}{1}+\frac{xx}{4}+\frac{x^3}{9}+\frac{x^4}{16}+\text{etc.}$ $+\frac{y}{1}+\frac{yy}{4}+\frac{y^3}{9}+\frac{y^4}{16}+\text{etc.}$ erit $=\frac{\pi\pi}{6}-lx.ly$; hincque statim sequitur tertius casus supra memoratus. Sumto enim $x=\frac{1}{2}$, erit quoque $y=\frac{1}{2}$, ideoque ambae hae series inter se aequales, unde sequitur fore
$$\frac{1}{1.2}+\frac{1}{4.2^2}+\frac{1}{9.2^3}+\frac{1}{16.2^4}+\text{etc.}=\frac{\pi\pi}{12}-\frac{1}{2}(l\tfrac{1}{2})^2=\frac{\pi\pi}{12}-\frac{1}{2}(l2)^2.$$
Praeterea vero, quoties fuerit $a+b=1$ ponaturque
$$A=\frac{a}{1}+\frac{aa}{4}+\frac{a^3}{9}+\text{etc.} \text{ et } B=\frac{b}{1}+\frac{b^2}{4}+\frac{b^3}{9}+\text{etc.}$$
semper erit $A+B=\frac{\pi\pi}{6}-la.lb$. Hinc ergo si alterius harum serierum summa aliunde esset cognita, etiam alterius summa innotesceret. Hocque est illud ipsum problema, quod jam olim tractavi.

Problema II.

Si fuerit $x-y=1$, binas illas formulas: $p=\int\frac{dx}{x}ly$ et $q=\int\frac{dy}{y}lx$ in series resolvere, ita ut hinc prodeat $p+q=lx.ly+C$.

Solutio.

§. 5. Cum hic sit $y=x-1$, erit
$$ly=l(x-1)=lx+l(1-\tfrac{1}{x})=lx-\frac{1}{x}-\frac{1}{2xx}-\frac{1}{3x^3}-\frac{1}{4x^4}-\text{etc.}$$
hincque $p=\int\frac{\partial x}{x}ly=\frac{1}{2}(lx)^2+\frac{1}{x}+\frac{1}{4x^2}+\frac{1}{9x^3}+\frac{1}{16x^4}+\text{etc.}$

Deinde, ob $x = 1 + y$, erit $lx = \frac{y}{1} - \frac{yy}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \text{etc.}$ ideoque $q = \int \frac{\partial y}{y} lx = \frac{y}{1} - \frac{y^2}{4} + \frac{y^3}{9} - \frac{y^4}{16} + \text{etc.}$ quamobrem habebimus: $p + q = lx \,.\, ly + C.$ Pro constante determinanda consideremus casum $y = 0$, quo fit $x = 1$ et $lx\, ly = 0$; tum igitur erit $p = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.} = \frac{\pi\pi}{6}$ et $q = 0$, unde definitur constans $C = \frac{\pi\pi}{6}$.

§. 6. Hic igitur iterum duas habemus series, quarum conjunctim summam assignare valemus:

$$\left.\begin{array}{l} \frac{1}{x} + \frac{1}{4x^2} + \frac{1}{9x^3} + \frac{1}{16x^4} + \text{etc.} \\ + \frac{y}{1} - \frac{yy}{4} + \frac{y^3}{9} - \frac{y^4}{16} + \text{etc.} \end{array}\right\} = \frac{\pi\pi}{6} - \frac{1}{2}(lx)^2 + lx\, ly$$
$$= \frac{\pi\pi}{6} + lx \,.\, l\frac{y}{\sqrt{x}}.$$

§. 7. Quod si ergo habeantur hae duae series:

$$A = \frac{a}{1} + \frac{a^2}{4} + \frac{a^3}{9} + \frac{a^4}{16} + \text{etc.} \quad \text{et}$$
$$B = \frac{b}{1} - \frac{b^2}{4} + \frac{b^3}{9} - \frac{b^4}{16} + \text{etc.}$$

ita ut sit $a = \frac{1}{x}$ et $b = y$, atque inter a et b haec detur relatio: $ab + a = 1$, erit $A + B = \frac{\pi\pi}{6} - la \,.\, lb\sqrt{a}$. Consideremus casum quo $b = a$ $(= -\frac{1 + \sqrt{5}}{2}$, ob $ab + a = 1)$, eritque $A + B = 2\left(\frac{a}{1} + \frac{a^3}{9} + \frac{a^5}{25} + \frac{a^7}{49} + \text{etc.}\right)$; quocirca, existente $a = \frac{\sqrt{5} - 1}{2}$, hujus seriei: $\frac{a}{1} + \frac{a^3}{9} + \frac{a^5}{25} + \text{etc.}$ summa erit $\frac{\pi\pi}{12} - \frac{1}{2} la \,.\, la\sqrt{a}$.

§. 8. Deinde etiam hic notatu dignus est casus, quo $b = -a$, atque adeo $A + B = 0$; hoc enim casu erit $\frac{\pi\pi}{6} = la \,.\, lb\sqrt{a}$. At quia $b = -a$, erit $-aa + a = 1$, hincque $a = \frac{1+\sqrt{-3}}{2}$ et $b = \frac{-1-\sqrt{-3}}{2}$. Jam cum sit $lb\sqrt{a} = \frac{1}{2}\,labb$, ob $bb = \frac{-1+\sqrt{-3}}{2}$ erit $abb = -1$, unde sequitur fore $\frac{\pi\pi}{6} = l\,\frac{1+\sqrt{-3}}{2}\,.\,l-1$, id quod egregie convenit cum expressione cognita peripheriae circuli per logarithmos imaginarios.

§. 9. Si poneremus hic $a = \frac{1}{2}$, foret $b = 1$, ideoque

$$B = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \text{etc. hincque}$$

$$A + B = \frac{1}{1.2} + \frac{1}{4.2^2} + \frac{1}{9.2^3} + \text{etc.} + \frac{\pi\pi}{12} = \frac{\pi\pi}{6} - \frac{1}{2}(l2)^2$$

unde prodiret tertius casus initio memoratus. At vero faciamus hic $b = \frac{1}{2}$, eritque $a = \frac{2}{3}$ et $lb\sqrt{a} = \frac{1}{2}\,lbba = \frac{1}{2}\,l\frac{1}{6} = -\frac{1}{2}\,l6$ et $la = -l\frac{3}{2}$, unde habebimus

$$\left.\begin{array}{l} A = \frac{2}{1.3} + \frac{2^2}{4.3^2} + \frac{2^3}{9.3^3} + \text{etc.} \\ B = \frac{1}{1.2} - \frac{1}{4.2^2} + \frac{1}{9.2^3} - \text{etc.} \end{array}\right\} = \frac{\pi\pi}{6} - \frac{1}{2}\,l\frac{3}{2}\,.\,l6.$$

Subtrahamus hinc ex problemate primo hanc aequationem:

$$\left.\begin{array}{l} \frac{1}{1.3} + \frac{1}{4.3^2} + \frac{1}{9.3^3} \text{ etc.} \\ + \frac{2}{1.3} + \frac{2^2}{4.3^2} + \frac{2^3}{9.3^3} \text{ etc.} \end{array}\right\} = \frac{\pi\pi}{6} - l3\,.\,l\frac{3}{2}.$$

et remanebit

$$\left.\begin{array}{l} \frac{1}{1.2} - \frac{1}{4.2^2} + \frac{1}{9.2^3} - \frac{1}{16.2^4} + \text{etc.} \\ - \frac{1}{1.3} - \frac{1}{4.3^2} - \frac{1}{9.3^3} - \frac{1}{16.3^4} - \text{etc.} \end{array}\right\} = l3\,.\,l\frac{3}{2} - \frac{1}{2}\,l\frac{3}{2}\,.\,l6 = \frac{1}{2}(l\frac{3}{2})^2.$$

Sicque nacti sumus hanc aequationem notatu dignam:

$$\frac{1}{1.2} - \frac{1}{4.2^2} + \frac{1}{9.2^3} - \text{etc.} = \frac{1}{2}(l\frac{3}{2})^2 + \frac{1}{1.3} + \frac{1}{4.3^2} + \frac{1}{9.3^3} + \text{etc.}$$

Ubi ratio peripheriae π penitus e calculo excessit. Verum eadem relatio sequenti modo facilius eruitur.

Alia solutio ejusdem problematis.

§. 10. Manente evolutione prioris partis p, altera pars q, ob $lx = l(1+y) = ly + l(1+\frac{1}{y})$, hinc $lx = ly + \frac{1}{y} - \frac{1}{2y^2} + \frac{1}{3y^3}$ etc. erit $q = \int \frac{\partial y}{y}(l\,x) = \frac{1}{2}(ly)^2 - \frac{1}{y} + \frac{1}{4y^2} - \frac{1}{9y^3} + \frac{1}{16y^4}$ — etc. Nunc igitur erit $p + q = lx\,.\,ly + C$; ubi constans C inde definiri potest, quod posito $y = 1$ fit $x = 2$, hincque $p = \frac{1}{2}(l2)^2 + \frac{\pi\pi}{12} - \frac{1}{2}(l2)^2 = \frac{\pi\pi}{12}$ et $q = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16}$ — etc. $= -\frac{\pi\pi}{12}$, quibus valoribus substitutis pro hoc casu prodit $p + q = 0 = 0 + C$, consequenter $C = 0$.

§. 11. Verum haec constans etiam alio modo definiri potest. Ponamus brev. gr. $X = \frac{1}{x} + \frac{1}{4xx} + \frac{1}{9x^3} - \frac{1}{16x^4} +$ etc. et $Y = \frac{1}{y} - \frac{1}{4y^2} + \frac{1}{9y^3} - \frac{1}{16y^4} +$ etc., ut habeamus $p = \frac{1}{2}(lx)^2 + X$ et $Q = \frac{1}{2}(ly^2) - Y$, hincque fiet

$$p + q = \frac{1}{2}(lx)^2 + \frac{1}{2}(ly)^2 + X - Y = lx\,.\,ly + C;$$

unde deducimus

$$Y - X = \frac{1}{2}(lx)^2 + \frac{1}{2}(ly)^2 - lx\,.\,ly - C = \frac{1}{2}(l\tfrac{x}{y})^2 - C,$$

ubi notandum est, esse $y = x - 1$. Jam ad constantem C definiendam consideretur casus $x = \infty$, quo fit $X = 0$ et $Y = 0$; praeterea vero $l\frac{x}{y} = 0$, quibus notatis erit $0 = -C$, ideoque $C = 0$.

§. 12. Hinc igitur nacti sumus duas series X et Y, quarum differentia per solos logarithmos exprimitur, cum sit $Y - X = \frac{1}{2}(l\frac{x}{y})^2 = \frac{1}{2}(l\frac{y+1}{y})^2$, ob $x = y + 1$. Ex hac forma, sumto $y = 2$, statim fluit relatio ante inventa:

$$\frac{1}{1 \cdot 2} - \frac{1}{4 \cdot 2^2} + \frac{1}{9 \cdot 2^3} - \frac{1}{16 \cdot 2^4} + \text{etc.}$$

$$= \frac{1}{2}(l\frac{3}{2})^2 + \frac{1}{1 \cdot 3} + \frac{1}{4 \cdot 3^2} + \frac{1}{9 \cdot 3^3} + \frac{1}{16 \cdot 3^4} + \text{etc.}$$

Simili autem modo nunc multo generalius habebimus:

$$\frac{1}{1 \cdot} - \frac{1}{4 \cdot y^2} + \frac{1}{9 \cdot y^3} - \frac{1}{16 \cdot y^4} + \text{etc.}$$

$$= \frac{1}{2}(l\frac{y+1}{y})^2 + \frac{1}{1(y+1)} + \frac{1}{4(y+1)^2} + \frac{1}{9(y+1)^3} + \text{etc.}$$

ubi loco y quicquid lubuerit accipere licet.

Problema III.

Si inter x et y haec detur relatio: $xy + x + y = c$, binas formulas $p = \int \frac{\partial x}{x} ly$ et $q = \int \frac{\partial y}{y} lx$ in series resolvere, ita ut hinc prodeat $p + q = lx \cdot ly + C$.

Solutio.

§. 13. Hinc igitur primo erit $y = \frac{c-x}{1+x}$, cujus logarithmus per duas series sequentes exprimitur:

$$ly = \begin{cases} lc - \frac{x}{c} - \frac{x^2}{2c^2} - \frac{x^3}{3c^3} - \frac{x^4}{4c^4} - \text{etc.} \\ -x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} - \text{etc.} \end{cases}$$

unde fit

$$p = \int \frac{\partial x}{x} ly = \begin{cases} lc \cdot lx - \frac{x}{c} - \frac{x^2}{4c^2} - \frac{x^3}{9c^3} - \frac{x^4}{16c^4} - \text{etc.} \\ \qquad - x + \frac{x^2}{4} - \frac{x^3}{9} + \frac{x^4}{16} - \text{etc.} \end{cases}$$

Simili modo, cum sit $x = \frac{c-y}{1+y}$, erit

$$q = \int \frac{\partial y}{y}\, lx = \begin{cases} lc\,.\,ly - \frac{y}{c} - \frac{y^2}{4c^2} - \frac{y^3}{9c^3} - \frac{y^4}{16c^4} - \text{etc.} \\ \qquad - \frac{y}{1} + \frac{y^2}{4} - \frac{y^3}{9} + \frac{y^4}{16} - \text{etc.} \end{cases}$$

Atque hinc erit $p + q = lx\,.\,ly + \mathrm{C}$.

§. 14. Pro constante definienda consideremus casum quo $x = 0$, ideoque $p = lc\,.\,lx$ et

$$q = \left.\begin{matrix} (lc)^2 - 1 - \frac{1}{4} - \frac{1}{9} - \frac{1}{16} - \text{etc.} \\ - \frac{c}{1} + \frac{c^2}{4} - \frac{c^3}{9} + \frac{c^4}{16} - \text{etc.} \end{matrix}\right\}$$

sive $q = (lc)^2 - \frac{\pi\pi}{6} - \frac{c}{1} + \frac{c^2}{4} - \frac{c^3}{9} + \frac{c^4}{16} - \text{etc.}$, unde aequatio nostra evadit $p + q = lc\,.\,lx + (lc)^2 - \frac{\pi\pi}{6} - \frac{c}{1} + \frac{c^2}{4} - \frac{c^3}{9} + \text{etc.}$ $= lc\,.\,lx + \mathrm{C}$, ubi ergo termini $lc\,.\,lx$ se mutuo destruunt, ita ut sit $\mathrm{C} = (lc)^2 - \frac{\pi\pi}{6} - \frac{c}{1} + \frac{c^2}{4} - \frac{c^3}{9} + \text{etc.}$

§. 15. Hic ergo quinque occurrunt series infinitae, quas sequenti modo indicemus:

$$\frac{c}{1} - \frac{c^2}{4} + \frac{c^3}{9} - \frac{c^4}{16} + \text{etc.} = \mathrm{O}$$
$$\frac{x}{c} + \frac{x^2}{4\,.\,c^2} + \frac{x^3}{9\,.\,c^3} + \frac{x^4}{16\,.\,c^4} + \text{etc.} = \mathrm{P}$$
$$\frac{x}{1} - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \text{etc.} = \mathrm{Q}$$
$$\frac{y}{c} + \frac{y^2}{4\,.\,c^2} + \frac{y^3}{9\,.\,c^3} + \frac{y^4}{16\,.\,c^4} + \text{etc.} = \mathrm{R}$$
$$\frac{y}{1} - \frac{y^2}{4} + \frac{y^3}{9} - \frac{y^4}{16} + \text{etc.} = \mathrm{S},$$

quibus litteris introductis nostra aequatio erit:

$$lc\,.\,lx - \mathrm{P} - \mathrm{Q} + lc\,.\,ly - \mathrm{R} - \mathrm{S} = lx\,.\,ly + (lc)^2 - \frac{\pi\pi}{6} - \mathrm{O},$$

unde sequitur fore:

$$\mathrm{O} - \mathrm{P} - \mathrm{Q} - \mathrm{R} - \mathrm{S} = lx\,.\,ly + (lc)^2 - lc\,.\,lx - lc\,.\,ly - \frac{\pi\pi}{6}.$$

quae expressio contrahitur in sequentem:

$O - P - Q - R - S = l\frac{x}{c} \cdot l\frac{y}{c} - \frac{\pi\pi}{6}$, sive mutatis signis:

$P + Q + R + S - O = \frac{\pi\pi}{6} - l\frac{x}{c} \cdot l\frac{y}{c}$.

§. 16. Hic casus satis memorabilis occurrit, quando $c = 1$, quia tum fit

$$P + Q = \frac{2x}{1} + \frac{2x^3}{9} + \frac{2x^5}{25} + \text{etc.}$$

$$\text{et } R + S = \frac{2y}{1} + \frac{2y^3}{9} + \frac{2y^5}{25} + \text{etc.};$$

tum vero $O = \frac{\pi\pi}{12}$, sicque inter binas series satis simplicem relationem sumus assecuti, quae est:

$$\left.\begin{array}{l} +\frac{x}{1} + \frac{x^3}{9} + \frac{x^5}{25} + \frac{x^7}{49} + \text{etc.} \\ +\frac{y}{1} + \frac{y^3}{9} + \frac{y^5}{25} + \frac{y^7}{49} + \text{etc.} \end{array}\right\} = \frac{\pi\pi}{8} - \frac{1}{2} lx \cdot ly,$$

ubi notandum est fore $xy + x + y = 1$, hinc sive $y = \frac{1-x}{1+x}$, sive $x = \frac{1-y}{1+y}$, cujus aliquot exempla evolvisse juvabit.

§. 17. 1°) Si $x = \frac{1}{2}$, erit $y = \frac{1}{3}$, unde sequitur aequatio:

$$\left.\begin{array}{l} \frac{1}{1.2} + \frac{1}{9.2^3} + \frac{1}{25.2^5} + \frac{1}{49.2^7} + \text{etc.} \\ + \frac{1}{1.3} + \frac{1}{9.3^3} + \frac{1}{25.3^5} + \frac{1}{49.2^7} + \text{etc.} \end{array}\right\} = \frac{\pi\pi}{8} - \frac{1}{2} l2 \cdot l3.$$

2°) Si $x = \frac{1}{4}$, erit $y = \frac{3}{5}$, ideoque

$$\left.\begin{array}{l} \frac{1}{1.4} + \frac{1}{9.4^3} + \frac{1}{25.4^5} + \frac{1}{49.4^7} + \text{etc.} \\ + \frac{3}{1.5} + \frac{3^3}{9.5^3} + \frac{3^5}{25.5^5} + \frac{3^7}{49.5^7} + \text{etc.} \end{array}\right\} = \frac{\pi\pi}{8} - \frac{1}{2} l4 \cdot l\frac{5}{3}.$$

3°) Quin etiam datur casus quo $x = y$, quod evenit ponendo $x = y = -1 + \sqrt{2} = a$; tum igitur fiet:

$$\frac{a}{1} + \frac{a^3}{9} + \frac{a^5}{25} + \frac{a^7}{49} + \text{etc.} = \frac{\pi\pi}{16} - \frac{1}{4}(la)^2.$$

§. 18. In genere igitur etiam, quicquid fuerit c, operae pretium erit casum perpendere, quo fit $x = y$, quod evenit si $x = y = -\frac{1+\sqrt{1+c}}{2} = a$; tum igitur erit:

$$P = R = \frac{a}{c} + \frac{a^2}{4.c^2} + \frac{a^3}{9.c^3} + \frac{a^4}{16.c^4} + \text{etc.}$$

$$Q = S = \frac{a}{1} - \frac{a^2}{4} + \frac{a^3}{9} - \frac{a^4}{16} + \text{etc.}$$

unde deducitur ista aequatio:

$$\left.\begin{array}{l} \frac{a}{1.c} + \frac{a^2}{4.c^2} + \frac{a^3}{9.c^3} + \frac{a^4}{16.c^4} + \text{etc.} \\ + \frac{a}{1} - \frac{a^2}{4} + \frac{a^3}{9} - \frac{a^4}{16} + \text{etc.} \end{array}\right\} = \frac{\pi\pi}{12} - \frac{1}{2}(l\frac{a}{c})^2 + \frac{1}{2}(\frac{c}{1} - \frac{c^2}{4} + \frac{c^3}{9} - \text{etc.}).$$

Hinc plurimas egregias relationes inter ternas hujusmodi series derivare licet, quae ergo evadunt rationales, quoties fuerit $1 + c$ quadratum.

§. 19. Plures alias relationes inter binos numeros x et y evolvere liceret, in hac scilicet forma generali contentas: $xy \pm \alpha x \pm \beta y = \gamma$, quae autem, posito $x = \beta t$ et $y = \alpha u$, in hanc simpliciorem mutatur: $tu \pm t \pm u = \frac{\gamma}{\alpha\beta}$, ubi tantum varietas signorum in computum venit. Verum quia hinc plerumque tres pluresve series reperiuntur, alteriori evolutioni hic non immoror, sed potissimum iis casibus inhaerebo, quibus relatio inter duas tantum hujusmodi series definitur, quos igitur in sequentibus theorematibus sum complexurus.

Theorema I.

§. 20. *Si habeantur hae duae series:*

$$X = \frac{x}{1} + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \text{etc. } et$$

$$Y = \frac{y}{1} + \frac{y^2}{4} + \frac{y^3}{9} + \frac{y^4}{16} + \text{etc.}$$

fueritque $x + y = 1$; *tum semper erit:*

$$X + Y = \frac{\pi\pi}{6} - lx \,.\, ly.$$

cujus theorematis demonstratio in §. 4. jam est tradita.

Corollarium I.

§. 21. Hic ante omnia manifestum est, summas harum serierum reales esse non posse, simulac vel x vel y unitatem superaverit. Summa quidem his casibus videtur in infinitum excrescere; verum ea fit adeo imaginaria, cum, ob y negativum, logarithmus y imaginarius evadat.

Corollarium II.

§. 22. Usus hujus theorematis potissimum iis casibus cernitur, quibus x parum ab unitate deficit, ideoque prior series X parum convergit; tum enim altera Y eo magis converget. Veluti si fuerit $x = \frac{9}{10}$, erit:

$$X = \frac{9}{10} + \frac{9^2}{4 \cdot 10^2} + \frac{9^3}{9 \cdot 10^3} + \frac{9^4}{16 \cdot 10^4} + \text{etc.},$$

series vix convergens, cujus tamen summa per nostrum theorema facile quam proxime assignari poterit. Cum enim sit $Y = \frac{1}{10} + \frac{1}{4 \cdot 10^2} + \frac{1}{9 \cdot 10^3} + \frac{1}{16 \cdot 10^4} + \text{etc.}$, quae series est maxime convergens, erit utique $X = \frac{\pi\pi}{6} - l10 \,.\, l\frac{10}{9} - Y$.

Corollarium III.

§. 23. Ita in genere, si statuamus $x = \frac{m}{m+n}$ et $y = \frac{n}{m+n}$, erit $X = \frac{m}{1(m+n)} + \frac{m^2}{4(m+n)^2} + \frac{m^3}{9(m+n)^3} + \text{etc.}$ et

$$Y = \frac{n}{1(m+n)} + \frac{n^2}{4(m+n)^2} + \frac{n^3}{9(m+n)^3} + \text{etc.};$$

tum igitur erit $X + Y = \frac{\pi\pi}{6} - l\frac{m+n}{m} \cdot l\frac{n+n}{n}$.

Theorema II.

§. 24. *Si habeantur hae duae series:*

$$X = \frac{1}{x} - \frac{1}{4xx} + \frac{1}{9x^3} - \frac{1}{16x^4} + \text{etc.}$$

$$Y = \frac{1}{y} + \frac{1}{4yy} + \frac{1}{9y^3} + \frac{1}{16y^4} + \text{etc.}$$

existente $y = x + 1$, *semper erit*

$$X - Y = \frac{1}{2}(l\tfrac{y}{x})^2 = \frac{1}{2}(l\tfrac{x+1}{x})^2,$$

cujus demonstratio colligitur ex §. 12, dummodo litterae x, y et X, Y, permutantur.

Corollarium I.

§. 25. Quia hic est $y = x + 1$, posterior series Y magis convergit quam prior X. Quin etiam, si prior series X fuerit adeo divergens, quod evenit, quando x est fractio unitate minor, posterior nihilominus manet convergens. Veluti si fuerit $x = \frac{1}{2}$, erit $y = \frac{3}{4}$, ipsae vero series erunt:

$$X = \frac{2}{1} - \frac{2^2}{4} + \frac{2^3}{9} - \frac{2^4}{16} + \frac{2^5}{25} - \text{etc.} \quad \text{et}$$

$$Y = \frac{2}{3} + \frac{2^2}{4 \cdot 3^2} + \frac{2^3}{9 \cdot 3^3} + \frac{2^4}{16 \cdot 3^4} + \text{etc.}$$

consequenter erit $X - Y = \frac{1}{2}(l\,3)^2$.

Quia vero posterior series y parum convergit, eam per theorema primum hoc modo reducimus:

$$\frac{2}{1\cdot 3}+\frac{2^2}{4\cdot 3^2}+\frac{2^3}{9\cdot 3^3}+\text{etc.}=\frac{\pi\pi}{6}-l3\,.\,l\tfrac{3}{2}-\frac{1}{1\cdot 3}-\frac{1}{4\cdot 3^2}-\frac{1}{9\cdot 3^3}-\text{etc.}$$

hincque habebimus hanc summationem:

$$\frac{2}{1}-\frac{2^2}{4}+\frac{2^3}{9}-\frac{2^4}{16}+\text{etc.}=\tfrac{1}{2}(l3)^2+\frac{\pi\pi}{6}-l3\,.\,l\tfrac{3}{2}-\left(\frac{1}{1\cdot 3}+\frac{1}{4\cdot 3^2}+\frac{1}{9\cdot 3^3}+\text{etc.}\right)$$

Corollarium II.

§. 26. Sumamus nunc in genere $x=\frac{1}{n}$, ut sit series summanda $X=\frac{n}{1}-\frac{n^2}{4}+\frac{n^3}{9}-\frac{n^4}{16}+\text{etc.}$, tum vero ob $y=\frac{1+n}{n}$ altera series erit $Y=\frac{n}{n+1}+\frac{nn}{4(n+1)^2}+\frac{n^2}{9(n+1)^2}$ etc. hincque $X=\frac{1}{2}(ln+1)^2+Y$. At vero per theorema I. est

$$Y=\frac{\pi\pi}{6}-l(n+1)\,.\,l\frac{n+1}{n}-\frac{1}{n+1}-\frac{1}{4(n+1)^2}-\frac{1}{9(n+1)^3}-\text{etc.}$$

quo valore substituto erit:

$$X=\tfrac{1}{2}(l(n+1))^2+\frac{\pi\pi}{6}-l(n+1).l\frac{n+1}{n}-\left(\frac{1}{n+1}+\frac{1}{4(n+1)^2}+\frac{1}{9(n+1)^3}+\text{etc.}\right)$$

quae expressio contrahitur in hanc:

$$\frac{n}{1}-\frac{n^2}{4}+\frac{n^3}{9}-\frac{n^4}{16}+\text{etc.}$$
$$=\tfrac{1}{2}l(n+1)\,.\,l\frac{nn}{n+1}+\frac{\pi\pi}{6}-\left(\frac{1}{n+1}+\frac{1}{4(n+1)^2}+\frac{1}{9(n+1)^3}+\text{etc.}\right).$$

Theorema III.

§. 27. *Si habeantur hae duae series:*

$$X=\frac{x}{1}-\frac{x^2}{4}+\frac{x^3}{9}-\frac{x^4}{16}+\text{etc. } et$$
$$X=\frac{1}{x}-\frac{1}{4x^2}+\frac{1}{9x^3}-\frac{1}{16x^4}+\text{etc.}$$
erit $X+Y=\frac{\pi\pi}{6}+\frac{1}{2}(lx)^2.$

Demonstratio in praecedentibus non continetur, verum ea hoc modo facile adornatur:

Cum per formulam integralem sit $X = \int \frac{\partial x}{x} l(1+x)$, loco x scribendo $\frac{1}{x}$ erit $Y = -\int \frac{\partial x}{x} l \frac{x}{1+x}$, sive

$$Y = -\int \frac{\partial x}{x} l(1+x) + \int \frac{\partial x}{x} lx,$$

hincque addendo $X + Y = \int \frac{\partial x}{x} lx = \frac{1}{2}(lx)^2 + C$; ubi constans ex casu $x = 1$ facillime definitur. Quia enim hoc casu fit tam X quam $Y = \frac{\pi\pi}{12}$, erit constans $C = \frac{\pi\pi}{6}$, ideoque $X + Y = \frac{\pi\pi}{6} + \frac{1}{2}(lx)^2$.

Corollarium I.

§. 28. Quod si ergo pro x numerus quantumvis magnus accipiatur, ope hujus theorematis summa seriei X, quae maxime est divergens, facillime assignatur, cum reducatur ad seriem Y, quae eo magis est convergens, quo magis prior divergit.

Corollarium II.

§. 29. Nunc vero, ope theorematis secundi, series $Y = \frac{1}{x} - \frac{1}{4x^2} + \frac{1}{9x^3} - \text{etc.}$ reducitur ad hanc formam:

$$Y = \frac{1}{2}\left(l\frac{x+1}{x}\right)^2 + \frac{1}{x+1} + \frac{1}{4(x+1)^2} + \frac{1}{9(x+1)^3} + \text{etc.}$$

quo valore substituto prodibit sequens aequatio:

$$\frac{x}{1} - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \text{etc.}$$

$$= \frac{\pi\pi}{6} + \frac{1}{2}(lx)^2 - \frac{1}{2}\left(l\frac{x+1}{x}\right)^2 - \left(\frac{1}{x+1} + \frac{1}{4(x+1)^2} + \frac{1}{9(x+1)^3} + \text{etc.}\right)$$

quae expressio cum superiori §. 26. egregie convenit, quia est $\frac{1}{2} l(x+1) \,.\, l\frac{xx}{x+1} = \frac{1}{2}(lx)^2 - \frac{1}{2}\left(l\frac{x+1}{x}\right)^2$, uti evolventi facile patebit.

Theorema IV.

§. 30. *Si habeantur hae series:*

$X = \frac{x}{1} + \frac{x^3}{9} + \frac{x^5}{25} + \text{etc.}$ *et* $Y = \frac{y}{1} + \frac{y^3}{9} + \frac{y^5}{25} + \text{etc.}$,
existente $xy + x + y = 1$, *sive* $x = \frac{1-y}{1+y}$, *vel* $y = \frac{1-x}{1+x}$,
erit $X + Y = \frac{\pi\pi}{8} - \frac{1}{2} lx \, . \, ly$.

Demonstratio manifesta est ex §. 16.

Corollarium I.

§. 31. Hic iterum, ut supra, observandum est, summas harum serierum fieri imaginarias, simulac litterae x et y unitatem superaverint. At si fuerit $x < 1$, tum semper alia series ejusdem formae exhiberi potest, cujus summa ab illa pendeat. Ita si fuerit $x = \frac{1}{2}$, erit $y = \frac{1}{3}$. At si x prope ad unitatem accedat, veluti $x = \frac{9}{10}$, altera series Y maxime converget.

Corollarium II.

§. 32. In his quatuor theorematibus omnes casus contineri videntur, quibus binas hujusmodi series inter se comparare licet. Ad quod ostendendum sequens theorema speciale subjungamus, quod demum per longas calculi ambages sum adeptus, quod autem nunc satis commode ex praecedentibus theorematibus deduci potest.

Theorema speciale.

§. 33. *Si habeantur hae series sibi affines:*

$A = \frac{1}{1.3} + \frac{1}{9.3^3} + \frac{1}{25.3^5} + \text{etc.}$ *et*

$B = \frac{1}{1.3} + \frac{1}{4.3^2} + \frac{1}{9.3^3} + \text{etc.};$ *tum erit*

$2A + B = \frac{\pi\pi}{6} - \frac{1}{2}(l\,3)^2.$

Demonstratio:

Cum ex theoremate primo, sumto $x = y = \frac{1}{2}$, sit $\frac{1}{1.2} + \frac{1}{4.2^2} + \frac{1}{9.2^3} + \text{etc.} = \frac{\pi\pi}{12} - \frac{1}{2}(l\,2)^2$, haec series sequenti modó resoluta repraesentari potest:

$$2\left(\frac{1}{1.2} + \frac{1}{9.2^3} + \frac{1}{25.2^5} + \text{etc.}\right) - 1\left(\frac{1}{1.2} - \frac{1}{4.2^2} + \frac{9}{9.2^3} - \text{etc.}\right) = \frac{\pi\pi}{12} - \frac{1}{2}(l2)^2.$$

Nunc vero per theorema IV., sumto $x = \frac{1}{2}$ et $y = \frac{1}{3}$, habemus hanc aequationem:

$$\frac{1}{1.2} + \frac{1}{9.2^3} + \frac{1}{25.2^5} + \text{etc.} = \frac{\pi\pi}{8} - \frac{1}{2}l2\,.\,l3 - \frac{1}{1.3} - \frac{1}{9.3^3} - \frac{1}{25.3^5} - \text{etc.}.$$

Deinde vero ex theoremate secundo, sumto $x = 2$ et $y = 3$, erit:

$$\frac{1}{1.2} - \frac{1}{4.2^2} + \frac{1}{9.2^3} - \frac{1}{16.2^4} + \text{etc.} = \frac{1}{2}(l\tfrac{3}{2})^2 + \frac{1}{1.3} + \frac{1}{4.3^2} + \frac{1}{9.3^3} + \text{etc.}$$

Substituantur jam hi valores loco illarum serierum, ac pro parte sinistra prodibit:

$$\left.\begin{matrix}\frac{\pi\pi}{4} - l2\,.\,l3 - 2\left(\frac{1}{1.3} + \frac{1}{9.3^3} + \frac{1}{25.3^5} + \text{etc.}\right) \\ - \frac{1}{2}(l\tfrac{3}{2})^2 - \left(\frac{1}{1.3} + \frac{1}{4.3^2} + \frac{1}{9.3^3} + \text{etc.}\right)\end{matrix}\right\} = \frac{\pi\pi}{12} - \frac{1}{2}(l\,2)^2.$$

Unde concludimus fore:

$$\left.\begin{matrix}2\left(\frac{1}{1.3} + \frac{1}{9.3^3} + \frac{1}{25.3^5} + \text{etc.}\right) \\ + 1\left(\frac{1}{1.3} + \frac{1}{4.3^2} + \frac{1}{9.3^3} + \text{etc.}\right)\end{matrix}\right\} = \frac{\pi\pi}{6} - l2\,.\,l3 - \frac{1}{2}(l\tfrac{3}{2})^2 + \frac{1}{2}(l\,2)^2$$

$$= \frac{\pi\pi}{6} - \frac{1}{2}(l\,3)^2 \;(\text{ob } (l\tfrac{3}{2})^2 = (l\,3)^2 - 2\,l\,2\,.\,l\,3 + (l\,2)^2).$$

§. 34. Quomodocunque autem theoremata hic data inter se combinentur, vix alia relatio inter binas hujusmodi series elici potest, multo minus autem inde ejusmodi series simplices eruere licet, quarum summa absolute exhiberi queat, praeter casus jam indicatos, quos igitur hic conjunctim ob oculos ponamus.

$$1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \tfrac{1}{25} + \text{etc.} = \tfrac{\pi\pi}{6}$$

$$1 - \tfrac{1}{4} + \tfrac{1}{9} - \tfrac{1}{16} + \tfrac{1}{25} - \text{etc.} = \tfrac{\pi\pi}{12}$$

$$\tfrac{1}{1.2} + \tfrac{1}{4.2^2} + \tfrac{1}{9.2^3} + \tfrac{1}{16.2^4} + \text{etc.} = \tfrac{\pi\pi}{12} - \tfrac{1}{2}(l2)^2$$

$$1 + \tfrac{1}{9} + \tfrac{1}{25} + \tfrac{1}{49} + \text{etc.} = \tfrac{\pi\pi}{8}$$

Praeterea vero adjungi potest adhuc ista series:

$$\tfrac{a}{1} + \tfrac{a^3}{9} + \tfrac{a^5}{25} + \tfrac{a^7}{49} + \text{etc.} = \tfrac{\pi\pi}{16} - \tfrac{1}{4}(la)^2$$

existente $a = \sqrt{2} - 1$. Quanquam autem in hac serie valor ipsius a sit irrationalis, ideoque quaevis potestas seorsim evolvi debere videatur, tamen numeratores etiam seriem recurrentem constituunt, in qua quilibet terminus per binos praecedentes definiri potest ope hujus formulae: $a^{n+4} = 6a^{n+2} - a^n$, cujus veritas inde elucet, quod sit, per a^n dividendo, $a^4 = 6aa - 1$. Quia enim $a = \sqrt{2} - 1$, erit $a^2 = 3 - 2\sqrt{2}$ et $a^4 = 17 - 12\sqrt{2}$, unde veritas fit manifesta.