SOLUTIO
TRIUM PROBLEMATUM DIFFICILIORU[M]
AD METHODUM TANGENTIUM INVERSAM PERTINENTIUM.

AUCTORE

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Cum Ellipsis ea gaudeat proprietate, ut, ductis ex ejus fo[cis] ad punctum quodcunque in curva duobus rectis, eae aequaliter [ad] curvam inclinentur, earumque summa simul ubique ejusdem sit qua[n]titatis: hinc formari poterunt duae quaestiones reciprocae haud [fa]cilis indaginis, quae ob artificia calculi in solvendo adhibita atte[n]tionem merere videntur. Eas igitur breviter hic exhibere a[ni]mus est.

Problema 1.

Tab. I. Fig. 4. *Datis duobus punctis A et B invenire lineam curvam FM ita comparatam ut, ductis ex singulis ejus punctis [...] rectis MA et MB, eae utrinque aequaliter ad curv[am] inclinentur.*

Solutio:

Sint rectae $AM = z$ et $BM = v$, vocenturque anguli $MAB = \Phi$, $MBA = \psi$ et anguli inclinationis $AMF = BMG = \omega$. Tum si co[n]sideretur aliud punctum curvae proximum m, ducta recta Am, d[i]missoque ex m in AM perpendiculo mu, erit angulus $MAm = \partial\Phi$, $Mu = -\partial z$, $mu = z\partial\Phi$, ideoque $\cot. mMu = \cot.\omega = \frac{Mu}{mu} = -\frac{\partial z}{z\partial\Phi}$.
Simili modo ex altera parte reperietur $\cot. BMG = \cot.\omega = -\frac{\partial v}{v\partial\psi}$.

ita ut haec prodeat aequatio: $\frac{\partial z}{z\partial\Phi} = \frac{\partial v}{v\partial\psi}$, sive $\frac{\partial z}{z}\partial\psi = \frac{\partial v}{v}.\partial\Phi$.

Porro ex triangulo A M B, posita recta A B $= c$; ob angulum AMB $= 180^\circ - (\Phi + \psi)$, erit $z = \frac{c \sin\psi}{\sin.(\Phi+\psi)}$ et $v = \frac{c \sin.\Phi}{\sin.(\Phi+\psi)}$.

Hinc fiet sumtis differentialibus logarithmicis

$$\frac{\partial z}{z} = \frac{\partial\psi}{\text{tag.}\,\psi} - \frac{(\partial\Phi + \partial\psi)}{\text{tag.}\,(\Phi+\psi)},$$

$$\frac{\partial v}{v} = \frac{\partial\Phi}{\text{tag.}\,\Phi} - \frac{(\partial\Phi + \partial\psi)}{\text{tag.}\,(\Phi+\psi)},$$

quibus substitutis aequatio illa hanc induet formam:

$$\frac{\partial\psi^2}{\text{tag.}\,\psi} - \frac{\partial\psi(\partial\Phi+\partial\psi)}{\text{tag.}\,(\Phi+\psi)} = \frac{\partial\Phi^2}{\text{tag.}\,\Phi} - \frac{\partial\Phi(\partial\Phi+\partial\psi)}{\text{tag.}\,(\Phi+\psi)},$$

sive $\frac{\partial\psi^2}{\text{tag.}\,\psi} - \frac{\partial\Phi^2}{\text{tag.}\,\Phi} = \frac{\partial\psi^2 - \partial\Phi^2}{\text{tag.}\,(\Phi+\psi)}$, quae transmutatur in hanc:

$$\partial\psi^2\left(\frac{\cos.\psi}{\sin.\psi} - \frac{\cos.(\Phi+\psi)}{\sin.(\Phi+\psi)}\right) = \partial\Phi^2\left(\frac{\cos.\Phi}{\sin.\Phi} - \frac{\cos.(\Phi+\psi)}{\sin.(\Phi+\psi)}\right), \text{ unde}$$

$$\frac{\partial\psi^2 \sin.(\Phi+\psi-\psi)}{\sin.\psi \sin.(\Phi+\psi)} = \frac{\partial\Phi^2 \sin.(\Phi+\psi-\Phi)}{\sin\Phi \sin.(\Phi+\psi)}$$

sive denique $\partial\psi^2 \sin.\Phi^2 = \partial\Phi^2 \sin.\psi^2$, ideoque $\frac{\partial\psi}{\sin.\psi} = \pm \frac{\partial\Phi}{\sin\Phi}$ unde integrando erit $l\,\text{tag.}\,\tfrac{1}{2}\psi = \pm\, l\,\text{tag.}\,\tfrac{1}{2}\Phi + l\text{C}$, ita ut duae nascantur solutiones, quarum prima ex aequatione $\text{tag.}\,\tfrac{1}{2}\psi = \text{C}\,\text{tag.}\,\tfrac{1}{2}\Phi$ est deducenda.

I. Ponatur $\text{tag.}\,\tfrac{1}{2}\Phi = \frac{t}{a}$ et $\text{tag.}\,\tfrac{1}{2}\psi = \frac{t}{b}$, fietque

$$\sin.\Phi = \frac{2at}{aa+tt}, \quad \cos.\Phi = \frac{aa-tt}{aa+tt}$$

$$\sin.\psi = \frac{2bt}{bb+tt}, \quad \cos.\psi = \frac{bb-tt}{bb+tt}, \text{ unde colligitur}$$

$$\sin.(\Phi+\psi) = \frac{2t(a+b)(ab-tt)}{(aa+tt)(bb+tt)}. \text{ Hinc fit}$$

$$z = \frac{c \sin.\psi}{\sin.(\Phi+\psi)} = \frac{bc(aa+tt)}{(a+b)(ab-tt)},$$

quo valore invento coordinatae pro curva quaesita facile determinantur, quae si vocentur AP $= x$, PM $= y$, erit

$$x = z\cos.\Phi = \frac{bc\,(aa - tt)}{(a+b)\,(ab - tt)},$$

$$y = z\sin.\Phi = \frac{2\,a\,b\,c\,t}{(a+b)\,(ab - tt)}.$$

Sit brevitatis gratia $\frac{bc}{a+b} = f$, eritque $x = \frac{f\,(aa - tt)}{ab - tt}$, unde $tt = \frac{a\,(af - bx)}{f - x}$, et $ab - tt = \frac{af\,(b - a)}{f - x}$, hincque colligitur $y = \frac{2}{b - a}\sqrt{a\,(f - x)(af - bx)}$, sive $yy = \frac{4a}{(b-a)^2}(f - x)\,(af - b$ aequatio pro Hyperbola.

II. Pro altero signo, iisdem denominationibus adhibitis, perietur:

$\sin.(\Phi + \psi) = \frac{2t\,(a - b)\,(ab + tt)}{(aa + tt)\,(bb + tt)}$, ex quo fit $z = \frac{bc\,(aa + tt)}{(a - b)\,(ab +}$ sicque habebimus coordinatas

$$AP = x = z\cos.\Phi = \frac{bc\,(aa - tt)}{(a - b)\,(ab + tt)}$$

$$PM = y = z\sin.\Phi = \frac{2\,a\,b\,c\,t}{(a - b)\,(ab + tt)}.$$

unde, posito ut supra, $\frac{bc}{a - b} = f$, erit

$$x = \frac{f\,(aa - tt)}{ab + tt} \quad \text{et} \quad y = \frac{2\,a\,f\,t}{ab + tt},$$

atque ob $tt = \frac{a\,(af - bx)}{f + x}$ et $ab + tt = \frac{af\,(a + b)}{f + x}$, aequatio inter co dinatas prodit haec: $yy = \frac{4a}{(a+b)^2}(f + x)\,(af - bx)$, pro Ellip

Problema 2.

Tab. I. Fig. 5. *Invenire lineam curvam, ad axem AO et punctum fixum A ferendam, ejusmodi ut sumto radio incidente AM, cui pondeat radius reflexus MO, summa amborum AM + sit ubique constans = a.*

Solutio:

Ducta ad curvam normali MN anguli AMN et OMN er inter se aequales. Hinc si, ut in problemate praecedente, voc tur anguli MAN $= \phi$, MCN $= \psi$, tum vero AMN $=$ OMN $= \omega$, erit $\psi = 180^\circ - \phi - 2\omega$. Sit AM $= z$, OM $= v$, eritque $z + v = a$, ideoque $v = a - z$, unde ex triangulo AMO erit $z : a - z = \sin.\psi : \sin.\phi$, consequenter $z = \frac{a \sin.\psi}{\sin.\phi + \sin.\psi}$. Porro, ob $z : \sin.\psi =$ AO $: \sin. 2\omega$, erit AO $= \frac{z \sin. 2\omega}{\sin.\psi} = \frac{a \sin. 2\omega}{\sin.\phi + \sin.\psi}$, ubi notetur esse $\sin.\psi = \sin.(\phi + 2\omega) = \sin.\phi \cos. 2\omega + \cos.\phi \sin. 2\omega$

Ex distantia z, cum angulo ϕ, prodit

$$\text{tag. AMF} = \text{cot. AMN} = -\frac{z \partial \phi}{\partial z},$$

ideoque $\text{tag.}\,\omega = -\frac{\partial z}{z \partial \phi}$ et $\frac{\partial z}{z} = -\partial\phi\, \text{tag.}\,\omega$, quae est aequatio problema determinans.

Pro ea evolvenda statuatur $\text{tag.}\,\phi = t$ et $\text{tag.}\,\omega = u$, eritque $\sin.\phi = \frac{t}{\sqrt{1+tt}}$, $\cos.\phi = \frac{1}{\sqrt{1+tt}}$, ut et $\sin.\omega = \frac{u}{\sqrt{1+uu}}$, $\cos.\omega = \frac{1}{\sqrt{1+uu}}$, unde fit $\sin. 2\omega = \frac{2u}{1+uu}$ et $\cos. 2\omega = \frac{1-uu}{1+uu}$; praeterea vero $\partial\phi = \frac{\partial t}{1+tt}$ et $\partial\omega = \frac{\partial u}{1+uu}$. Ex his valoribus colligitur $\sin.\psi = \frac{t(1-uu)+2u}{(1+uu)\sqrt{1+tt}}$, ideoque $\sin.\phi + \sin.\psi = \frac{2(t+u)}{(1+uu)\sqrt{1+tt}}$, unde porro fit $z = \frac{at(1-uu)+2au}{2(t+u)}$ et AO $= \frac{au\sqrt{1+tt}}{t+u}$, hinc

$$\frac{\partial z}{z} = -\partial\phi\,\text{tag.}\,\omega = -\frac{u\partial t}{1+tt} = \frac{t\partial u(1-2tu-uu) - u\partial t(1+uu)}{(t+u)(2u+t)(1-uu)}.$$

Si haec aequatio inter t et u evolvatur, prodit:

$$t\partial u(1 - 2tu - uu) = \frac{u\partial t(1-tu)(1-2tu-uu)}{1+tt},$$

quae, cum habeat divisorem $1 - 2tu - uu$, duplicem subministrat solutionem, quarum altera in aequatione $1 - 2tu - uu = 0$, altera in aequatione $t\partial u = \frac{u\partial t(1-tu)}{1+tt}$ continetur.

Ex priore aequatione prodit $t = \frac{1-uu}{2u}$, hoc est

$$\text{tag.}\,\phi = \frac{1 - \text{tag.}\,\omega^2}{2\,\text{tag.}\,\omega} = \cot. 2\omega,$$

unde concluditur fore $2\omega = 90^\circ - \phi$, ideoque $\psi = 90^\circ$. Erit igitur

$z = \frac{a\sin\psi}{\sin.\phi + \sin.\psi} = \frac{a}{1+\sin.\phi}$, sive $z = a - z\sin.\phi$.

Positis jam $AO = x$, $OM = z\sin.\phi = y$, erit

$$z = \sqrt{xx+yy} = a - z\sin.\phi = a - y,$$

sive $aa - 2ay = xx$; et posito $\frac{1}{2}a - y = v$, erit $xx = 2av$, qu...

Tab. I. Fig. 6. aequatio est pro Parabola, cujus parameter $= 2a$. Sit CA = ... erit A focus Parabolae CMB et CA axis: Constat autem si AM ... Am sint radii incidentes, radios reflexos MO, mo fore axi paral... los atque angulos AMC = BMO, ut et AmC = Bmo.

Evolvamus alteram aequationem $t\partial u = \frac{u(1-tu)\partial t}{1+tt}$, quae ... separabilitatem reducetur ponendo $t = \frac{p-u}{1+pu}$, unde differentia... do fit elementum $\partial t = \frac{\partial p(1+uu) - \partial u(1+pp)}{(1+pu)^2}$, tum vero

$$1 + tt = \frac{(1+uu)(1+pp)}{(1+pu)^2},$$

hincque colligitur $\frac{\partial t}{1+tt} = \frac{\partial p}{2+pp} - \frac{\partial u}{1+uu}$. Porro est $1 - tu = \frac{1+\ldots}{1+\ldots}$ unde facta substitutione obtinetur haec aequatio:

$$\frac{(p-u)\partial u}{1+pu} = \frac{u(1+uu)}{1+pu}\left(\frac{\partial p}{1+pp} - \frac{\partial u}{1+uu}\right),$$

sive $p\partial u = \frac{u(1+uu)\partial p}{1+pp}$, seu $\frac{\partial u}{u(1+uu)} = \frac{\partial p}{p(1+pp)}$, cujus aequat... nis, penitus separatae, integrale est $l\frac{u}{\sqrt{1+uu}} = lC + l\frac{p}{\sqrt{1+\ldots}}$ ejusque evolutio, nisi ad angulos recurrere liceret, non parum ... ret molesta. Cum autem posuerimus $t = \frac{p-u}{1+pu}$, erit

$$p = \frac{t+u}{1-tu} = \frac{\text{tag.}\,\phi + \text{tag.}\,\omega}{1 - \text{tag.}\,\phi\,\text{tag.}\,\omega} = \text{tag.}\,(\phi+\omega)$$

ideoque $\frac{p}{\sqrt{1+pp}} = \sin.(\phi+\omega)$, unde ob $\frac{u}{\sqrt{1+uu}} = \sin.\omega$ ...

Fig. 5. $\sin.\omega = C\sin.(\phi+\omega)$. Cum igitur in figura sit ang... $MNO = \phi+\omega$, erit $C = \frac{\sin.\omega}{\sin.(\phi+\omega)} = \frac{AN}{AM}$, nec minus erit C = ... et componendo $C = \frac{AN+ON}{AM+OM} = \frac{AO}{a}$, ideoque $AO = aC$, hoc ... constans. Punctum igitur O erit fixum, ex qua conditione sta... manifesto sequitur curvam esse sectionem coni, ita ut praeter...

rabolam, Hyperbolam et Ellipsin nullae aliae curvae dentur problemati satisfacientes.

Posterior aequatio $t\partial u = \frac{u(1-tu)\partial t}{1+tt}$ etiam sequenti modo resolvi potest. Reducatur ea primo ad hanc formam:

$$t\partial u - u\partial t + t^3\partial u + tuu\partial t = 0.$$

Ponatur $u = pt$ atque ob $\partial u = p\partial t + t\partial p$ prodibit haec aequatio:

$$tt(1+tt)\partial p + pt^3(1+p)\partial t = 0, \text{ sive}$$

$$\frac{\partial p}{p(1+p)} = -\frac{t\partial t}{1+tt} \text{ sive } \frac{\partial p}{p} - \frac{\partial p}{1+p} + \frac{t\partial t}{1+tt} = 0,$$

unde fit integrando $lp - l(1+p) + l\sqrt{1+tt} = lC$ et ad numeros descendendo $\frac{p\sqrt{1+tt}}{1+p} = C$, unde colligitur $p = \frac{C}{\sqrt{1+tt}-C}$ et $u = \frac{Ct}{\sqrt{1+tt}-C}$, hincque $t+u = \frac{t\sqrt{1+tt}}{\sqrt{1+tt}-C}$. Supra autem invenimus $AO = \frac{au\sqrt{1+tt}}{t+u}$, unde concluditur fore $AO = aC$, ideoque constantem ut supra, ita ut inde iterum sectio conica oriatur.

Sin autem aequationem inter coordinatas eruere atque inde naturam curvae concludere velimus, ex valore modo ante invento $u = \frac{Ct}{\sqrt{1+tt}-C}$ quaeratur $1 - uu = \frac{1+tt+CC(1-tt)-2C\sqrt{1+tt}}{(\sqrt{1+tt}-C)^2}$, atque ob $t+u = \frac{t\sqrt{1+tt}}{\sqrt{1+tt}-C}$, substitutione facta colligitur

$$z = \frac{at(1-uu)+2au}{2(t+u)} = \frac{a(1-CC)\sqrt{1+tt}}{2(\sqrt{1+tt}-C)},$$

sive posito brevitatis gratia $\frac{a(1-CC)}{2} = b$, erit $z = \frac{b\sqrt{1+tt}}{\sqrt{1+tt}-C}$.

Quod si jam introducantur coordinatae orthogonales $AN = x = z\cos.\Phi$ et $MN = y = z\sin.\Phi$, ob $\text{tag}.\Phi = \frac{y}{x} = t$ erit $\sqrt{1+tt} = \frac{\sqrt{x^2+y^2}}{x} = \frac{z}{x}$.

Hinc prodit $z = \frac{b\sqrt{1+tt}}{\sqrt{1+tt}-C} = \frac{bz}{z-Cx}$, sive $z - Cx = b$ et $z = b+Cx$, quo valore substituto in aequatione $\sqrt{xx+yy} = z$, ea abibit in istam: $yy + (1-CC)xx = 2bCx + bb$, quae est pro Ellipsi, si $C < 1$, at vero pro Hyperbola, si $C > 1$.

Alia solutio ejusdem problematis.

Maneant omnes denominationes, ut in praecedentibus sunt s
bilitae, et cum tota solutio his duabus formulis innitatur: tag. $\omega =$ —
et $\frac{z}{a-z} = \frac{\sin.\psi}{\sin.\phi}$, ponatur $\cot.\phi = v$, ut sit $v = \frac{1}{t}$ atque $\partial\phi = -\frac{\partial}{1+}$
unde fit $\frac{\partial z}{z} = -\partial\phi$ tag. $\omega = -u\partial\phi$, hoc est $\frac{\partial z}{z} = \frac{u\partial v}{1+vv}$. Alte
aequatio $\frac{z}{a-z} = \frac{\sin.\psi}{\sin.\phi}$, ob

$$\sin.\psi = \sin.(\phi + 2\omega) = \sin.\phi \cos.2\omega + \cos.\phi \sin.2\omega,$$

fit $\frac{z}{a-z} = \cos.2\omega + \cot.\phi \sin.2\omega = \frac{1-uu+2vu}{1+uu}$, unde collig
$v = \frac{2z-a(1-uu)}{2u(a-z)}$, hincque

$$\partial v = \frac{2au(1+uu)\partial z + (a-z)(2a(1+uu)-4z)\partial u}{4uu(a-z)^2} \quad \text{et}$$

$$1 + vv = \frac{(1+uu)(aa(1+uu)-4z(a-z))}{4uu(a-z)^2}.$$

Habebimus igitur

$$\frac{\partial v}{1+vv} = \frac{2au(1+uu)\partial z + 2(a-z)(a(1+uu)-2z)\partial u}{(1+uu)(aa(1+uu)-4z(a-z))} = \frac{\partial z}{uz}.$$

Quod si jam differentialia ∂z et ∂u separentur, prodibit seque
aequatio:

$$\partial z(1+uu)(a-2z)\left(2z-a(1+uu)\right) = 2zu(a-z)\partial u\left(2z-a(1+uu\right.$$

quae, cum habeat divisorem, scil. $2z - a(1+uu)$, duas pra
bebit solutiones, quarum prior ex aequatione $2z = a(1+u$
altera ex aequatione $\frac{\partial z(a-2z)}{z(a-z)} = \frac{2u\partial u}{1+uu}$ erit petenda.

Haec posterior aequatio integrata dat $lz(a-z) = lC + l(1+u$
sive in numeris $az - zz = C(1+uu)$, unde si in expressione s
pra pro $1+vv$ data loco $az - zz$ hic valor $C(1+uu)$ subst
tuatur, orietur sequens expressio: $1 + vv = \frac{(1+uu)^2(aa-4C)}{4uu(a-z)^2}$,
ut, ob $\cot.\phi = v$ et $\sin.\phi = \frac{1}{\sqrt{1+vv}}$, fiat $\sin.\phi = \frac{2u(a-z)}{(1+uu)\sqrt{aa-4C}}$

Tab. I. Fig. 5. Hinc cum sit $AO : \sin.2\omega = MO : \sin.\phi$, erit

$$AO = \frac{(a-z)\sin.2\omega}{\sin.\phi} = \frac{2u(a-z)}{(1+uu)\sin.\phi} = \sqrt{aa-4C};$$

unde iterum patet, intervallum AO esse constans ideoque punctum O fixum, ex quo statim sequitur sectio conica.

Altera aequatio $2z = a(1+uu)$ dat $a-z = \frac{a(1-uu)}{2}$, unde $\frac{z}{a-z} = \frac{1+uu}{1-uu}$ atque $v = \cot.\Phi = \frac{2z - a(1-uu)}{2u(a-z)} = \frac{(1+uu)-(1-uu)}{u(1-uu)}$; sive $\cot\Phi = \frac{2u}{1-uu} = \text{tag}.\, 2\omega$, unde concluditur fore $90^\circ - \Phi = 2\omega$, sive $90^\circ = \Phi + 2\omega$, quo, ut ante, parabola indicatur.

Cum invenerimus $z(a-z) = C(1+uu) = \frac{C}{\cos.\omega^2}$, erit $z\cos.\omega \times (a-z)\cos.\omega = C$. Ducatur recta PQ, curvam in M tangens, et ex A et O in hanc tangentem demittantur perpendicula AP, OQ, eritque $AP = z\cos.\omega$ et $OQ = (a-z)\cos.\omega$, unde patet rectangulum ex his perpendiculis AP . OQ fore constans. Constat autem, in omnibus sectionibus conicis, quarum foci in A et O, rectangulum AP . OQ aequale esse quadrato semiaxis conjugati, unde semiaxis conjugatus sectionis conicae, quam hic eruimus, erit $= \sqrt{C}$. Tab. I. Fig. 5.

Tertia solutio sine calculo expedita.

Consideretur curvae punctum M, ejusque proximum m, ex quo radius reflexus mo cadat in axis punctum o, et cum requiratur ut sit tam $AM + MO = a$, quam $Am + mo = a$, erit $Am - AM = MO - mo$. Jam ex M in Am demittatur perpendiculum Mp, similique modo ex m in MO perpendiculum mq; et cum sit angulus $Mmp = mMq$, erunt triangula Mmp et mMq inter se aequalia, ob communem hypothenusam, ideoque $Mq = mp$. Atqui est $mp = Am - AM$ et $Mq = MO - mo$; tum vero $Mq = MO - Oq$, unde sequitur $Oq = mo$, id quod duplici modo fieri potest: 1°) quando omnes radii reflexi ad axem sunt perpendiculares, qui casus statim dat Parabolam. Praeterea vero fiet 2°) $qO = mo$, si punctum o cadet in O, sive quando O est punctum fixum, qui casus statim perducit ad Ellipsin vel Hyperbolam. Fig. 7.

Problema.

Invenire curvam LMN, in cujus tangentes MT si ex datis Fig. 8.

duobus punctis A et B demittantur perpendicula A… BG, eorum rectangulum sit constans, hoc est AF . BG…

Solutio.

Bisecto intervallo AB in C sit $CA = CB = b$, ac pon… $CP = x$, $PM = y$, eritque tag. $MTP = -\frac{\partial y}{\partial x} = -p$, po… $\partial y = p\partial x$; tum vero habebimus $PT = -\frac{y}{p}$ et $CT = \frac{px - y}{p}$, un… colligitur $AT = \frac{px - y - bp}{p}$, hincque $BT = \frac{px - y + bp}{p}$. C… jam sit $AF = AT . \sin. T$ et $BG = BT . \sin. T$, ob $\sin. T = \frac{…}{\sqrt{1+…}}$ habebimus $AF . BG = \frac{(px - y)^2 - bbpp}{pp} \times \frac{pp}{1 + pp} = cc$, sive

$$(px - y)^2 - bbpp = cc\,(1 + pp),$$

unde, posito brevitatis gratia $bb + cc = aa$, haec oritur aequati…

$$(y - px)^2 = cc + aapp, \text{ sive } y - px = \sqrt{cc + aapp}.$$

Ista aequatio, ob $p = \frac{\partial y}{\partial x}$, est differentialis ideoque integra… debere videtur: interim tamen hic ope differentiationis integrale e… potest. Cum enim sit $\partial y = p\partial x$, differentiatione facta prodit

$$x\,\partial p = -\frac{aap\,\partial p}{\sqrt{cc + aapp}},$$

quae aequatio, cum divisorem habeat ∂p, subministrat statim so… tionem ex aequatione $\partial p = 0$ petendam, unde fit p constans, pu… $p = \alpha$, ex quo colligitur $\partial y = \alpha \partial x$, ideoque $y = \alpha x + \beta$, qu… aequatio est pro linea recta.

Altera solutio ex aequatione $x = \frac{-aap}{\sqrt{cc + aapp}}$ erit deducend… ex qua fit $y = px + \sqrt{cc + aapp} = \frac{cc}{\sqrt{cc + aapp}}$. Hinc patet fo… $\frac{xx}{aa} + \frac{yy}{cc} = 1$, quae aequatio est pro Ellipsi, quoties cc est quantit… positiva, sive quoties $a > b$; at pro Hyperbola quoties $a < b$.

Quodsi autem aequatio $(y - px)^2 = cc + aapp$ evolvatur e… loco p scribatur $\frac{\partial y}{\partial x}$, ita ut prodeat

$$yy\partial x^2 - 2xy\partial x\partial y + xx\partial y^2 = cc\partial x^2 + aa\partial y^2$$

Quodsi vero haec aequatio more solito tractetur, ob

$$\partial y^2 (xx - aa) = 2xy\partial x\partial y + (cc - yy)\partial x^2, \text{ erit}$$

$$(xx - aa)\partial y = xy\partial x \pm \partial x\sqrt{(xxyy + (cc - yy)(xx - aa))}, \text{ sive}$$

$$\partial y = \frac{xy\partial x \pm \partial x\sqrt{ccxx + aayy - aacc}}{xx - aa}$$

cujus resolutio non parum difficultatis habet.

Ponatur $a = 1$, $b = 1$, $c = 1$, erit aequatio illa

$$(xx - 1)\partial y = xy\partial x \pm \partial x\sqrt{xx + yy - cc}.$$

Sit porro $\sqrt{xx + yy - cc} = z$ et $y = ux$, atque ob $\partial y = u\partial x + x\partial u$ emerget sequens aequatio:

$$u\partial x(xx - 1) + x\partial u(xx - 1) = uxx\partial x + z\partial x, \text{ sive}$$

$$x\partial u(xx - 1) - u\partial x = z\partial x.$$

Cum igitur sit $xx(1 + uu) - 1 = zz$, erit $xx = \frac{zz + 1}{uu + 1}$, unde nostra aequatio: $\partial u(xx - 1) - u\frac{\partial x}{x} = z\frac{\partial x}{x}$, ob $xx - 1 = \frac{zz - uu}{uu + 1}$ et $\frac{\partial x}{x} = \frac{z\partial z}{1 + zz} - \frac{u\partial u}{1 + uu}$, hanc induet formam:

$$\frac{\partial u(zz - uu)}{1 + uu} = \frac{z(u + z)\partial z}{1 + zz} - \frac{u(u + z)\partial u}{1 + uu}$$

quae manifesto reducitur ad hanc:

$$\frac{(zz + uz)\partial u}{1 + uu} = \frac{(zz + uz)\partial z}{1 + zz}.$$

Haec aequatio factores habet z et $u + z$, quorum uterque dat solutionem. Primo enim prodit aequatio $zz = xx + yy - 1 = 0$, sive $xx + yy = 1$, cujus natura neminem latet. Secundo fit

$$z + u = \sqrt{xx + yy - 1} + \frac{y}{x} = 0,$$

hoc est $xx(xx + yy) = xx + yy$, unde fit $x = -1$ et $x = -y$, pro recta. Dividendo autem aequationem illam per factorem communem colligitur $\frac{\partial u}{1 + uu} = \frac{\partial z}{1 + zz}$, unde integrando

$$\text{A tag.}\, u = \text{A tag.}\, z + C, \text{ sive A tag.}\, z = \text{A tag.}\, u - \text{A tag.}\, n$$

hoc est $\text{A tag.}\, z = \text{A tag.}\, \frac{n + u}{1 - nu}$, hincque $z = \frac{n + u}{1 - nu}$, sive

$$\sqrt{xx + yy - 1} = \frac{nx + y}{x - ny},$$

ergo $xx + yy = \frac{(1 + nn)(xx + yy)}{(x - ny)^2}$, consequenter $(x - ny)^2 = 1$ vel $x - ny = \sqrt{1 + nn}$, iterum pro recta. Hac autem met[...] uti non licet simulac littera p ad altiores potestates ascendit.

Aequatio autem generalis, quae integrationem per differenti[...] nem admimittit, est, quando, posito $\frac{\partial y}{\partial x} = p$, formula $px - y$ cu[...] que functioni ipsius p aequatur. Posita enim hac functione Π, $\Pi = px - y$, quae aequatio differentiata dat $\partial \Pi = x \partial p = \Pi$[...] unde factor $\partial p = 0$ ostendit, semper lineam rectam satisfa[...] Praeterea vero habetur haec solutio: $x = \Pi'$ et $y = p\Pi' - \Pi$[...]