	T		T		T
1. 2. 3. 4. 5. 6	1	1. 3. 5. 7. 9	2	$1.\ 1+a.\ \ 1+2a.\ \ 1+3a$	a
1. 3. 6. 10. 15. 21	3	1. 4. 9. 16. 25	4	$1.\ 2+a.\ \ 3+3a.\ \ 4+6a$	$a+2$
1. 4. 10. 20. 35. 56	5	1. 5. 14. 30. 55	6	$1.\ 3+a.\ \ 6+4a.\ \ 10+10a$	$a+4$
1. 5. 15. 35. 70	7	1. 6. 20. 50. 105	8	$1.\ 4+a.\ \ 10+5a.\ \ 20+15a$	$a+6$
				$1.\ 5+a.\ \ 15+6a.\ \ 35+21a$	$a+8$

(*Krafft.*)

Omnes illae superiores series numerorum figuratorum sequenti forma generali comprehendi possunt

$$1;\quad \frac{n+a}{1};\quad \frac{(n+1)(n+2a)}{1.2};\quad \frac{(n+1)(n+2)(n+3a)}{1.2.3};\quad \frac{(n+1)(n+2)(n+3)(n+4a)}{1.2.3.4}\quad \text{etc.}$$

pro qua superior littera T fit $= a+2n-2$.

A. m. T. I. p. 234. 235.

C. Analysis Diophantea.

a) *Quaestiones ad resolutionem unius aequationis ducentes.*

37.

(*J. A. Euler.*)

Problema. Si fuerit $x^3 = m$, et proposita sit formula $axx+bx+c$, invenire multiplicatorem $pxx+qx+r$, ut productum

$$(axx+bx+c)(pxx+qx+r)$$

fiat numerus absolutus non amplius involvens x, posito scilicet $x^3 = m$.

Solutio. Productum ergo erit

$$\begin{aligned} &mapx + (aq+bp)m + (ar+bq+cp)\,xx \\ &+(br+cq)\,x + cr. \end{aligned}$$

Debet ergo poni $br+cq+map = 0$ et $ar+bq+cp = 0$, tum enim productum erit

$$m(aq+bp)+cr.$$

Fit autem

$$p = \frac{-br-cq}{ma} = \frac{-ar-bq}{c},$$

unde fit $bcr+ccq = maar+mabq$; hinc $mabq-ccq = bcr-maar$,

consequenter $$\frac{q}{r} = \frac{bc-maa}{mab-cc}.$$

Capiatur ergo $q = bc-maa$, $r = mab-cc$, erit $p = ac-bb$, ita ut multiplicator quaesitus sit

$$(ac-bb)\,xx+(bc-maa)\,x+mab-cc;$$

ac tum productum erit

$$3mabc-m^2a^3-mb^3-c^3.$$

A. m. T. I. p. 50. 51.

38.

Problema. Invenire numeros x et y, ut fiat $xy(xx-yy) = \alpha nn$, existente α numero primo: ubi hi casus sunt notandi:

I. Si $\alpha = 7$, sumatur $x = 16$ et $y = 9$, tum enim fit $xy(xx-yy) = 7.\ 16.\ 9.\ 25$.

II. Si $\alpha = 13$, sumatur $x = 325$ et $y = 36$, erit $xy(xx-yy) = 13.\ 25.\ 36.\ 361.\ 289$.

III. Si $\alpha=23$, sumatur $x=156^2$ et $y=133^2$, erit $xy(xx-yy)=23.156^2.133^2.289.42025$.

IV. Ut $\alpha=41$, capiatur $x=21^2$ et $y=20^2$, erit $xy(xx-yy)=41.21^2.20^2.29^2$.

V. Ut fiat $\alpha=31$, sumatur $x=40^2$ et $y=9^2$, fit enim

$$xy(xx-yy)=31.9^2.40^2.7^2.41^2.$$

VI. In genere si capiatur $x=4ppqq$ et $y=(pp-qq)^2$, fiet

$$xy(xx-yy)=(pp-qq+2pq)(2pq-pp+qq).\square.$$

Unde si sit $2pq+pp-qq=aa$, fit $\alpha=2pq-pp+qq$, at illa formula $2pq+pp-qq$ fit quadratum sumendo $q=2rs$ et $p=2rr+ss-2rs$.

VII. Deinde vero si sumatur $x=(2pp+qq)^2$ et $y=(2pp-qq)^2$, fiet

$$xy(xx-yy)=8pp.qq(8p^4+2q^4)\square=(4p^4+q^4)\square$$

unde fit

$$\alpha\square=4p^4+q^4=(2pp+2pq+qq)(2pp-2pq+qq).$$

Unde si fuerit $2pp+2pq+qq=\square$, tunc erit $\alpha=2pp-2pq+qq$. At illud evenit

si $p=2rs$ et $q=rr-ss-2rs$, unde fit $\alpha=pp+(p-q)^2$.

VIII. Ex casu VII, si $p=5$ et $q=7$, capiatur $x=99^2$ et $y=1$, erit $\alpha=29$ et $xy(xx-yy)=29\square$; vel si capiatur $x=29.13^2$ et $y=70^2$.

A. m. T. I. p. 21. 22.

39.

(*Lexell.*)

Problema. Invenire numeros x, y, z, ut fiat $\alpha xx+\beta yy=\gamma zz$, siquidem cognitus fuerit casus

$$\alpha ff+\beta gg=\gamma hh.$$

Solutio. Statuatur $\alpha xx+\beta yy=(\alpha ff+\beta gg)(\alpha pp+\beta qq)^2$, tum enim erit

$$\alpha xx+\beta yy=\gamma hh(\alpha pp+\beta qq)^2,\quad \text{sicque erit}\quad z=h(\alpha pp+\beta qq);$$

illud autem hoc modo per factores praestetur. Sit $x\sqrt{\alpha}+y\sqrt{-\beta}=(f\sqrt{\alpha}+g\sqrt{-\beta})(p\sqrt{\alpha}+q\sqrt{-\beta})^2$, tum enim sponte fit $x\sqrt{\alpha}-y\sqrt{-\beta}=(f\sqrt{\alpha}-g\sqrt{-\beta})(p\sqrt{\alpha}-q\sqrt{-\beta})^2$, quarum formularum productum ipsa est aequatio supposita. Prior autem evoluta dat

$$x\sqrt{\alpha}+y\sqrt{-\beta}=\alpha fpp\sqrt{\alpha}-\beta fqq\sqrt{\alpha}-2\beta gpq\sqrt{\alpha}$$
$$+\alpha gpp\sqrt{-\beta}-\beta gqq\sqrt{-\beta}+2\alpha fpq\sqrt{-\beta}$$

$$x=f(\alpha pp-\beta qq)-2\beta gpq$$
$$y=g(\alpha pp-\beta qq)+2\alpha fpq$$

ac tum erit

$$z=h(\alpha pp+\beta qq).$$

Verum haec solutio nondum est generalis, eodem modo enim ponere potuissemus

$$\alpha xx+\beta yy=(\alpha ff+\beta gg)(pp+\alpha\beta qq)^2,$$

unde fit $z=h(pp+\alpha\beta qq)$. Pro hoc ergo casu statuatur

$$x\sqrt{\alpha}+y\sqrt{-\beta}=(f\sqrt{\alpha}+g\sqrt{-\beta})(p+q\sqrt{-\alpha\beta})^2,$$

cujus evolutio praebet

$$x=f(pp-\alpha\beta qq)-2g\beta pq$$
$$y=g(pp-\alpha\beta qq)+2f\alpha pq.$$

Verum ne hi ambo quidem casus solutionem praebent generalem, cum sine dubio ejusmodi casus dentur, quibus z non per h fit divisibile, quare pro solutione generali statuatur

$$\alpha xx+\beta yy=(\alpha ff+\beta gg)(\alpha npp+\beta nqq)^2,\quad \text{unde fit}\quad z=hn(\alpha pp+\beta qq),$$

ubi forte n potest esse fractio denominatoris h. Statuatur igitur

$$x\sqrt{\alpha}+y\sqrt{-\beta}=(f\sqrt{\alpha}+g\sqrt{-\beta})(p\sqrt{\alpha n}+q\sqrt{-\beta n})^2,$$

cujus evolutio praebet

$$x = f(\alpha npp - \beta nqq) - 2\beta ngpq, \quad y = g(\alpha npp - \beta nqq) + 2\alpha nfpq.$$

Videamus igitur an esse possit $n = \frac{1}{h}$, manentibus x et y integris. Cum igitur sit

$$x = \frac{f(\alpha pp - \beta qq) - 2\beta gpq}{h}, \quad y = \frac{g(\alpha pp - \beta qq) + 2\alpha fpq}{h},$$

quod evenit si p et q ita sumantur, ut $fq + gp$ fiat per h divisibile.

(*W. L. Krafft.*)

Problematis supra propositi solutio facillime sequenti modo absolvetur, siquidem constet unus casus, quo sit $\alpha ff + \beta gg = \gamma hh$, ubi scilicet $x = f$, $y = g$ et $z = h$. Statuamus $x = fp + \beta gq$ et $y = gp - \alpha fq$, tum enim erit

$$\alpha xx + \beta yy = pp(\alpha ff + \beta gg) + \alpha\beta qq(\alpha ff + \beta gg) = \gamma hh(pp + \alpha\beta qq).$$

Sicque aequatio adhuc resolvenda erit $hh(pp + \alpha\beta qq) = zz$, ita ut $pp + \alpha\beta qq$ debeat reddi quadratum, quod fit capiendo $p = rr - \alpha\beta ss$ et $q = 2rs$, tum enim fit

$$pp + \alpha\beta qq = (rr + \alpha\beta ss)^2.$$

Ideoque $z = h(rr + \alpha\beta ss)$. Ipsarum vero x et y valores erunt

$$x = f(rr - \alpha\beta ss) + 2\beta grs, \quad y = g(rr - \alpha\beta ss) - 2\alpha frs,$$

ubi numeri r et s pro lubitu assumi possunt. (Conf. Comment. arithm. T. I. p. 556.)

A. m. T. I. p. 95. 96. 98. 99.

40.

(*J. A. Euler.*)

Theorema. Si fuerint $naa + pbb = \square = cc$ et $nff + qgg = \square = hh$, tum semper assignare licet x et y ut sit $nxx + pqyy = \square = zz$.

Demonstratio. Cum sit $pbb = cc - naa$ et $qgg = hh - nff$, erit productum

$$pqbbgg = (cc - naa)(hh - nff) = (ch + naf)^2 - n(ah + fc)^2,$$

unde manifestum est fore $n(ah + fc)^2 + pqbbgg = (ch + naf)^2$, sicque erit

$$x = ah + fc, \quad y = bg \quad \text{et} \quad z = ch + naf.$$

A. m. T. I. p. 130.

41.

(*N. Fuss I.*)

Problema. Resolvere aequationem $\lambda zz = \mu xx + \nu yy$, ex cognito casu $\lambda cc = \mu aa + \nu bb$.

Solutio. A priore aequatione in cc ducta subtrahatur posterior in zz ducta, eritque

$$0 = \mu(ccxx - aazz) + \nu(ccyy - bbzz),$$

$$\text{sive} \quad \mu(ccxx - aazz) = \nu(bbzz - ccyy), \quad \text{hinc} \quad \frac{\mu(cx + az)}{bz - cy} = \frac{\nu(bz + cy)}{cx - az}.$$

Utraque haec fractio statuatur $= \frac{p}{q}$ et ex priore elicitur

$$z = \frac{\mu cqx + pcy}{bp - \mu aq} \quad \text{et ex altera} \quad z = \frac{cpx - \nu cqy}{\nu bq + ap},$$

qui duo valores inter se aequati dant

$$\frac{y}{x} = \frac{\mu\nu bcqq + 2\mu acpq - bcpp}{\mu\nu acqq - 2\nu bcpq - acpp}.$$

Statuatur ergo $\quad x = \mu\nu aqq - 2\nu bpq - app \quad$ et $\quad y = \mu\nu bqq + 2\mu apq - bpp,$

$$\text{sive} \quad x = a(\mu\nu qq - pp) - 2\nu bpq \quad \text{et} \quad y = b(\mu\nu qq - pp) + 2\mu apq.$$

Cum autem sit
$$\frac{z}{c}(bp-\mu aq)=\mu qx+py=\mu\mu\nu aq^3-\mu\nu bpqq+\mu appq-bp^3$$
$$=\mu aq(\mu\nu qq+pp)-bp(\mu\nu qq+pp)=(\mu\nu qq+pp)(\mu aq-bp)$$
hinc $\frac{z}{c}=-(\mu\nu qq+pp)$ et $z=-c(\mu\nu qq+pp)$.

Alia Solutio. Quia semper f, g, h invenire licet, ut sit $hh=ff+\mu\nu gg$, per hanc aequationem multiplicetur cognita $\lambda cc=\mu aa+\nu bb$ eritque
$$\lambda cchh=\mu aaff+\mu\nu\nu bbgg+\nu bbff+\mu\mu\nu aagg=\mu(af+\nu bg)^2+\nu(bf-\mu ag)^2.$$
Cum igitur esse debeat $\lambda zz=\mu xx+\nu yy$, capi poterit $z=ch$ deinde $x=af+\nu bg$ et $y=bf-\mu ag$, at vero ut fiat $hh=ff+\mu\nu gg$, debet esse
$$f=\mu\nu qq-pp \quad\text{et}\quad g=2pq, \quad\text{eritque}\quad h=\mu\nu qq+pp,$$
consequenter formula proposita ita resolvetur
$$x=a(\mu\nu qq-pp)+2\nu bpq \quad\text{et}\quad y=b(\mu\nu qq-pp)-2\mu apq \quad\text{et}\quad z=c(\mu\nu qq+pp).$$

Corollarium. Haec solutio duobus modis variari potest, prouti aequationes propositae aliter disponuntur, scil. primo $\mu xx=\lambda zz-\nu yy$ et $\mu aa=\lambda cc-\nu bb$. Ad quem casum solutio praecedens revocabitur

si loco $\lambda,\ \mu,\ \nu,\ z,\ x,\ y,\ c,\ a,\ b$

ponatur $\mu,\ \lambda,\ -\nu,\ x,\ z,\ y,\ a,\ c,\ b.$

Unde si loco p et q scribamus r et s obtinebimus
$$z=c(-\lambda\nu ss-rr)-2\nu brs,\quad y=b(-\lambda\nu ss-rr)-2\lambda crs,\quad x=a(-\lambda\nu ss+rr).$$

Eodem modo si formulae datae ita disponantur $\nu yy=\lambda zz-\mu xx$ et $\nu bb=\lambda cc-\mu aa$, unde

si loco $\lambda,\ \mu,\ \nu,\ z,\ x,\ y,\ c,\ a,\ b$

ponatur $\nu,\ \lambda,\ -\mu,\ y,\ z,\ x,\ b,\ c,\ a$

tum loco p et q, t et u, obtinebitur
$$z=c(-\lambda\mu uu-tt)-2\mu atu,\quad x=a(-\lambda\mu uu-tt)-2\lambda ctu,\quad y=b(-\lambda\mu uu+tt).$$

Has igitur tres solutiones ita aspectui opponamus:

Solutiones	x	y	z
I	$a(\mu\nu qq-pp)+2\nu bpq$,	$b(\mu\nu qq-pp)-2\mu apq$,	$c(\mu\nu qq+pp)$
II	$a(rr-\lambda\nu ss)$	$b(rr+\lambda\nu ss)+2\lambda crs$,	$c(rr+\lambda\nu ss)+2\nu brs$
III	$a(tt+\lambda\mu uu)+2\lambda ctu$,	$b(tt-\lambda\mu uu)$,	$c(tt+\lambda\mu uu)+2\mu atu$
I	$6qq-pp+6pq$,	$6qq-pp-4pq$,	$6qq+pp$
II	$rr-15ss$,	$rr+15ss+10rs$,	$rr+15ss+6rs$
III	$tt+10uu+10tu$,	$tt-10uu$,	$tt+10uu+4tu$.

Ubi notatu dignum, quod ternae formulae in qualibet columna eosdem numeros praebere queant, dummodo fuerit $\lambda cc=\mu aa+\nu bb$.

Exemplum. Sit proposita haec formula $5zz=2xx+3yy$, ut sit $\lambda=5$, $\mu=2$ et $\nu=3$, tum vero quia $5.1^2=2.1^2+3.1^2$, erit $c=1$, $a=1$ et $b=1$, unde ternae nostrae solutiones in tabella hic supra apponamus. Hinc si $p=1$ et $q=1$, erit $x=11$, $y=1$ et $z=7$; si $p=1$ et $q=-1$, erit $x=\pm 1$, $y=9$ et $z=7$, qui valores satisfaciunt. Sit porro pro secunda solutione $r=1$ et $s=1$ eritque $x=\pm 14$ seu ± 7, $y=26$ seu 13 et $z=22$ seu 11. Unde fit $5zz=2xx+3yy$ sive $605=98+507$. Sit $r=1$ et $s=-1$, ut sit $x=14$ seu 7, $y=6$ seu 3 et $z=10$ seu 5. Pro tertia sit $t=1$ et $u=1$ et erit $x=21$ seu 7, $y=\pm 9$ seu 3 et $z=15$ seu 5. Sit $t=1$ et $u=-1$ fietque $x=1$, $y=\pm 9$ et $z=7$.

A. m. T. I. p. 299. 300.

42.

(J. A. Euler.)

Criterium ad dignoscendum, utrum hujusmodi aequatio $fxx+gyy=hzz$ *sit possibilis, nec ne?*

Si est possibilis, casu $h=a$, tunc etiam erit possibilis casu $h=\frac{a(pp+fg)}{qq}$: hic scilicet pro p ejusmodi numerus sumi debet, ut $pp+fg$ divisorem habeat a, fuerit nempe $pp+fg=ab$ et sumto $q=a$ etiam casu h erit possibilis. Tum vero pro b eodem modo operatio instituatur, sicque continuo ad minores numeros pervenietur, donec tandem judicium fiat facile.

Exemplum. I. Sit $7xx+113yy=114zz$, quae aequatio an sit possibilis, quaeritur. Hic est f $g=113$, et quaeritur an sit possibilis casu $h=114=2.3.19$? Statuatur ergo

$$h=\frac{114(pp+791)}{qq} \text{ et sumto } p=3 \text{ et } q=40, \text{ prodit casus } h=\frac{114.800}{1600}=57.$$

II. Nunc iterum fiet $h=\frac{57(pp+791)}{qq}$ et fiat $pp+791$ divisibile per 19, quod si fieri potest, dabitur casus quo $p<19$. Ut, ex. gr. $pp+12$ fiat divisibile per 19, debet esse $p=8$, unde $h=\frac{855}{3.19}=15$.

III. Quaestio ergo huc est reducta, an aequatio $7xx+113yy=15zz$ sit possibilis? quae hoc modo repraesentetur $15zz-7xx=113yy$, ubi $f=15$, $g=-7$, $fg=-105$ et $h=113$. Nunc fiat $h=\frac{113(pp-105)}{qq}$ reddatur $pp-105$ divisibile per 113, quod fit sumendo $p=52$, tum autem fiat $h=\frac{113.2599}{\square}=\frac{113.113.23}{\square}$ ergo quadrato sublato fit $h=23$ et quaestio huc est reducta, an aequatio $15zz-7xx=23yy$ sit possibilis.

IV. Fiat ergo $h=\frac{23(pp-105)}{\square}$ sitque $pp-13$ per 23 divisibile, sive $pp=23n+13$, quod fit si $n=1$ et $p=6$, ergo $h=\frac{-23.69}{\square}=-3$. Habetur igitur haec aequatio

$$15zz-7xx=-3yy, \text{ sive } 7xx-3yy=15zz,$$

ubi $f=7$, $g=-3$ et $fg=-21$, $h=15$.

V. Fiat nunc $h=\frac{15(pp-21)}{\square}$. Sumatur $p=4$, erit $h=\frac{-15.5}{\square}=-3$, ergo aequatio $7xx-3yy=$ quod actu evenit si $x=0$ et $y=z$, atque hinc sequitur ipsam aequationem propositam esse possibilem.

Nota. Revera autem est possibilis: si enim capiatur $z=2$ et $y=1$, fit

$$7xx+113=456 \text{ sive } 7xx=343 \text{ et } xx=49=\square.$$

Ita semper aequatio si fuerit possibilis, ad talem formam reduci poterit $axx+byy=azz$, cui manifesto satisfit sumendo $y=0$ et $z=x$.

(Krafft.)

Judicium hoc reddi potest adhuc facilius hoc modo:

Cum sit $h=\frac{2.3.19(p^2+791)}{q^2}$, capiatur p ita, ut p^2+791 divisibile fiat per $2.3.19$. Primo autem fit divisibile per 2, si $p=2n+1$; at vero per 3 fit divisibile, si $p=3n\mp1$. Utrumque igitur obtinetur $p=6n\mp1$. Restat, ut p^2+791 sit per 19 divisibile; quod fit, si p^2 per 19 divisum relinquat 7, sive debet esse $p^2=19n+7$, ergo $19n+7$ debet esse quadratum, quod fit, si $n=3$, eritque $p=8$. In genere ergo fiet si $p=19n\mp8$; hoc est casibus $p=8$, $p=11$, $p=27$, $p=30$, $p=46$, $p=49$, $p=65$, etc. inter quos numeros, reperitur statim 11, qui est formae $6n\mp1$. Sumatur ergo $p=11$, eritque $p^2+791=912=$ ergo $h=\frac{114^2.8}{\square}$ et sublatis quadratis $h=2$. Res ergo eo redit, an sit $7x^2+113y^2=2z^2$. Quia hic est h sumatur iterum $h=\frac{2(p^2+791)}{q^2}$. Ponatur $p=7$, erit $h=\frac{2.840}{q^2}=105$. Si sumsissemus $p=3$, prod

$\frac{2.800}{qq} = 1$, et jam quaeritur, utrum possit esse $7x^2 + 113y^2 = z^2$. Sumatur ergo $h = \frac{1(p^2+791)}{qq}$ et sumatur $p = 0$, erit $h = \frac{7.113}{qq}$, et cum quaestio sit de forma $7x^2 + 113y^2 = 7.113z^2$, debet esse $x = 113v$, ideoque $7.113v^2 + y^2 = 7z^2$. Felicissime succedit, si in aequatione $h = p^2 + 791$ capiatur $p = 7.4$. Tum erit

$$h = \frac{7^2.16 + 7.113}{qq} = \frac{7.225}{qq} = 7,$$

ergo ventum est ad $7x^2 + 113y^2 = 7z^2$, quod fit si $y = 0$ et $x = z$, ergo proposita aequatio est possibilis.

Haec solutio isti innititur principio: si fuerit $fx^2 + gy^2 = hz^2$, multiplicetur utrinque per $p^2 + fgq^2$ fietque

$$h(p^2 + fgq^2)z^2 = fp^2x^2 + gp^2y^2 + f^2gq^2x^2 + fg^2q^2y^2 = f(px + gqy)^2 + g(py - fqx)^2.$$

Ergo si ponatur $x' = px + gqy$ et $y' = py - fqx$, erit $fx'^2 + gy'^2 = h(p^2 + fgq^2)z^2$; adeoque si aequatio proposita fuerit possibilis, etiam haec erit possibilis et vicissim.

Jam sumto $q = 1$, habebitur praecedens forma $h(p^2 + fg)$. Si nunc p ita sumi potest, ut $p^2 + fg$ divisorem habeat h, quod semper eveniet valore $p < \frac{1}{2}h$, et ponatur $p^2 + fg = hh'$, ita ut loco h habeatur h^2h', sive omisso quadrato simpliciter h'. Sicque loco h prodiit novus valor h' illo multo minor; cum enim sit $p < \frac{1}{2}h$, erit $hh' < \frac{1}{4}h^2 + fg$, ideoque $h' < \frac{1}{4}h + \frac{fg}{h}$. Sin autem pro p talis valor non detur, indicio id erit, aequationem propositam esse impossibilem; non autem hoc judicium inverti potest; dantur enim casus, quibus aequatio nihilominus est impossibilis; veluti evenit in hoc exemplo $2x^2 + 3y^2 = 7z^2$, ubi $f = 2$, $g = 3$ et $h = 7$. Hinc novus valor orietur $h = 7(p^2 + 6)$ et sumto $p = 1$ fit $h = 1$, unde novus valor erit $h = 1(p^2 + 6)$, qui dat valores 7, 10, 15, etc., qui autem omnes nullo modo satisfaciunt; nam facile ostendi potest, aequationem $2x^2 + 3y^2 = z^2$ esse impossibilem, sive $2x^2 + 3y^2$ quadratum esse non posse, vel enim x est divisibile per 3, vel non. Priori casu y non erit divisibile per 3, quia alioquin tota aequatio per 9 dividi posset, et posito $x = 3v$, formula erit $18v^2 + 3y^2$, quae divisibilis est per 3, non vero per 9, ideoque quadratum esse nequit. At si $x = 3v \pm 1$, erit x^2 numerus formae $3n + 1$, ideoque $2x^2 = 3n + 2$ et ipsa formula $2x^2 + 3y^2$ erit numerus formae $3n + 2$, quae forma quadratum esse nequit.

Simili modo judicari poterit, utrum hujusmodi aequatio generalior $fx^2 + gxy + hy^2 = kz^2$ possibilis sit nec ne. Multiplicetur enim utrinque per $p^2 + gpq + fhq^2$, ut habeatur

$$(p^2 + gpq + fhq^2)(fx^2 + gxy + hy^2) = k(p^2 + gpq + fhq^2)z^2,$$

ubi notandum, prius productum semper reduci posse ad formam $fX^2 + gXY + hY^2$, quod cum non tam facile appareat, per factores irrationales sequenti modo ostendetur.

Quaerantur factores formulae $fx^2 + gxy + hy^2$, quod fit ponendo hanc formulam $= 0$ et radicem extrahendo, unde fit $x = \frac{-gy \mp y\sqrt{(g^2 - 4fh)}}{2f}$, unde factores erunt

$$\frac{1}{4f}(2fx + gy + y\sqrt{(g^2 - 4fh)})(2fx + gy - y\sqrt{(g^2 - 4fh)})$$

sive $\frac{1}{f}(fx + \frac{1}{2}gy + y\sqrt{(\frac{1}{4}g^2 - fh)})(fx + \frac{1}{2}gy - y\sqrt{(\frac{1}{4}g^2 - fh)})$,

et posito brevitatis gratia $\frac{1}{4}g^2 - fh = l$, ut fiat

$$fx^2 + gxy + hy^2 = \frac{1}{f}(fx + \frac{1}{2}gy + y\sqrt{l})(fx + \frac{1}{2}gy - y\sqrt{l}).$$

Simili modo erit $$p^2 + gpq + fhq^2 = (p + \frac{1}{2}gq + q\sqrt{l})(p + \frac{1}{2}gq - q\sqrt{l}) \text{ et}$$

$$fX^2 + gXY + hY^2 = \frac{1}{f}(fX + \frac{1}{2}gY + Y\sqrt{l})(fX + \frac{1}{2}gY - Y\sqrt{l});$$

haec ergo forma aequalis esse debet producto ex binis praecedentibus, quod fiet aequando alterutrum factorem producto ex binis praecedentibus, scilicet

$$fX+\frac{1}{2}gY+Y\sqrt{l}=(fx+\frac{1}{2}gy+y\sqrt{l})(p+\frac{1}{2}gq+q\sqrt{l});$$

sic enim sumto $\sqrt{l}$ negative, sponte fiet

$$fX+\frac{1}{2}gY-Y\sqrt{l}=(fx+\frac{1}{2}gy-y\sqrt{l})(p+\frac{1}{2}gq-q\sqrt{l});$$

sufficiet ergo alterutram ita evolvisse, ut membra rationalia et irrationalia seorsim inter se aequentur. Tum igitur fiet

$$fX+\frac{1}{2}gY=(fx+\frac{1}{2}gy)(p+\frac{1}{2}gq)+lgy$$

$$=pfx+\frac{1}{2}gpy+\frac{1}{2}fgqx+\frac{1}{2}g^2qy-fhqy$$

$$Y=fqx+ggy+py,$$

qui posterior valor in priore substitutus praebet

$$fX=pfx-fhqy, \quad \text{hinc} \quad X=px-hqy \quad \text{et} \quad Y=fqx+ggy+py.$$

Hoc igitur demonstrato ex dato valore k alius investigetur k', ut sit $k'=k(p^2+gpq+fhq^2)$, omissis factoribus quadratis, capiatur autem $g=1$, ut fiat $k'=k(p^2+gp+fh)$, et si aequatio est possibilis, loco p semper ejusmodi valorem reperire licet, ut formula $p^2+gp+fh$ factorem habeat k, quae posita $=kk'$ dabit novum valorem k'; quod si succedit, talis valor ipsius p semper dabitur minor, quam $\frac{1}{2}h$, dum scilicet p tam negative quam positive accipiatur, et sic valor k' multo minor erit quam k, unde continuo ad minores valores pervenietur, donec judicium facile reddatur.

Res exemplo illustretur: $5x^2+16xy+7y^2$, ubi $f=5$, $g=16$, $h=7$. Quaeramus casum possibilem, quo $k=7$, quippe qui oritur, si $x=1$ et $y=1$, ita ut sit $5x^2+16xy+7y^2=7z^2$, qui autem maxime est obvius sumendo $x=0$ et $y=z$. Ergo alium eligamus sitque $5x^2+16xy+7y^2=59z^2$ ut sit $k=59$. Jam quaeratur $k'=k(p^2\mp16p+35)$ et capiatur p ita, ut factor 59 tollatur, quod fit si $p=10$, $k'=59.295=5.59^2$, unde $k'=5$ qui casus est obvius sumendo $y=0$ et $z=x$.

Problema. Invenire numeros f et g, ut fiat $fx^2+gy^2=p^2+fg$.

Solutio. Erit ergo $fg-fx^2-gy^2=-p^2$; addatur x^2y^2 eritque $(f-y^2)(g-x^2)=x^2y^2-p^2$. Fiat $f-y^2=xy-p$, erit $g-x^2=xy+p$, ideoque $f=y^2+xy-p$ et $g=x^2+xy+p$, unde si f detur, ob $p=y^2+xy-f$, erit $g=x^2+y^2+2xy-f$, sive $f+g=(x+y)^2=\Box$. Quoties ergo $f+g$ fuerit quadratum, problemati satisfit; satisfiet ergo quoque, dummodo fuerit $fm^2+gn^2=\Box$.

Theorema. Si fuerit $fx^2+gy^2=sz^2$ casu, quo $s=h$; tum etiam aequatio subsistere potest, quoties fuerit $s=h\mp4nfg$, dummodo hic numerus fuerit primus.

Hujus theorematis demonstratio etiamnum desideratur.

Exemplum. Sit $2x^2+3y^2=sz^2$ quod fieri potest si $s=5$. Idem ergo praestari potest si fuerit $s=5+24n$, unde hi numeri primi oriuntur: 5, 29, 53, 101, 149, 173, 197, 269, etc. Cum ergo sit $2x^2+3y^2=101z^2$, ita ut in superiori calculo sit $h=101$; erit $s=101(p^2+6)$. Fiat p^2+6 per 101 divisibile, sive $p^2=101n-6$, unde nascetur haec progressio arithmetica

0	1	2	3	4	5	6	
−6	95	196	297	398	499	600	etc.

ex qua vero illi numeri n valores excluduntur, qui habent sequentes formas

$$3\alpha+1, \quad 4\alpha, \quad 4\alpha+1, \quad 5\alpha+1, \quad 5\alpha+2.$$

Casui nostro satisfit si $p=14$, unde fit $s=2$; qui vero casus $2x^2+3y^2=2z^2$ est obvius; fit enim $y=0$.

eodem casu fit $s=149$; unde alius $149(p^2+6)$, ideoque $149n-6$ debet esse quadratum; unde excluduntur:

$$3\alpha+1,\quad 4\alpha,\quad 4\alpha+1,\quad 5\alpha+1,\quad 5\alpha+2,$$

remanent pro n ergo 3α, $3\alpha+2$, etc. et in numeris 3, 14, 15, 23, ubi $p=21$ satisfacit, seu $n=3$; $s=149.3.149=3$, unde iterum nascitur casus obvius. Omnes autem numeri primi pro s, quibus formula $2x^2+3y^2=sz^2$ subsistere potest, continentur in his duabus formulis $24n+5$ et $24n+11$, quibus adjungi debent 2 et 3 et praeterea nulli alii satisfaciunt, ita, ut satisfacientes ordine sint:

2, 3, 5, 11, 29, 53, 59, 83, 101, 107, 131, 149, 173, 179, 197.

Aliud judicium, utrum talis aequatio $fx^2+gy^2=hz^2$ *sit possibilis.*

Dividantur omnia quadrata per numerum h et notentur residua, quae sint 1, a, b, c, d, etc. et quadratum x^2 det residuum a, y^2 vero det b, sicque formula fx^2+gy^2 dabit residuum $af+bg$, quod cum per h debeat esse divisibile, fieri poterit $af+bg=0$, ideoque $b=\frac{-af}{g}$; ergo quodvis residuum si per $\frac{-f}{g}$ multiplicetur, iterum erit residuum. Quia autem $\frac{-f}{g}$ est fractus, ejus loco scribatur $\frac{nh-f}{g}$, ubi n ita sumatur, ut $nh-f$ fiat divisibile per g et quotus sit k, qui si inter residua reperiatur, aequatio erit possibilis; sin secus, impossibilis. Sic proposita aequatione $2x^2+3y^2=29z^2$, ubi $f=2$, $g=3$ et $h=29$, quaerantur residua quadratorum per 29 divisorum, quae sunt numero 14, nempe:

1, 4, 9, 16, 25, 7, 20, 6, 23, 13, 5, 28, 24, 22.

Quaeratur ergo $\frac{29n-2}{3}=9$ posito $n=1$. Quia ergo 9 inter residua occurrit, haec forma est possibilis. Sin autem proponatur $2x^2+3y^2=17z^2$, quadrata per 17 divisa dant residua

1, 4, 9, 16, 8, 2, 15, 13.

Nunc debet esse $\frac{17n-2}{3}=$ numero integro 5, qui cum non sit inter residua, indicat aequationem esse impossibilem.

Hoc vero judicium non certum videtur, nam si aequatio hac forma exhibeatur $17z^2-2x^2=3y^2$, ubi $f=17$, $g=-2$ et $h=3$, residuum quadratorum est unicum 1; at vero $\frac{3n-17}{-2}=7$, si $n=1$, et denuo per 3 dividendo prodit 1, quod est residuum, et tamen aequatio est impossibilis.

Notari meretur aequatio $7x^2+2y^2=23z^2$, quia ipse numerus 23 non in forma $7a^2+2b^2$ continetur, siquidem a et b sint integri; at si $a=\frac{1}{3}$ et $b=\frac{10}{3}$, fit utique $\frac{207}{9}=23$. Per regulam primam autem ex 23 prodit alius $23(p^2+14)$. Sumatur $p=3$ proditque unitas.

(*J. A. Euler.*)

Ut dubium circa criterium postremum tollatur, observandum est primum, criterium eo redire, num inter residua quadratorum per h divisorum occurrat numerus $-fg$, sive $nh-fg$, qui si non occurrat, aequatio $fxx+gyy=hzz$ certe est impossibilis; sin autem occurrat, plus inde non sequitur, quam vel hanc ipsam aequationem $fxx+gyy=hzz$, vel istam $xx+fgyy=hzz$ esse possibilem; unde fieri potest, ut prior non sit possibilis, tum autem certo posterior fit possibilis.

A. m. T. I. p. 201—207.

43.

(*W. L. Krafft.*)

Problema. Formulam mx^3+n quadratum reddere ex casu cognito $ma^3+n=bb$.

Solutio. Ponatur $x=a+y$ et formula proposita fiet

$$bb+3maay+3mayy+my^3=\square,$$

cujus radix ponatur $b+\frac{3maa}{2b}y$; hujus quadratum

$$bb+3maay+\frac{9mma^4}{4bb}yy$$ praebet

$$3a+y=\frac{9ma^4}{4bb}=\frac{9a(bb-n)}{4bb}$$

unde $y=-\frac{3a}{4}-\frac{9an}{4bb}$, ergo $x=\frac{a}{4}-\frac{9an}{4bb}$. Sin autem radix ponatur $b+\frac{3maa}{2b}y+pyy$, erit

$$3mayy+my^3=2bpyy+\frac{9mma^4}{4bb}yy+\frac{3mpaay^3}{b}+ppy^4.$$

Jam fiat $3ma=2bp+\frac{9mma^4}{4bb}=2bp+\frac{9am(bb-n)}{4bb}$, ergo

$$2bp=3ma-\frac{9am(bb-n)}{4bb}=\frac{3ma}{4}+\frac{9amn}{4bb},\quad \text{ergo}$$

$$p=\frac{3ma}{8b}+\frac{9amn}{8b^3}=\frac{3ma}{8b}\left(1+\frac{3n}{bb}\right).$$

Superest haec aequatio $1=\frac{3paa}{b}+\frac{ppy}{m}$, ergo

$$\frac{ppy}{m}=1-\frac{3paa}{b}=1-\frac{9ma^3}{8bb}-\frac{27a^3mn}{8b^4}=-\frac{1}{8}+\frac{9n}{8bb}-\frac{27n}{8bb}+\frac{27nn}{8b^4}$$

$$=-\frac{1}{8}-\frac{9n}{4bb}+\frac{27nn}{8b^4},\ \text{ergo}\quad y=\frac{m}{8pp}\left(-1-\frac{18n}{bb}+\frac{27nn}{b^4}\right).$$

(*Lexell.*)

Annotatio ad superiorem formulam $mx^3+n=\square$, *ubi ex dato casu* $ma^3+n=bb$ *ope transformationis elicuimus novum casum.*

Omni attentione dignum videtur, quod si n fuerit numerus quadratus $=kk$, ex casu cognito $ma^3+kk=bb$ immediate duo alii elici queant hoc modo. Ponatur $x=az$, ut habeatur $ma^3z^3+kk=\square$, hoc est

$$(bb-kk)z^3+kk=\square=yy,\quad \text{ita ut sit}\quad (bb-kk)z^3=yy-kk=(y+k)(y-k).$$

Resolvetur ergo formula $(bb-kk)z^3$ etiam in duos factores, quod duplici modo fieri potest:

I. Sit unus factor $(b+k)zz=y+k$, eritque alter $(b-k)z=y-k$, haec aequatio ab illa subtracta relinquit $(b+k)zz-(b-k)z=2k$, cujus una radix manifesto est $z=1$, unde pro altera fit

$$z=\frac{-2k}{b+k},\quad \text{ergo}\quad x=\frac{-2ak}{b+k}.$$

II. Sit jam prior factor $(b-k)zz=y-k$, et alter dabit $(b+k)z=y+k$, unde fit differentia

$$(b-k)zz-(b+k)z=-2k,$$

cujus una radix est $z=1$, et altera $\quad z=\frac{2k}{b-k}\quad$ et $\quad x=\frac{2ak}{b-k}$.

Unde conficimus istud egregium THEOREMA:

Si formula mx^3+kk fuerit quadratum casu $x=a$, ita ut sit $ma^3+kk=bb$, tum etiam quadratum erit his duobus casibus:

$$\text{primo:}\quad x=\frac{-2ak}{b+k},\quad \text{et altero:}\quad x=\frac{+2ak}{b-k}.$$

Exempli gratia, cum formula $90x^3+1$ fiat quadratum casu $x=\frac{1}{2}$; fiet enim $90\cdot\frac{1}{8}+1=\frac{49}{4}$, ubi est $a=\frac{1}{2}$, $k=1$ et $b=\frac{7}{2}$, bini casus derivati erunt $x=-\frac{2}{9}$ et $x=+\frac{2}{5}$; ex priore enim fit $\frac{-10.8}{81}+1=\frac{1}{81}$, et ex posteriore $\frac{90.8}{5^3}+1=\frac{18.8}{25}+1=\frac{169}{25}$.

A. m. T. I. p. 120. 121.

44.

Omni attentione digna est haec formula $181^2+7=32^3$; nam etsi $32=5^2+7$, tamen nullo modo est $181+\sqrt{-7}=(5+\sqrt{-7})^3$, verum tamen est

$$181+\sqrt{-7}=\left(\frac{1+3\sqrt{-7}}{8}\right)(5+\sqrt{-7})^3.$$

Notandum autem est $\frac{1+3\sqrt{-7}}{8}\cdot\frac{1-3\sqrt{-7}}{8}=1$; unde patet evolutionem illam per factores imaginarios profundiorem investigationem requirere.

Problema. Invenire in integris quadratum et cubum, quorum differentia sit valde parva, veluti

$$32^3-181^2=7 \quad \text{et} \quad 253^2-40^3=9.$$

Cum proxime esse debeat $x^2=y^3$, ponatur $x=p^3+a$, hincque fit

$$y=(p^3+a)^{\frac{2}{3}}=pp+\frac{2}{3}\cdot\frac{a}{p}-\frac{aa}{9p^4}\text{ etc.}$$

unde si p et a ita sumantur, ut haec formula proxime aequetur numero integro y, problema erit solutum; veluti sumto $a=3$ et $p=2$, formula illa dat $y=5$ et $x=11$, fit autem 11^2 proxime $=5^3$.

A. m. T. I. p. 127.

45.

(*J. A. Euler.*)

Si debeat esse $13xx+12=\square$, valores pro x erunt

$$1,\ 2,\ 13,\ 23,$$

quorum ordo ita se habet

$$-23,\quad -2,\quad +1,\quad +13,\ \ldots\ p,\quad q,\quad r$$

existente $r=11q-p$, ubi numerus 11 inde oritur, quod sit $\frac{11}{2}=\sqrt{(13\cdot\frac{9}{4}+1)}$.

Si debeat esse $5xx+44=\square$, valores pro x erunt

$$1,\quad 2,\quad 5,\quad 7,\quad 14,\quad 19,\quad 37,\quad 50,$$

quorum ordo ita se habet

$$-50,\quad -19,\quad -7,\quad -2,\quad +1,\quad +5,\quad +14,\quad +37,\ \ldots\ p,\quad q,\quad r$$

et $r=3q-p$, propter $\frac{3}{2}=\sqrt{(5\cdot\frac{1}{4}+1)}$.

Ut $3xx-143=\square$ debet esse $x=7,\ 8,\ 9,\ 12,\ 16,\ 23,\ 28$, quae multitudo est notatu digna et inde venit quod $143=11.13$.

A. m. T. I. p. 135. 136.

46.

Problema. Datis numeris a et b, invenire omnes numeros x, ut haec formula $ax+b$ fiat quadratum.

Ponatur $x=ayy+2py+q$ et quadratum esse debet

$$aayy+2apy+aq+b=\square,\text{ quod fit, si } p=\sqrt{(aq+b)}.$$

Cognito ergo unico casu, qui sit $aq+b=pp$, erit

$$x=ayy\pm 2py+q,$$

quae formula omnes soluliones continet.

Exemplum. Sit $a=7$ et $b=2$, et formula nostra $7x+2$. Quia $7.1+2=3^2$, erit $q=1$ et $p=3$; ergo omnes casus sunt $x=7yy\pm 6y+1$, quae formula praebet hos numeros pro x:

Existente y	prodit x
0	1
1	$7\pm 6+1$; 2 vel 14
2	29 ± 12; 17 vel 41
3	64 ± 18; 46 vel 82
4	113 ± 24; 89 vel 137
etc.	etc.

Lex progressionis valorum pro x:

1, 2, 14, 17, 41, 46, 82, 89, 137, etc.

Diff. 1 12 3 24 5 36 7 48 9 60 etc.

A. m. T. I. p. 214.

47.

Zwei Trigonalzahlen zu finden, deren Produkt wieder eine Trigonalzahl sei.

Also
$$\frac{xx+x}{2}\cdot\frac{yy+y}{2}=\frac{zz+z}{2},\quad\text{oder}\quad x(x+1)\,y(y+1)=2z(z+1),\quad\text{oder}$$
$$pq\,.\,x(x+1)\,y(y+1)=2z(z+1)\,.\,pq.$$

Nun mache man $px(y+1)=2qz$ und $qy(x+1)=p(z+1)$, so wird aus dem ersten Satze
$$z=\frac{px(y+1)}{2q},\quad\text{und aus dem andern:}\quad z=\frac{qy(x+1)}{p}-1.$$

Daher
$$2qqy(x+1)-2pq=ppx(y+1),$$
welches sich auch so darstellen lässt
$$xy(2qq-pp)+2qqy-2pq-ppx=0.$$

Es sei nun $2qq-pp=a$, so ist
$$axy+2qqy-ppx-2pq=0,\quad\text{und hieraus}\quad y=\frac{ppx+2pq}{ax+2qq},$$
also
$$ay=\frac{appx+2apq}{ax+2qq}\quad\text{und}\quad ay-pp=\frac{2apq-2ppqq}{ax+2qq}.$$

Es sei nunmehr $2ppqq-2apq=2pq(pq-a)=fg$, so haben wir $pp-ay=\frac{fg}{ax+2qq}$. Nun setze man $ax+2qq=f$, so wird $pp-ay=g$, folglich $y=\frac{pp-g}{a}$ und $x=\frac{f-2qq}{a}$, wo leicht zu machen, dass $a=1$ sei.

Exempel. Man nehme $p=7$, $q=5$, $a=1$, so ist $fg=34.70=4.5.7.17$. Daher wird
$$x=f-50\quad\text{und}\quad y=49-g.$$

Es sei $g=20$, $f=119$, so ist $\quad x=69\quad$ und $\quad y=29$;

also die zwei Trigonalzahlen 35.69 und 15.29. Da nun $z=\frac{px(y+1)}{2}=7.69.3=1449$, so ist
$$\frac{zz+z}{2}=1449.725,\quad\text{welches in der That}=69.35.29.15\ \text{ist.}$$

Es sei ferner $f = 85 = 5.17$, so wird $g = 4.7 = 28$, $x = 35$, $y = 21$; mithin die beiden Trigonalzahlen 35.18 und 11.21. Es ist aber $z = 7.7.11 = 539$, also

$$\frac{zz+z}{2} = 539.270 = 35.18.11.21.$$

A. m. T. I. p. 254.

48.

(*N. Fuss I.*)

Problema. Invenire numeros integros x et y, ut fiat $axx - byy = A$.

Solutio. Primo notandum est hoc fieri non posse, nisi fuerit $A = aff - bgg$; deinde quaerantur per problema Pellianum numeri m et n, ut fiat $mm = abnn + 1$, sive $m = \sqrt{(abnn+1)}$. Cum ergo sit $mm - abnn = 1$, erit quoque $(mm - abnn)^{\lambda} = 1$; ponere igitur licebit

$$axx - byy = (aff - bgg)(mm - abnn)^{\lambda},$$

quae forma in factores irrationales resoluta dabit

$$x\sqrt{a} + y\sqrt{b} = (f\sqrt{a} + g\sqrt{b})(m + n\sqrt{ab})^{\lambda},$$

quod posterius productum evolvatur et termini signo $\sqrt{a}$ affecti aequentur ipsi $x\sqrt{a}$; reliqui termini signo $\sqrt{b}$ affecti aequentur ipsi $y\sqrt{b}$, hocque modo tam x quam y per numeros integros determinabitur, quod exemplo illustremus:

Exemplum. Quaerantur numeri x et y, ut fiat $3xx - yy = 2$, ubi $a = 3$ et $b = 1$; erit autem $2 = 3ff - gg$ sumendo $f = 1$ et $g = \pm 1$; quia igitur $ab = 3$, fiet $m = \sqrt{(3nn+1)}$. Sumendo $n = 1$ et $m = 2$, unde nostra formula erit $x\sqrt{3} + y = (\sqrt{3} + 1)(2 + \sqrt{3})^{\lambda}$

$$\begin{array}{llll}
\text{si} & \lambda = 0 & \text{erit} & x\sqrt{3} + y = \sqrt{3} + 1 \\
 & \lambda = 1 & \text{"} & x\sqrt{3} + y = 3\sqrt{3} + 5 \\
 & \lambda = 2 & \text{"} & x\sqrt{3} + y = 11\sqrt{3} + 19 \\
 & \lambda = 3 & \text{"} & x\sqrt{3} + y = 41\sqrt{3} + 71 \\
 & \lambda = 4 & \text{"} & x\sqrt{3} + y = 153\sqrt{3} + 265.
\end{array}$$

Ceterum valores tam ipsius x quam ipsius y constituunt series recurrentes, quarum ultimus quisque terminus per $2m$ multiplicatus demto penultimo, praebet sequentem; sic in nostro exemplo, ubi $2m = 4$, litterae x et y ita procedant

$$x = 1,\ 3,\ 11,\ 41,\ 153$$
$$y = 1,\ 5,\ 19,\ 71,\ 265.$$

Occasione problematis Pelliani, seu formulae $m = \sqrt{(abnn+1)}$, praeter casus, ubi est vel $ab = \alpha\alpha \pm 1$, vel $ab = \alpha\alpha \pm 2$, etiam sequentes casus generaliores locum habent, scilicet si fuerit $ab = \alpha\alpha\beta\beta \pm \beta$, fiet $nn = 4\alpha\alpha$ et $n = 2\alpha$ et $m = 2\alpha\alpha\beta \pm 1$. Deinde si fuerit $ab = \alpha\alpha\beta\beta \pm 2\beta$, erit $n = \alpha$ et $m = \alpha\alpha\beta \pm 1$.

A. m. T. I. p. 277.

49.

Problema. Invenire duos numeros p et q, ut fiat $(pp+1)^2 + (qq+1)^2 = \square$.

Solutio. Ponatur $pp + 1 = xx - yy$ et $qq + 1 = 2xy$ eritque $pp = xx - yy - 1$ et $qq = 2xy - 1$. Sit jam $p = x - z$, erit $2xz - zz = yy + 1$, unde fit

$$x = \frac{zz+yy+1}{2z}, \quad \text{hinc} \quad qq = \frac{yzz+y^3+y}{z} - 1.$$

Statuatur nunc $y = nz$ fietque $qq = nzz + n^3zz + n - 1$. Capiatur ergo $n = 2$, ut sit $y = 2z$, erit $qq = 10z^2 + 1$ cui satisfit:

1) Si $z = \frac{2}{3}$; tum erit $qq = \frac{49}{9}$, ergo $q = \frac{7}{3}$. Porro $y = \frac{4}{3}$, unde $x = \frac{29}{12}$ et $p = \frac{7}{4}$. Tum igitur formul

$$\left(\frac{49}{16}+1\right)^2 + \left(\frac{49}{9}+1\right)^2, \quad \text{erit quadratum radicis} \quad xx+yy = \frac{841}{144} + \frac{16}{9} = \frac{1097}{144}.$$

2) Sumatur $z = \frac{2}{9}$, erit $qq = \frac{121}{81}$, hinc $q = \frac{11}{9}$; tum vero $y = \frac{4}{9}$ et $x = \frac{5zz+1}{2z} = \frac{101}{36}$, ergo

$$p = \frac{101}{81} - \frac{2}{9} = \frac{83}{81}.$$

3) Sumatur $z = 6$, erit $qq = 361$ et $q = 19$; porro $y = 12$ et $x = \frac{181}{12}$ et $p = \frac{109}{12}$; ergo

$$\left(\frac{109^2}{144}+1\right)^2 + (361+1)^2 = \left(\frac{181^2}{144}+144\right)^2.$$

A. m. T. I. p. 279.

50.

Problema difficillimum,

quo quatuor biquadrata quaeruntur, quorum summa itidem sit biquadratum,

sive ut sit $A^4 + B^4 + C^4 + D^4 = E^4$,

siquidem fieri potest, sequenti modo tractandum videtur:

Statuatur scilicet $A^2 = \frac{pp+qq+rr-ss}{n}$; $B^2 = \frac{2ps}{n}$; $C^2 = \frac{2qs}{n}$ et $D^2 = \frac{2rs}{n}$; tum enim fiet

$$E^2 = \frac{pp+qq+rr+ss}{n},$$

ita, ut haec quinque formulae quadrata reddi debeant. Incipiamus a prima et ultima, ita, ut reddi debeat

$$\frac{pp+qq+rr}{n} \pm \frac{ss}{n} = \square.$$

Cum igitur sit $aa + bb \pm 2ab = \square$, statuatur

$$\frac{pp+qq+rr}{n} = aa+bb \quad \text{et} \quad \frac{ss}{n} = 2ab,$$

fit ergo $ss = 2nab = \square$. Sit $2n = \alpha\beta$ et statuatur $a = \alpha ff$ et $b = \beta gg$ eritque

$$ss = \alpha\alpha\beta\beta ffgg, \quad \text{ergo} \quad s = \alpha\beta fg = 2nfg;$$

tum vero esse debet $\frac{pp+qq+rr}{n} = \alpha\alpha f^4 + \beta\beta g^4$.

Pro reliquis conditionibus faciamus $\frac{2ps}{n} = 4pfg = 4xx$, unde $p = \frac{xx}{fg}$, porro $\frac{2qs}{n} = 4qfg = 4yy$, hinc $q = \frac{yy}{fg}$, pro quarta $\frac{2rs}{n} = 4rfg = 4zz$, habemus $r = \frac{zz}{fg}$. Hinc superest ista aequatio

$$\frac{pp+qq+rr}{n} = \alpha\alpha f^4 + \beta\beta g^4, \quad \text{sive} \quad pp+qq+rr = \frac{1}{2}\alpha\beta(\alpha\alpha f^4 + \beta\beta g^4),$$

et loco p, q, r valores inventos substituendo

$$\frac{x^4}{f^2g^2} + \frac{y^4}{f^2g^2} + \frac{z^4}{f^2g^2} = \frac{1}{2}\alpha\beta(\alpha\alpha f^4 + \beta\beta g^4), \quad \text{sive} \quad x^4+y^4+z^4 = \frac{1}{2}\alpha\beta ffgg(\alpha\alpha f^4 + \beta\beta g^4).$$

Quodsi jam hic restituamus $\alpha ff = a$ et $\beta gg = b$, colligitur $x^4 + y^4 + z^4 = \frac{1}{2} ab\,(aa + bb)$. Quaeritur ergo num huic aequationi satisfieri possit; tum autem fiet

$$n = \frac{1}{2}\alpha\beta, \quad s = 2nfg \text{ vel } \alpha\beta fg, \quad \text{et} \quad p = \frac{xx}{fg}, \quad q = \frac{yy}{fg} \quad \text{et} \quad r = \frac{zz}{fg},$$

atque hinc porro $\quad A = a - b$ et $E = a + b$, $B = 2x$, $C = 2y$ et $D = 2z$,

verum ne his quidem ambagibus est opus, cum enim fieri debeat $B^4 + C^4 + D^4 = E^4 - A^4$. Statuatur

$$A = a - b \text{ et } E = a + b, \quad \text{fietque} \quad E^4 - A^4 = 8a^3 b + 8ab^3 = 8ab\,(aa + bb),$$

ergo utrinque per 16 dividendo prodit

$$\frac{1}{2}\,ab\,(aa + bb) = \frac{B^4 + C^4 + D^4}{16} = x^4 + y^4 + z^4.$$

A. m. T. I. p. 281.

51.

Notatu digna est haec formula: $1 + z - z^3$, quae fit quadratum sumto $z = \frac{11}{9}$, qui tamen valor per regulas vulgares non elicitur. Hinc ista quaestio:

Numerum 2 dividere in duas partes x et $2 - x$, quarum productum $2x - xx$ sit numerus formae $z^3 - z$; tum enim erit $1 - 2x + xx = 1 + z - z^3$, ideoque $1 - x = \sqrt{(1 + z - z^3)}$. Sumto ergo $z = \frac{11}{9}$, erit $1 - x = \frac{17}{27}$, hinc $x = \frac{10}{27}$, et altera pars $2 - x = \frac{44}{27}$, quarum productum est

$$2x - xx = \frac{440}{27^2}, \text{ at vero } z^3 - z = \frac{1331}{729} - \frac{11}{9} = \frac{440}{27^2}.$$

A. m. T. I. p. 295.

52.

Theorema I. Si p denotet numerum primum quemcunque, talis aequatio $z^3 = py^3 \pm ppx^3$ semper est impossibilis.

Demonstratio. Quia enim esse deberet z^3 divisibile per p, ideoque $z = pA$, unde fieret

$$ppA^3 = y^3 \pm px^3, \quad \text{sive} \quad y^3 = ppA^3 \mp px^3;$$

foret igitur etiam y divisibilis per p. Sit ergo $y = pB$, unde fieret $ppB^3 = pA^3 \mp x^3$, hinc ergo etiam x divisibile esse debet per p; hincque ponatur $x = pC$; unde fiet $ppC^3 = A^3 - pB^3$; foret igitur eodem modo $A = pD$, foretque $ppD^3 = pC^3 + B^3$, tum vero etiam B per p divisibilis esse deberet, porro etiam C, D, etc. in infinitum. Hoc ergo modo singulae litterae z, y, x non solum per p, sed etiam per pp, per p^3 atque adeo per p^∞ deberent esse divisibiles; quod cum sit absurdum, veritas theorematis est evicta.

Theorema II. Si numeri a, b, c fuerint primi ad p, ita ut nullus eorum per p sit divisibilis, tum etiam haec aequatio $az^3 \pm bpy^3 \pm cppx^3 = 0$ semper est impossibilis.

Demonstratio. Quia enim a per p non est divisibile, z deberet esse divisibile, tum vero etiam pari modo y et x, sicque ad eandem aequationem perveniretur, unde patet impossibilitas, uti casu praecedente.

Corollarium. Eadem demonstratio quoque habet locum, si p fuerit productum ex duobus vel pluribus numeris primis diversis, veluti si sit $p = 2.3$, vel 3.5, vel $3.5.7$, etc.

Theorema III. Si p sit vel numerus primus, vel productum ex aliquot numeris primis diversis, aliqui vero numeri $a, b, c, d,$ etc. sint numeri ad p primi, tum etiam haec aequatio semper est impossibilis:

$$az^4 \pm bpy^4 \pm cppx^4 \pm dp^3v^4 = 0.$$

Quia ob rationes superiores singuli numeri $z, y, x, v,$ etc. non solum per p, sed per omnes potestates ipsius p deberent esse divisibiles. Taliaque theoremata ad potestates altiores extendi possunt.

NB. Hae autem demonstrationes vim perderent suam, si esset $p = 1$, quia omnes plane numeri divisibiles sunt per omnes potestates ipsius 1.

A. m. T. II. p. 10. 11.

53.

Problema. Reddere hanc formulam quadratum: $(A + Bz)(a + bz + czz + dz^3)$.

Solutio. Statuatur hoc quadratum $= (A + Bz)^2 (p + qz)^2$ fietque

$$\left.\begin{matrix} App \\ -a \\ \end{matrix}\quad\begin{matrix} +2Apq \\ +Bpp \\ -b \end{matrix}\right\} z \quad \left.\begin{matrix} +Aqq \\ +2Bpq \\ -c \end{matrix}\right\} zz \quad \left.\begin{matrix} +Bqq \\ -d \end{matrix}\right\} z^3 = 0,$$

ubi duae solutiones sunt considerandae, primo si $pp = \frac{a}{A}$, sumatur $q = \frac{b - Bpp}{2Ap}$, tum erit $z = \frac{c - Aqq - 2Bpq}{Bqq - d}$. Altera solutio locum habet si $qq = \frac{d}{B}$; tum sumatur

$$p = \frac{c - Aqq}{2Bq} \quad \text{eritque} \quad z = \frac{a - App}{2Apq + Bpp - b}.$$

Problema. Si proposita fuerit haec formula $(A + Bz + Czz)(a + bz + czz)$, eam reddere quadratum.

Solutio. Ponatur hoc quadratum $= pp(a + bz + czz)^2$ fietque

$$A + Bz + Czz = ppa + ppbz + ppczz.$$

Hic ergo si fuerit $pp = \frac{A}{a}$, statim fit $z = \frac{ppb - B}{C - cpp}$. Secundo si fuerit $pp = \frac{C}{c}$, erit $z = \frac{app - A}{B - bpp}$.

In genere autem si satisfaciat valor $z = f$, quo casu fit $pp = \frac{A + Bf + Cff}{a + bf + cff}$, tum semper alius valor potest inveniri; quoniam enim habetur haec aequatio quadratica

$$zz - \frac{B - bpp}{C - cpp} z + \frac{A - app}{C - cpp} = 0,$$

eaque per hypothesin radicem habeat $z = f$; erit quoque $z = g$ existente

$$\text{tam} \quad f + g = \frac{bpp - B}{C - cpp} \quad \text{quam} \quad fg = \frac{A - app}{C - cpp},$$

unde duplici modo alter valor g reperitur. Hoc adhuc clarius ita ostendi potest. Cum esse det

$$A + Bz + Czz = pp(a + bz + czz),$$
$$\text{tum vero} \quad A + Bf + Cff = kk(a + bf + cff),$$

semper alius valor pro z assignari potest, existente pariter $p = k$. Dividatur prior aequatio per posteriorem fietque

$$\frac{A + Bz + Czz}{A + Bf + Cff} = \frac{a + bz + czz}{a + bf + cff};$$

subtrahendo utrinque unitatem et dividendo per $z - f$ prodit

$$\frac{B + C(z + f)}{A + Bf + Cff} = \frac{b + c(z + f)}{a + bf + cff},$$

unde f facile definitur.

Hac methodo insuper duo alii valores reperiri possunt. Cum enim sit

$$\frac{A + Bz + Czz}{A + Bf + Cff} = \frac{a + bz + czz}{a + bf + cff},$$

multiplicetur utrinque per $\frac{f}{z}$, ut habeatur

$$\frac{Af + Bfz + Cfzz}{Az + Bfz + Cffz} = \frac{af + bfz + cfzz}{az + bfz + cffz}$$

et sublata unitate erit
$$\frac{A(f-z) + Cfz(z-f)}{Az + Bfz + Cffz} = \frac{a(f-z) + cfz(z-f)}{az + bfz + cffz},$$

sive
$$\frac{A - Cfz}{A + Bf + Cff} = \frac{a - cfz}{a + bf + cff},$$

unde tertius desumitur valor. Porro multiplicetur utrinque per $\frac{ff}{zz}$, ut sit

$$\frac{Aff + Bffz + Cffzz}{Azz + Bfzz + Cffzz} = \frac{aff + bffz + cffzz}{azz + bfzz + cffzz}$$ et unitate utrinque sublata $$\frac{A(f+z) + Bfz}{A + Bf + Cff} = \frac{a(f+z) + bfz}{a + bf + cff}.$$

Horum valorum quilibet pro f assumtus praebebit duos novos valores, ita ut hoc modo infiniti valores reperiri queant. Sit $A = 2$, $B = 3$, $C = -1$, $a = 3$, $b = -1$, $c = 2$, ut debeat esse $\frac{2 + 3z - zz}{3 - z + 2zz} = \square$. Cui primo satisfacit $z = 1$. Sit ergo $f = 1$, erit secundo $\frac{3 - z - 1}{4} = \frac{2(z+1) - 1}{4}$, unde $z = 1$; tertio $2 + z = 3 - 2z$, unde $z = \frac{1}{3}$; quarto $2 + 2z + 3z = 3 + 3z - z$, unde $z = \frac{1}{3}$.

Interim tamen haec methodus nihil plane juvat, sed tantum duos valores ostendit. Cum enim f sit numerus definitus, aequatio $\frac{A + Bz + Czz}{A + Bf + Cff} = \frac{a + bz + czz}{a + bf + cff}$ manifesto est aequatio quadratica determinata, quae tantum duos admittit valores. Interim tamen haec methodus cum successu adhibitur in resolutione hujus formulae simplicioris $a + bz + czz = pp$, casusque constet, quo $a + bf + cff = kk$, erit igitur $\frac{a + bz + czz}{a + bf + cff} = \frac{pp}{kk}$. Jam sumatur primo $pp = kk$, fietque $b + c(z + f) = 0$, unde $z = \frac{-b - cf}{c}$. Deinde sumatur $\frac{pp}{kk} = \frac{zz}{ff}$ eritque

$$ppff = kkzz, \quad \text{ideoque} \quad aff + bffz + cffzz = azz + bfzz + cffzz,$$

unde fit
$$a(f + z) + bfz = 0 \quad \text{atque} \quad z = \frac{-af}{a + bf},$$

qui posterior valor loco f assumtus denuo novum valorem praebet, et ita porro.

Verum hoc casu solutio generalis ita reperiri potest. Sumatur $p = k + \nu(z - f)$, ut sit

$$\frac{a + bz + czz}{a + bf + cff} = \frac{kk + 2k\nu(z - f) + \nu\nu(z - f)^2}{kk}.$$

Subtrahatur utrinque unitas eritque

$$\frac{b + c(z + f)}{kk} = \frac{2k\nu + \nu\nu(z - f)}{kk},$$ unde reperitur $$z = \frac{2k\nu - f\nu\nu - b - cf}{c - \nu\nu},$$

unde prior oritur posito $\nu = 0$, posterior vero posito $\nu = \frac{k}{f}$.

A. m. T. II. p. 155. 156.

54.

Theorema. Si fuerit p numerus primus formae $4n + 1$, semper dabitur numerus x, minor quam n, ut fiat $px - 1 = \square$.

Demonstratio. Cum sit $p = 4n + 1$, erit $p = aa + bb$. Jam quaeratur fractio $\frac{c}{d}$ proxime aequalis fractioni $\frac{a}{b}$, ita, ut sit $ad - bc = \pm 1$, eritque $x = cc + dd$. Semper enim numeri c et d infra semisses numerorum a et b assignari possunt; tum autem erit $px - 1 = (ac + bd)^2$. Erit enim

$$px = (aa + bb)(cc + dd) = (ac + bd)^2 + (ad - bc)^2$$

at $ad - bc = \pm 1$ per hypothesin, unde $px - 1 = (ac + bd)^2$.

Exemplum. Sit $p = 193$, erit $a = 12$ et $b = 7$, porro $c = 5$ et $d = 3$, unde fit

$$x = 34 \quad \text{eritque} \quad px - 1 = 81^2.$$

A. m. T. II. p. 167.

55.

Theorema numericum profundissimae indaginis.

Si m, n et z denotent numeros integros positivos, tum ista formula

$$4mnz - m - n$$

nunquam evadere potest quadratum.

Hoc theorema inde est derivatum, quod inter divisores formae $mxx + yy$ occurrat formula $4mz + 1$, unde sequitur, formulam $4mz - 1$ nunquam esse posse divisorem illius formae $mxx + yy$, vel saltem hujus $m + yy$, unde haec aequatio

$$(4mz - 1)n = m + yy$$

semper erit impossibilis. Sit $\sqsupset\!\sqsubset$ signum impossibilitatis eritque $(4mz - 1)n - m \sqsupset\!\sqsubset yy$ sive $(4mz - 1)n - m \sqsupset\!\sqsubset \square$. Verum hoc fundamentum nondum est rigide demonstratum, ideoque demonstratio hujus theorematis plurimum desideratur. Interim tamen evidens est ejus veritas casibus, quibus est $m + n = 4i + 2$, quia tum fit $4mnz - 4i - 2$ numerus impariter par, a quadrato abhorrens. Dein etiam casu $m + n = 4i + 1$, quia tum prodit forma $4mnz - 4i - 1$, sive forma $4A - 1$, quae nunquam esse potest quadratum. Demonstrandi igitur tantum restant duo casus, alter, quo $m + n = 4i$, alter vero, quo $m + n = 4i + 3$, vel $4i - 1$. Pro casu priore $m + n = 4i$ sumi poterit $m = 2i - k$ et $n = 2i + k$, unde erit $(2i - k)(2i + k)z - i \sqsupset\!\sqsubset \square$, sive $(4ii - kk)z - i \sqsupset\!\sqsubset \square$. Pro altero casu, quo $m + n = 4i - 1$ sumi poterit

$$m = 2i - k \quad \text{et} \quad n = 2i + k - 1, \quad \text{eritque} \quad 4(2i - k)(2i + k - 1)z - 4i + 1 \sqsupset\!\sqsubset \square,$$

sive hoc modo

$$((4i - 1)^2 - (2k - 1)^2)z - 4i + 1 \sqsupset\!\sqsubset \square.$$

Hinc innumerae formae speciales derivari possunt, veluti ex priore forma $(4ii - kk)z - i \sqsupset\!\sqsubset \square$, unde casu $i = 1$

prodit $4z - 1 \sqsupset\!\sqsubset \square$, $3z - 1 \sqsupset\!\sqsubset \square$, qui per se sunt manifesti

casu $i = 2$: $16z - 2 \sqsupset\!\sqsubset \square$, $15z - 2 \sqsupset\!\sqsubset \square$, $12z - 2 \sqsupset\!\sqsubset \square$, $7z - 2 \sqsupset\!\sqsubset \square$

casu $i = 3$: $36z - 3 \sqsupset\!\sqsubset \square$, $35z - 3 \sqsupset\!\sqsubset \square$, $32z - 3 \sqsupset\!\sqsubset \square$, $27z - 3 \sqsupset\!\sqsubset \square$, $20z - 3 \sqsupset\!\sqsubset \square$, $11z - 3 \sqsupset\!\sqsubset \square$

casu $i = 4$: $64z - 4 \sqsupset\!\sqsubset \square$, $63z - 4 \sqsupset\!\sqsubset \square$, $60z - 4 \sqsupset\!\sqsubset \square$, $55z - 4 \sqsupset\!\sqsubset \square$, $48z - 4 \sqsupset\!\sqsubset \square$, $39z - 4 \sqsupset\!\sqsubset \square$, $28z - 4 \sqsupset\!\sqsubset \square$, $15z - 4 \sqsupset\!\sqsubset \square$.

Hic autem veritas singularum ostendi potest, at vero ex principiis diversissimis. Eodem modo formulae speciales ex altero casu oriundae

$$((4i - 1)^2 - (2k - 1)^2)z - 4i + 1 \sqsupset\!\sqsubset \square$$

sunt

	casu $i = 1$	casu $i = 2$	casu $i = 3$
	$8z - 3 \sqsupset\!\sqsubset \square$	$48z - 7 \sqsupset\!\sqsubset \square$	$120z - 11 \sqsupset\!\sqsubset \square$
		$40z - 7 \sqsupset\!\sqsubset \square$	$112z - 11 \sqsupset\!\sqsubset \square$
		$24z - 7 \sqsupset\!\sqsubset \square$	$96z - 11 \sqsupset\!\sqsubset \square$
			$72z - 11 \sqsupset\!\sqsubset \square$
			$40z - 11 \sqsupset\!\sqsubset \square$

A. m. T. II. p. 211. 212.

56.

Proposita hac formula ad quadratum reducenda: $(qq-pp)^2+(ppqq-1)^2=\square$, statuatur $q=p+z$, et perveniet ad aequationem, unde per regulas cognitas reperitur

$$z=\frac{-4p(p^4-1)}{3p^4-1},\quad \text{unde fit}\quad q=\frac{-p(p^4-3)}{3p^4-1},$$

ubi p pro lubitu accipi potest, si modo excludantur casus $p=0$ et $p=\pm 1$. Ita sumto $p=2$ fit $q=\frac{26}{47}$. Si $p=3$ fit $q=\frac{117}{121}$. Si $p=\frac{3}{2}$ erit $q=\frac{99}{454}$. Ita sumto $p=2$ et $q=\frac{26}{47}$ erit $pp-qq=\frac{120.68}{47^2}$, et ob $pq=\frac{52}{47}$ erit $ppqq-1=\frac{5.99}{47^2}$. Quadratum ergo fieri debet $120^2.68^2+5^2.99^2=15^2(8^2.68^2+33^2)$, quod continetur in forma $4aabb+(aa-bb)^2$. Fit enim $8.68=2ab$ ergo $ab=4.68=16.17$ et $33=aa-bb=(a+b)(a-b)$.

Proposita tali formula $qq(pp-1)^2+pp(qq-1)^2=\square$, duplex solutio institui potest:

prior: ponatur $q=np+n-1$, tum enim erit $q+1=(p+1)n$,

altera: ponatur $q=\frac{np+1}{p+n}$, tum enim erit $q+1=\frac{(n+1)(p+1)}{p+n}$ et $q-1=\frac{(n-1)(p-1)}{p+n}$.

Praeterea notetur, hanc formam ad praecedentem $(qq-pp)^2+(ppqq-1)^2$ reduci ponendo $p=fg$ et $q=\frac{f}{g}$, unde etiam solutio praecedentis formulae hic adhiberi potest. Fluunt autem istae formulae ex solutione hujus problematis $aa+bb=\square$, $aa+cc=\square$, $bb+cc=\square$. Primo enim sumatur $b=\frac{pp-1}{2p}.a$, erit $aa+bb=\left(\frac{pp+1}{2p}\right)^2 aa$ et $c=\frac{qq-1}{2q}.a$. Tertia formula evadet $qq(pp-1)^2+pp(qq-1)^2$. Altera solutio ita se habet: Sumatur $a=2fg$ et $b=ff-gg$, satisfiet primae conditioni. Pro secunda statuatur

$$c=ffgg-1;\quad \text{erit enim}\quad aa+cc=2ffgg+f^4g^4+1.$$

Tertia ergo postulat, ut sit

$$(ff-gg)^2+(ffgg-1)^2=\square.$$

A. m. T. III. p. 9.

57.

Problema. Formulam $2x^4-y^4=zz$ ad hanc $8p^4+q^4=rr$ reducere.

Solutio. Ponatur $2x^4+y^4=\nu$, erit $\nu\nu-z^4=8x^4y^4$, unde fit $8x^4y^4+z^4=\nu\nu$, sicque erit $p=xy$, $q=z$ et $r=\nu=2x^4+y^4$.

Generalius ergo hoc fieri potest, nempe si $\alpha x^4-\beta y^4=zz$ posito $\alpha x^4+\beta y^4=\nu$, erit

$$\nu\nu-z^4=8\alpha\beta x^4y^4,\quad \text{ideoque}\quad \nu\nu=z^4+8\alpha\beta x^4y^4.$$

Problema. Formulam $8p^4+q^4=rr$ ad formam $2x^4-y^4=zz$ reducere.

Solutio. Cum ergo sit $8p^4=rr-q^4=(r+qq)(r-qq)$, manifestum est esse q et r numeros impares. Hinc sequitur, numerorum $r+qq$ et $r-qq$ alterum fore impariter parem, alterum pariter parem, unde nascuntur duo casus:

I. Sit $r+qq$ impariter par $=2\alpha$, alter vero $r-qq$ pariter par $=4\beta$; erit ergo $8p^4=8\alpha\beta$, ideoque $\alpha\beta=p^4$, unde quia α et β sunt primi inter se, uterque debet esse biquadratum. Sit ergo $\alpha=s^4$ et $\beta=t^4$, fiet $p=st$ et $r+qq=2s^4$ et $r-qq=4t^4$, unde oritur $2qq=2s^4-4t^4$, sive $q^2=s^4-2t^4$.

II. Sit $r-qq$ impariter par $=2\alpha$ et $r+qq$ pariter par $=4\beta$, eritque $8p^4=8\alpha\beta$, ideoque $p^4=\alpha\beta$. Sit nunc $\alpha=s^4$ et $\beta=t^4$, eritque $p=st$; ac nunc $r+qq=4t^4$ et $r-qq=2s^4$, ideoque $qq=2t^4-s^4$. Posteriore ergo tantum casu reductio praescripta fieri potest. Interim tamen formula $f^4+8g^4=hh$ semper ad formam $2x^4-y^4$ reduci potest. Quod si enim sumatur $x=f^3+2fgg-gh$ et $y=f^3-4fgg+gh$, semper erit $2x^4-y^4=zz$, existente $z=f^6+f^4gg+24ffg^4-8g^6-6f^3gh$.

Analysis, qua haec reductio est inventa. Posito $2x^4 - y^4 = zz$, debet esse

$$xx = pp + qq, \quad yy = pp + 2pq - qq, \text{ tum enim fiet } z = qq + 2pq - pp.$$

Hic ergo p et q ita definiri debent, ut xx et yy fiant quadrata, quod sequenti modo praestari potest. Cum sit $yy - xx = 2q(p - q) = (y + x)(y - x)$, jam statuatur $y + x = \frac{2a}{b} \cdot q$ et $y - x = \frac{b}{a}(p - q)$. Sic enim fiet $yy - xx = 2q(p - q)$. Addantur jam quadrata, fiet

$$2yy + 2xx = \frac{4aa}{bb} qq + \frac{bb}{aa} pp - \frac{2bb}{aa} pq + \frac{bb}{aa} qq.$$

At vero ex primis formulis fiet $2yy + 2xx = 4pp + 4pq$, qui valor illi aequatus et multiplicatione facta per $aabb$ dat

$$(b^4 - 4aabb)pp - 2pq(b^4 + 2aabb) + (b^4 + 4a^4)qq = 0.$$

Hinc radicem extrahendo fit $\frac{p}{q} = \dots\dots$

Theorema. Si fuerit $ma^4 - nb^4 = cc$, inde assignari potest talis forma $x^4 - mny^4 = zz$.

Demonstratio. Posito enim $ma^4 + nb^4 = \Delta$, erit $\Delta\Delta = c^4 + 4mna^4b^4$. At in altera formula si ponatur $xx = pp + mnqq$ et $yy = 2pq$, fiet $z = pp - mnqq$. Jam statuatur $p = rr$ et $q = 2ss$, ut fiat $y = 2rs$, hinc fiet $xx = r^4 + 4mns^4$. Facta ergo comparatione erit $x = \Delta$, $r = c$, $s = ab$, unde fit $y = 2abc$, $z = c^4 - 4mna^4b^4$. Hinc ergo necesse est ut fiat

$$x^4 - mny^4 = zz.$$

A. m. T. III. p. 129. 131.

58.

Observatio. Ut formula $\frac{pq(pp - qq)}{rs(rr - ss)}$ fiat quadratum, sumatur

$$\begin{array}{llll} & p = aa + bb & & r = aa + bb \\ & q = 2aa - bb & & s = 2bb - aa \\ \text{hinc} & p + q = 3aa & & r + s = 3bb \\ & p - q = 2bb - aa & & r - s = 2aa - bb \end{array}$$

substituendo fit formula $$\frac{pq(pp - qq)}{rs(rr - ss)} = \frac{aa}{bb}.$$

Aliter, sumi etiam potest

$$\begin{array}{llll} & p = bb - 2aa & & r = 2bb - 4aa \\ & q = 6aa & & s = bb + 4aa \\ \text{hinc} & p + q = bb + 4aa & & r + s = 3bb \\ & p - q = bb - 8aa & & r - s = bb - 8aa \end{array}$$

ac substituendo: $$\frac{pq(pp - qq)}{rs(rr - ss)} = \frac{aa}{bb}.$$

Ita sumi potest $p = 7$, $q = 6$, $r = 14$ et $s = 13$, eritque $pq(pp - qq) = 546$, $rs(rr - ss) = 4914$

hinc formula $$= \frac{546}{4914} = \frac{1}{9} = \left(\frac{1}{3}\right)^2.$$

A. m. T. I. p. 295.

59.

Evolutio generalior formulae: $$\frac{pq(pp - qq)}{rs(rr - ss)} = \square = n.$$

Hic ponatur $q = \alpha t$ et $s = \beta t$, tum vero $p = \nu r$, unde reperitur

$$\frac{tt}{rr}=\frac{\alpha\nu^3-n\beta}{\alpha^3\nu-n\beta^3}=\left(\frac{\nu}{\alpha}-z\right)^2,$$

unde oritur haec aequatio

$$0=n\beta-n\beta^3 zz+\frac{2n\beta^3\nu z}{\alpha}+\alpha^3\nu zz-2\alpha\alpha\nu\nu z-\frac{n\beta^3\nu\nu}{\alpha\alpha},$$

unde patet si $\nu=0$, fore $z=\pm\frac{1}{\beta}$; at si $z=0$, tum erit $\nu=\pm\frac{\alpha}{\beta}$, unde sequentes valores inveniuntur ope illarum formularum

$$z+z'=\frac{2\nu}{\alpha},\quad \nu+\nu'=\frac{\alpha z(2n\beta^3+\alpha^3 z)}{n\beta^3+2\alpha^3 z}.$$

At vero si sumamus $\nu=\infty$, erit

$$z=\frac{-n\beta^3}{2\alpha^4},\quad \nu=\frac{4\alpha^8-nn\beta^8}{3n\alpha^3\beta^5}.$$

Quia hic litteras α et β pro lubitu assumere licet, fortasse hinc novi valores eliciuntur, quos praecedens methodus non dat.

Si ex. gr. $\alpha=1$ et $\beta=2$, ut sit $q=t$ et $s=2t$, tum vero

$$\frac{p}{r}=\nu \quad\text{et}\quad \frac{t}{r}=\nu-z,\quad\text{erit}\quad z+z'=2\nu \quad\text{et}\quad \nu+\nu'=\frac{z(z+16)}{2z+8}.$$

Hinc sumto $\nu=0$, erit $z=\pm\frac{1}{2}$; at si $z=0$ erit $\nu=\pm\frac{1}{2}$. Praeterea si $\nu=\infty$, erit $z=-4$ et sequens $\nu=-\frac{21}{8}$, ex quibus casibus sequentes valores oriuntur

$$\nu=\infty,\quad z=-4,\quad \nu=-\frac{21}{8},\quad z=-\frac{5}{4},\quad \nu=-\frac{8}{11},\quad z=-\frac{9}{44},\quad \nu=\frac{403}{8.167},$$

tum vero

$$z=0,\quad \nu=\frac{1}{2},\quad z=1,\quad \nu=\frac{6}{5},\quad z=\frac{7}{5},\quad \nu=\frac{19}{18},$$

porro

$$z=0,\quad \nu=-\frac{1}{2},\quad z=-1,\quad \nu=2,\quad z=-3,\quad \nu=-\frac{35}{2},\quad z=-32,\quad \nu=\frac{117}{14},\quad \text{etc.}$$

$$\nu=0,\quad z=\frac{1}{2},\quad \nu=\frac{11}{12},\quad z=\frac{4}{3},\quad \nu=\frac{5}{4},\quad z=\frac{7}{6},\quad \nu=\frac{64}{93},\quad \text{etc.}$$

$$\nu=0,\quad z=-\frac{1}{2},\quad \nu=-\frac{31}{23},\quad z=-\frac{12}{7},\quad \nu=-\frac{17}{4}.$$

Casu $n=1$, valores ν et z ita se habebunt:

$$\nu=0,\quad 1,\quad 0,\quad 1,\quad \frac{8}{7},\quad 1,\quad \frac{104}{105},\quad 1$$

$$z=1,\quad 1,\quad -1,\quad 3,\quad -\frac{5}{7},\quad \frac{19}{7},\quad -\frac{11}{15},$$

tum vero

$$z=0,\quad 2,\quad -\frac{4}{5},\quad \frac{14}{5},\quad -\frac{8}{11},\quad 0,\quad -2,\quad 4,\quad -\frac{2}{3},\quad \frac{8}{3}$$

$$\nu=1,\quad \frac{3}{5},\quad 1,\quad \frac{57}{55},\quad 1,\quad -1,\quad 1,\quad \frac{5}{3},\quad 1,\quad \frac{55}{57}.$$

Tum

$$\nu=\infty,\quad 1,\quad \frac{7}{8},\quad 1,\quad \frac{105}{104},\quad 1,\quad \frac{1455}{1456},\quad 1,\quad \frac{11.19.97}{16.7.181}=\frac{20271}{20272}$$

$$z=-\frac{1}{2},\quad \frac{5}{2},\quad -\frac{3}{4},\quad \frac{11}{4},\quad -\frac{19}{26},\quad \frac{71}{28},\quad -\frac{41}{56},\quad \frac{153}{56}.$$

Ita ex casu $\nu=\frac{1455}{1456}=\frac{3.5.97}{16.7.13}$ valores pro p, q, r, s erunt: $p=3.5.97$, $q=2521$, $r=16.7.13$, $s=2521$, $p+q=8.7.71$, $p-q=2.13.41$, $r+s=41.97$, $r-s=3.5.71$, ubi factores utrinque se mutuo destruunt.

A. m. T. III. p. 143.

60.

Problema. Invenire duo triangula rectangula in numeris, quorum areae

$$A = pq(pp-qq) \quad \text{et} \quad B = rs(rr-ss)$$

datam inter se teneant rationem scil. $a:b$, ita ut sit $\frac{A}{B} = \frac{a}{b}$.

Hoc problema methodo directa frustra tractatur, unde ad solutiones particulares confugere necesse est cujusmodi sunt sequentes:

I. Sumatur $r = p$ et $s = p - q$, erit $r + s = 2p - q$ et $r - s = q$, unde fit

$$B = p(p-q)(2p-q)q$$

hincque $\frac{A}{B} = \frac{p+q}{2p-q} = \frac{a}{b}$, ergo $bp + bq = 2ap - aq$, ideoque $\frac{p}{q} = \frac{a+b}{2a-b}$, unde haec solutio nascitur

$$p = a + b \qquad r = a + b$$
$$q = 2a - b \qquad s = 2b - a.$$

II. Sit $r = 2p$ et $s = p + q$, erit $r + s = 3p + q$ et $r - s = p - q$, hinc $B = 2p(p+q)(3p+q)(p-q)$, ergo

$$\frac{A}{B} = \frac{q}{6p+2q} = \frac{a}{b}, \text{ sicque erit } bq = 6ap + 2aq, \text{ indeque } \frac{p}{q} = \frac{b-2a}{6a},$$

quocirca capiatur

$$p = b - 2a \qquad r = 2b - 4a$$
$$q = 6a \qquad s = b + 4a.$$

III. Sit $r = 2p$ et $s = p - q$, erit $r + s = 3p - q$, et $r - s = p + q$, hinc

$$\frac{A}{B} = \frac{q}{6p-2q} = \frac{a}{b}, \text{ sicque } bq = 6ap - 2aq, \text{ ergo } \frac{p}{q} = \frac{b+2a}{6a},$$

quocirca capiatur pro ista solutione

$$p = b + 2a \qquad r = 2b + 4a$$
$$q = 6a \qquad s = b - 4a.$$

IV. Sit $r = p + q$ et $s = p$, erit $r + s = 2p + q$ et $r - s = q$, hincque

$$\frac{A}{B} = \frac{p-q}{2p+q} = \frac{a}{b}, \text{ unde colligitur ob } bp - bq = 2ap + aq, \quad \frac{p}{q} = \frac{a+b}{b-2a}$$

ergo capiantur

$$p = a + b \qquad r = 2b - a$$
$$q = b - 2a \qquad s = a + b.$$

V. Sit $r = p + q$ et $s = q$, erit $r + s = p + 2q$ et $r - s = p$, unde fit

$$\frac{A}{B} = \frac{p-q}{p+2q} = \frac{a}{b}, \text{ inde } bp - bq = ap + 2aq, \text{ hinc } \frac{p}{q} = \frac{2a+b}{b-a},$$

quocirca capere debemus

$$p = 2a + b \qquad r = a + 2b$$
$$q = b - a \qquad s = b - a.$$

VI. Sit $r = p + q$ et $s = 2q$, erit $r + s = p + 3q$ et $r - s = p - q$, unde fit

$$\frac{A}{B} = \frac{p}{2p+6q} = \frac{a}{b}, \text{ unde ob } bp = 2ap + 6aq, \text{ erit } \frac{p}{q} = \frac{6a}{b-2a},$$

ideoque hic capere oportet

$$p = 6a \qquad r = 4a + b$$
$$q = b - 2a \qquad s = 2b - 4a.$$

VII. Sit $r = p - q$ et $s = q$, erit $r + s = p$ et $r - s = p - 2q$ et erit

$$\frac{A}{B} = \frac{p+q}{p-2q} = \frac{a}{b}, \text{ ideoque } bp + bq = ap - 2aq, \text{ hinc } \frac{p}{q} = \frac{b+2a}{a-b},$$

itaque ut capiatur necesse est

$$p = b + 2a \qquad r = 2b + a$$
$$q = a - b \qquad s = a - b.$$

VIII. Sit $r=p-q$ et $s=2q$, erit $r+s=p+q$ et $r-s=p-3q$, hincque

$$\frac{A}{B}=\frac{p}{2p-6q}=\frac{a}{b},\quad \text{unde ob}\quad bp=2ap-6aq\quad \text{invenitur}\quad \frac{p}{q}=\frac{6a}{2a-b},$$

ideoque capere oportet

$$p=6a \qquad r=b+4a$$
$$q=2a-b \qquad s=4a-2b.$$

IX. Sit $r=q$ et $s=p-q$, erit $r+s=p$ et $r-s=2q-p$, unde fit

$$\frac{A}{B}=\frac{p+q}{2q-p}=\frac{a}{b},\quad \text{ideoque}\quad bp+bq=2aq-ap,\quad \text{ergo}\quad \frac{p}{q}=\frac{2a-b}{b+a},$$

quocirca sumatur

$$p=2a-b \qquad r=b+a$$
$$q=b+a \qquad s=a-2b.$$

X. Sit denique $r=2q$ et $s=p-q$, erit $r+s=p+q$ et $r-s=3q-p$, unde reperitur

$$\frac{A}{B}=\frac{p}{6q-2p}=\frac{a}{b};\quad \text{hinc ob}\quad bp=6aq-2ap,\quad \text{erit}\quad \frac{p}{q}=\frac{6a}{b+2a},$$

quo notato manifestum est, ut esse debeat

$$p=6a \qquad r=2b+4a$$
$$q=b+2a \qquad s=4a-b.$$

Has jam omnes solutiones in sequenti tabella uni conspectui exponamus.

	p	q	r	s
I	$a+b$	$2a-b$	$a+b$	$2b-a$
II	$b-2a$	$6a$	$2b-4a$	$b+4a$
III	$b+2a$	$6a$	$2b+4a$	$b-4a$
IV	$a+b$	$b-2a$	$2b-a$	$a+b$
V	$2a+b$	$b-a$	$a+2b$	$b-a$
VI	$6a$	$b-2a$	$4a+b$	$2b-4a$
VII	$b+2a$	$a-b$	$2b+a$	$a-b$
VIII	$6a$	$2a-b$	$b+4a$	$4a-2b$
IX	$2a-b$	$b+a$	$b+a$	$a-2b$
X	$6a$	$b+2a$	$2b+4a$	$4a-b$

Hic numeri p et q dicuntur genitores trianguli A, et r et s genitores trianguli B, de quibus notandum, si qui eorum prodeant negativi, eos in positivos converti posse, dummodo majores litteris p et r, minores vero litteris q et s tribuantur. Quo observato aliquot exempla evolvamus:

Exemplum 1. Sit $a=1$ et $b=1$, exclusis triangulis inter se similibus, oritur haec una solutio:

$$p=6 \qquad r=5$$
$$q=1 \qquad s=2.$$

Exemplum 2. Sit $a=2$ et $b=1$ et solutiones orientur in hac tabella contentae

p	q	r	s
12	3	9	6
12	5	10	7
5	1	4	1.

Deletis autem iis casibus, qui bis occurrunt, sequens tabella exhibet solutiones diversas:

	p	q	r	s	
I	$a+b$	$2a-b$	$a+b$	$2b-a$	α
V	$2a+b$	$b-a$	$a+2b$	$b-a$	β
II	$b-2a$	$6a$	$2b-4a$	$b+4a$	γ
III	$b+2a$	$6a$	$2b+4a$	$b-4a$	δ
α	$3a$	$2b-a$	$3b$	$2a-b$	1
β	$a+2b$	$3a$	$3b$	$b+2a$	5
γ	$b+4a$	$b-8a$	$3b$	$8a-b$	2
δ	$b+8a$	$b-4a$	$3b$	$8a+b$	3

Inter has octo solutiones quaelibet habet suam sociam, quae scilicet ex numeris genitoribus $p+q$ et $p-q$ nascitur, quas igitur paribus litteris graecis insignivimus.

Exemplum 1. Sit $a=2$ et $b=1$, octo solutiones ita se habebunt:

	p	q	r	s
α	3	3	3	0
α	6	0	3	3
β	5	1	4	1
β	6	4	5	3
γ	12	3	9	6
γ	15	9	15	3
δ	12	5	10	7
δ	17	7	17	3

Exemplum 2. Sit $a=3$ et $b=2$, et oriuntur solutiones in hac tabula expressae:

	p	q	r	s
α	5	4	5	1
α	9	1	6	4
β	8	1	7	1
β	9	7	8	6
γ	{18	4	14	8}
	{9	2	7	4}
γ	11	7	11	3
δ	{18	8	16	10}
	{9	4	8	5}
δ	13	5	13	3

Exemplum 3. Si $a=1$ et $b=1$, sequentes oriuntur solutiones:

	p	q	r	s
α	2	1	2	1
α	3	1	3	1
β	3	0	3	0
β	3	3	3	3
γ	6	1	5	2
γ	7	5	7	3
δ	6	3	6	3
δ	9	3	9	3

si ambarum arearum productum AB debeat esse quadratum, tantum sumi oportet pro numeris a et b quadrata; sit igitur $a=4$ et $b=1$ eruntque solutiones

	p	q	r	s
α	7	5	5	2
α	12	2	7	3
β	9	3	6	3
	3	1	2	1
β	4	2	3	1
γ	24	7	17	14
γ	31	17	31	3
δ	24	9	18	15
	8	3	6	5
δ	11	5	11	1

A. m. T. I. p. 296—298.

Nota Editorum. Huic praecedenti fragmento in *Adversariorum* Tomo I Patris manu inscriptum est: «Omnia haec jam redacta» (*Dieses ist schon ausgeführt*); cum tamen in nullo cognitorum Euleri operum has investigationes detegere nobis contigerit, quae hanc ob rem et in recentissima editione *Commentationum arithmeticar.* desunt, esse utique potest eas in quapiam rarissima seu oblita collectione typis expressas reperiri. Hic saltem sufficiet remittere lectorem ad commentationem, cujus fragmentum supra in pag. 101 hujusce tomi Opp. posthum. reperitur, et in qua idem fere, aut simile argumentum tractatum fuisse videtur.

61.

Theorema. Haec formula $aax^4+bxxyy+ccy^4$, quadrato aequanda, semper reduci potest ad productum quatuor factoribus simplicibus constans, pariter quadrato aequandum.

Demonstratio. Formula proposita aequetur huic quadrato $\left(axx+cyy.\frac{p}{q}\right)^2$, fietque

$$bqqxx+ccqqyy=2acpqxx+ccppyy,\quad \text{unde fit}\quad \frac{xx}{yy}=\frac{cc(q+p)(q-p)}{q(2acp-bq)}=\Box.$$

Quadratum ergo esse debet $(q+p)(q-p)q(2acp-bq)$. Simili modo, si radicem illius formulae posuissemus

$$cyy+axx.\frac{r}{s},\quad \text{prodiisset}\quad \frac{xx}{yy}=\frac{s(2acr-bs)}{aa(s+r)(s-r)};$$

quadratum ergo debet esse $(s+r)(s-r)s(2acr-bs)$. Deinde, quia per utramque positionem est

$$axx+cyy.\frac{p}{q}=cyy+axx.\frac{r}{s},\quad \text{erit}\quad \frac{xx}{yy}=\frac{cs(q-p)}{aq(s-r)},$$

ideoque debet esse $cs(q-p).aq(s-r)=\Box$.

Corollarium. Pro $\frac{xx}{yy}$ nacti sumus sequentes tres valores:

$$\frac{cc(q+p)(q-p)}{q(2acp-bq)},\quad \frac{s(2acr-bs)}{aa(s+r)(s-r)},\quad \frac{cs(q-p)}{aq(s-r)},$$

ex quorum comparatione relatio inter rationes $r:s$ et $p:q$ deduci potest. Erit enim

$$r:s=\left(1+\frac{b}{ac}\right)q-p:q+p;\quad \text{vel erit etiam}\quad p:q=\left(1+\frac{b}{ac}\right)s-r:s+r.$$

A. m. T. III. p. 136.

62.

Problema. Invenire quatuor quadrata aa, bb, cc, dd, ut haec fractio fiat quadratum $\frac{aacc - bbdd}{aadd - bbcc}$.

Solutio duplex dari potest: Pro priore ponatur $c = ab$, $d = aa - 2bb$, eritque

$$ac + bd = 2b\,(aa - bb), \quad ac - bd = 2b^3, \quad ad + bc = a\,(aa - bb), \quad ad - bc = a\,(aa - 3bb),$$

ergo fieri debet $\frac{4b^4}{aa\,(aa - 3bb)} = \square$, sive tantum $aa - 3bb = \square$. Pro altera solutione fiat $a = cc + 2dd$, b … tum enim fiet $ac + bd = c\,(cc + 3dd)$, $ac - bd = c\,(cc + dd)$, $ad + bc = 2d\,(cc + dd)$, $ad - bc = 2d^3$, ergo fieri deb… $\frac{cc\,(cc + 3dd)}{4d^4} = \square$, sive tantum $cc + 3dd = \square$.

A. m. T. III. p. 150.

63.

Problema. Ad quadratum reducere hanc formulam $\frac{aabb - ccdd}{aacc - bbdd}$.

Solutio. Ponatur $b = ad$, erit formula $\frac{dd\,(a^4 - cc)}{aa\,(cc - d^4)} = \frac{a^4 - cc}{cc - d^4}$. Ponatur porro $c = aa - 2dd$, erit formula

$$\frac{4aadd - 4d^4}{a^4 - 4aadd + 3d^4},$$ quae demto quadrato in numeratore fit $$\frac{aa - dd}{a^4 - 4aadd + 3d^4} = \frac{1}{aa - 3dd}.$$

Sumatur $a = pp + 3qq$ et $d = 2pq$, erit forma $\frac{1}{(pp - 3qq)^2}$, hincque porro prodit

$$c = p^4 - 2ppqq + 9q^4 \quad \text{et} \quad b = 2pq\,(pp + 3qq).$$

Hic quaelibet positio solutionem suppeditat praecedentis problematis (*).

Sit $p = 1$ et $q = 1$, erit $a = 4$, $b = 8$, $c = 8$, $d = 2$.

Sit $p = 2$ et $q = 1$, erit $a = 7$, $b = 28$, $c = 17$, $d = 4$,

unde oritur solutio supra data problematis praecedentis.

Sit $p = 1$ et $q = 2$, erit $a = 13$, $b = 52$, $c = 137$, $d = 4$.

Sit $p = 3$ et $q = 1$, erit $a = 12$, $b = 72$, $c = 72$, $d = 6$.

Sit $p = 2$ et $q = 3$, erit $a = 31$, $b = 372$, $c = 673$, $d = 12$.

Sequenti autem modo praecedens problema ad praesens reducitur: Cum esse debeat $A^4 - B^4 = C^4 - D^4$, ponatur $A + B = \alpha x$ et $A - B = \beta y$, tum vero $C + D = \gamma x$, $C - D = \delta y$,

$$\text{fiet} \quad \alpha\beta\,(\alpha\alpha xx + \beta\beta yy) = \gamma\delta\,(\gamma\gamma xx + \delta\delta yy).$$

Hinc oritur $\frac{xx}{yy} = \frac{\gamma\delta^3 - \alpha\beta^3}{\alpha^3\beta - \gamma^3\delta}$, unde haud difficulter superior derivatur.

A. m. T. III, p. 161. 162.

(*) Resolutio hujus aequationis $A^4 - B^4 = C^4 - D^4$. *Comment. arithm* T. I p. 473.

64.

Theorema. Si fuerit $X = (a - bx)^2 (p - qx)^2 + (c - dx)^2 (r - sx)^2 - nn\,(a - bx)^2 (c - dx)^2$, statim sex valores habentur, quibus X fit quadratum.

Primo enim fiet $X = (a - bx)^2 (p - qx)^2$, si fuerit $c - dx = 0$, ideoque $x = \frac{c}{d}$, et si fuerit

$$(r - sx)^2 - nn\,(a - bx)^2 = 0, \quad \text{hoc est} \quad r - sx = \pm n\,(a - bx).$$

Simili modo fiet $X = (c - dx)^2 (r - sx)^2$ faciendo $a - bx = 0$, seu $x = \frac{a}{b}$; tum vero si $p - qx = \pm n\,(c - dx)$.

A. m. T. III. p. 166.

65.

Theorema. Ut in quadrilatero, circulo inscripto, quatuor latera a, b, c, d cum ambabus diagonalibus x et y numeris rationalibus exprimantur, necesse est, ut hoc productum $(ab+cd)(ac+bd)(ad+bc)$ reddatur quadratum. Quod fiet sumtis quinque numeris pro lubitu f, g, h, p, q, si capiatur

$$a=fgh(qq-pp),\quad b=g(fp+gq)^2-hhqq,\quad c=2fghpq+h(ff+gg-hh)qq,\quad d=f(gp+fq)^2-hhqq,$$

tum enim erit

$$x=f(fg(pp+qq)+(ff+gg-hh)pq)$$
$$y=g(fg(pp+qq)+(ff+gg-hh)pq).$$

Sin autem insuper requiratur, ut etiam area quadrilateri fiat rationalis, tum hanc formulam quadratum esse oportet

$$(a+b+c-d)(a+b+d-c)(a+c+d-b)(b+c+d-a),$$

quod autem per illas formulas nullo modo effici potest. At vero sequens problema generaliter resolvi potest:

Dato circulo polygonum quotcunque laterum inscribere, cujus omnia latera una cum omnibus diagonalibus, atque adeo area numeris rationalibus exprimantur.

Solutio. (Fig. 1.) Posito radio circuli $=1$ sint arcus $AB=2A$, $BC=2B$, $CD=2C$, etc., eritque latus

$$AB=2\sin A,\quad BC=2\sin B,\quad CD=2\sin C,\quad \text{etc.}$$

Tantum ergo opus est, ut horum angulorum sinus sint rationales, simulque etiam cosinus, ut etiam diagonales fiant rationales, si quidem est

$$AC=2\sin(A+B)=2\sin A\cos B+2\cos A\sin B.$$

At vero si fuerit $\sin A=\frac{2ab}{aa+bb}$, erit $\cos A=\frac{aa-bb}{aa+bb}$; tales igitur formulae pro sinibus et cosinibus accipiantur, hocque modo non solum omnia latera, sed etiam diagonales fient rationales atque adeo area, cum posito centro O sit area $\Delta AOB=\sin A\cos A$, quod de omnibus reliquis valet. Possunt enim singuli hi anguli $2A$, $2B$, etc. usque ad ultimum pro lubitu assumi, ultimi vero sinus erit sinus summae reliquorum, et cosinus $=-$ cosinui summae reliquorum.

A. m. T. III. p. 159. 160.

66.

(*Lexell.*)

Theorema. Si a fuerit numerus quicunque non quadratus, et b et c numeri quicunque ad illum primi, tum ista formula

$$a(bbx^4+aaccy^4)$$

nunquam esse potest quadratum.

Demonstratio. Hic assumi potest numerum a etiam per nullum quadratum esse divisibilem, si enim esset $a=\alpha ff$, quadratum esse deberet $\alpha(bbx^4+\alpha\alpha ccf^4y^4)$, ubi si loco fy scribatur y, habetur formula prior, quin etiam numeri x et y sunt primi inter se. Quoniam igitur a est factor nostrae formae, necesse est, ut alter factor $bbx^4+aaccy^4$ etiam habeat factorem a, sed pars posterior jam habet factorem a, ergo pars prior erit divisibilis per a, ex quo x factorem habebit a, ideoque y non erit divisibile per a. Ponatur ergo $x=az$, atque nunc haec forma quadratum esse debebit

$$a(bba^4z^4+aaccy^4),\quad \text{seu}\quad a(bbaaz^4+ccy^4).$$

Quod ob eandem rationem fieri nequit, nisi y esset divisibile per a, qui casus cum jam sit exclusus, formula nostra nullo modo quadratum esse poterit.

A. m. T. I. p. 51.

67.

Varia conamina aequationis $a^\lambda+b^\lambda=c^\lambda$ impossibilitatem casu $\lambda>2$ demonstrandi.

1. (*Lexell.*)

Theorema. Non dantur tres numeri x, y, z, ut fiat $xxy+xzz+yyz=0$.

Sumi potest numeros x, y, z communem divisorem non habere; si enim haberent, per divisionem ex hac aequatione tolleretur; interim tamen bini communem divisorem habere debent. Hinc ponatur a maximus communis divisor numerorum x et y, b ipsorum x et z, et c ipsorum y et z, atque tum bini horum a, b, c erunt inter se primi. Ponatur igitur $x = ap$, $y = aq$, eruntque p et q primi inter se. Deinde sit $x = br$ et $z = bs$, denique $y = ct$ et $z = cu$, ita ut sit $x = ap = br$, $y = aq = ct$, $z = bs = cu$, quibus valoribus substitutis formula nostra est:

$$abcprt + abcpsu + abcqts = 0, \quad \text{sive} \quad prt + psu + qst = 0.$$

Cum autem sit $ap = br$, sive $\frac{p}{r} = \frac{b}{a}$, erit $p = lb$, $r = la$; deinde $\frac{q}{t} = \frac{c}{a}$, unde $q = mc$ et $t = ma$, et $\frac{s}{u} = \frac{c}{b}$, $s = nc$, $u = nb$, ita ut sit $x = lab$, $y = mac$, $z = nbc$; ubi notandum numeros mc, lb esse inter se primos, nec non lb et nc, et ma et nc. Aequatio autem nostra hanc habebit formam:

$$llmaab + nnlbbc + mmncca = 0.$$

Hinc ergo lb divisor esse deberet membri $mmncca$, quod ob conditiones memoratas esse nequit.

2. (*J. A. Euler.*)

NB. Haec demonstratio non succedit. Caeterum hoc theorema huc redit, ut demonstretur esse non posse $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 0$. Hoc autem sequenti modo demonstrari posse videtur:

Posito $z = \frac{xy}{v}$, et nostra aequatio fiet $\frac{x}{y} + \frac{v}{x} + \frac{y}{v} = 0$, quae forma similis est propositae. Cum numeri x, y, z sint inaequales, sit z maximus, y medius et x minimus, sive negative, sive positive. Jam cum sit $z = \frac{xy}{v}$, manifestum est fore $v < x$. Unde patet, si terni numeri z, y et x satisfecerint, tum etiam hos y, x et v satisfacturos, quorum y jam erit maximus, x medius et v minimus. Ponatur jam $y = \frac{xv}{u}$, eritque $u < v$, quare etiam hi tres numeri x, v et u satisfacerent. Si porro ponatur $x = \frac{uv}{t}$, erit $t < u$, atque etiam hi v, u et t satisfacerent. Hocque modo continuo ad numeros minores perveniretur; quare cum in minimis hujusmodi numeri non dentur, etiam in maximis tales non dantur. Manifestum vero est hos numeros semper fore integros.

Corollarium. Hoc modo demonstrari posset fieri non posse $\frac{x}{y} + \frac{y}{x} = 0$. Si enim $y > x$ et ponatur $y = \frac{xx}{v}$, erit $v < x$, et tum prodit $\frac{v}{x} + \frac{x}{v} = 0$, quae posito denuo $x = \frac{vv}{u}$ daret $u < v$ et $\frac{v}{u} + \frac{u}{v} = 0$; quo pacto iterum ad numeros continuo minores perveniretur.

Eodem modo etiam demonstrari potest esse non posse $\frac{v}{x} + \frac{x}{y} + \frac{y}{z} + \frac{z}{v} = 0$. Posito enim $z = \frac{vy}{u}$, (ubi si z fuerit numerus maximus et v minimus, $u <$ erit v) similis aequatio prodit scilicet $\frac{u}{v} + \frac{v}{x} + \frac{x}{y} + \frac{y}{u} = 0$ ex numeris minoribus formata; hocque modo continuo minores invenire liceret.

Corollarium 2. Cum igitur aequatio $xxy + xzz + yyz = 0$ sit impossibilis, inde vero prodeat

$$z = \frac{-yy \pm \sqrt{(y^4 - 4x^3y)}}{2x},$$

sequitur formulam $y^4 - 4x^3y$ quadratum nunquam esse posse.

Corollarium 3. Ex aequatione supra allata pro quatuor numeris sequitur

$$vvyz + xxzv + yyxv + zzxy = 0$$

hinc

$$vv = \frac{-(xxz + yyx)v - zzxy}{yz}$$

et

$$v = \frac{-x(xz + yy) \pm \sqrt{(xx(xz + yy)^2 - 4z^3yyx)}}{2yz},$$

unde haec formula $xx(xz + yy)^2 - 4z^3yyx$ nunquam quadratum fieri potest.

Sit verbi gratia $z = x$ et haec formula fiet

$$xx(xx+yy)^2 - 4x^4yy, \quad \text{vel} \quad (xx+yy)^2 - 4xxyy$$

NB. Verum nostrum theorema in casu quatuor numerorum non amplius locum habet, quia utique in minimis numeris casus dantur possibiles: veluti si fuerit $z = x$ et $y = -v$. Quod ergo de quatuor numeris hic dictum est, neutiquam valet.

Theorema. Neque summa neque differentia duorum cuborum potest esse cubus.

Demonstratio I. Si p, q et r denotent numeros integros, sive positivos sive negativos, demonstrandum est hanc aequationem nullo modo subsistere posse:

$$p^3 + q^3 + r^3 = 0.$$

Tum enim dividendo per pqr foret

$$\frac{pp}{qr} + \frac{qq}{pr} + \frac{rr}{pq} = 0, \quad \text{ideoque etiam} \quad \frac{ppq}{qqr} + \frac{qqr}{prr} + \frac{prr}{ppq} = 0$$

atque hinc etiam si ponamus $ppq = x$, $qqr = y$ et $rrp = z$, foret

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 0.$$

Hoc autem nunquam fieri posse ante est demonstratum.

Demonstratio II. Demonstrabo hic hanc formulam $ab(a \pm b)$ cubum esse non posse. Primo enim numeri a et b non solum integri sed etiam primi inter se assumi possunt. Quare cum hi tres factores a, b et $a \pm b$ sint inter se primi, unusquisque foret cubus, unde posito $a = x^3$, $b = y^3$, foret $x^3 \pm y^3 =$ cubo. Quod autem formula $ab(a \pm b)$ cubus esse nequit, ita ostendo: Si esset cubus, ejus radix statui posset $\frac{m(a \pm b)}{n}$. Tum ergo foret $\quad ab(a \pm b) = \frac{m^3(a \pm b)^3}{n^3}$, vel $n^3 ab = m^3 (a \pm b)^2 = m^3 (aa \pm 2ab + bb)$.

Hoc enim si esset, numeri a et b forent inaequales. Sit igitur a major et b minor, et ponatur $a = \frac{bb}{c}$, eritque

$c < b$, tum autem foret $\frac{n^3 b^3}{c} = m^3 \left(\frac{b^4}{cc} \pm \frac{2b^3}{c} + bb\right)$, sive $n^3 bc = m^3 (bb \pm 2bc + cc)$, ubi $b > c$,

ergo si porro ponatur $b = \frac{cc}{d}$, erit $c > d$, hincque iterum foret $n^3 cd = m^3 (cc \pm 2cd + dd)$; hocque modo continuo ad numeros minores perveniretur. Unde quia res in minimis numeris non succedit, etiam in maximis succedere non posset.

NB. Hic vero vitium ingens inest, quoniam ob numeros a et b inter se primos, c non est integer, neque etiam sequentes d, e, etc. Quocirca ex parvitate horum numerorum nihil concludi potest. Interim tamen etiam ne prior demonstratio valet, etsi enim omnes tres numeri non habent communem divisorem, tamen bini quivis necessario communem habent factorem. Quamobrem ex aequalitate $\frac{y}{z} = \frac{-xx - zy}{xy}$ concludi nequit, esse z partem ipsius xy, quia fortasse fractio $\frac{y}{z}$ ad minores terminos reduci potest, cujus demum denominator divisor esse debet ipsius xy.

A. m. T. I. p. 51—54.

3. (*Lexell.*)

Theorema Fermatii, quo neque summa cuborum potest esse cubus, neque summa duarum potestatum quintarum potestas quinta esse potest, nec in genere summa duarum potestatum altiorum similis potestas altior, facile ita transformari potest, ut certae formulae quadrata esse nequeant. Si enim $a^5 + b^5 = c^5$, ponatur

$$x + y = a^5 \quad \text{et} \quad x - y = b^5, \quad \text{foretque} \quad 2x = a^5 + b^5 = c^5 \quad \text{et} \quad xx - yy = a^5 b^5 \quad \text{et} \quad 4xx = c^{10};$$

hinc igitur foret $\frac{xx - yy}{4xx} = \frac{a^5 b^5}{c^{10}}$, ideoque potestas quinta, pro qua scribatur

$$\frac{z^5}{x^5}, \text{ seu } \frac{xz^5}{x^6}, \text{ ita ut foret } \frac{xx-yy}{4xx}=\frac{xz^5}{x^6};$$

multiplicetur per $4x^6$, fietque

$$x^6-x^4yy=4xz^5, \quad \text{sive} \quad x^6-4xz^5=x^4yy=\square.$$

Quare si demonstrari posset formulam x^6-4xy^5 quadratum esse non posse, simul demonstratum est formulam a^5+b^5 potestatem quintam esse non posse. Si enim esset x^6-4xy^5 quadratum, ob factores $x(x^5-4y^5)$ inter se primos, uterque quadratum esse deberet. Sit igitur $x=pp$, et alter factor $p^{10}-4y^5$ deberet esse quadratum, puta qq; foret ergo

$$p^{10}-qq=4y^5=4r^5s^5=(p^5+q)(p^5-q), \quad \text{ideoque} \quad p^5+q=2r^5 \quad \text{et} \quad p^5-q=2s^5,$$

unde addendo foret

$$2p^5=2r^5+2s^5, \quad \text{sive} \quad p^5=r^5+s^5.$$

Simili modo formula $a^3+b^3=c^3$ transformabitur in hanc aequivalentem $x^4-4xy^3=\square$. Hoc postremum theorema etiam hoc modo repraesentari potest, ut nunquam fieri queat

$$x^3+(x+a)^3=(x+b)^3, \quad \text{ubi manifesto} \quad b>a.$$

Foret ergo

$$x^3=(x+b)^3-(x+a)^3=3(b-a)xx+3(bb-aa)x+b^3-a^3,$$

demonstrandum ergo est hanc aequationem nunquam habere radicem rationalem. Ad hoc observetur, cum posterius membrum factorem habeat $b-a$, etiam x^3 talem factorem habere debet, et perspicuum est $b-a$ vel esse cubum, vel noncuplum cubi.

Sit primo $b-a=f^3$, et erit $x^3=3f^3xx+3f^3(b+a)x+f^3(bb+ab+aa)$.

Ponatur ergo $x=fy$, eritque $y^3=3ffyy+3(b+a)fy+bb+ab+aa$, ideoque y debet esse factor formulae $bb+ab+aa$.

Sit secundo $b-a=9f^3$, et ultimum membrum fieret (ob $b=9f^3+a$)

$$9^3f^9+3.9^2af^6+3.9aaf^3=27f^3(27f^6+9af^3+aa),$$

unde fit

$$x^3=27f^3xx+27f^3(b+a)x+27f^3(27f^6+9af^3+aa).$$

Ponatur $x=3fy$,* erit

$$y^3=9ffyy+3fy(b+a)+27f^6+9af^3+aa.$$

Pro utroque casu limites assignari possunt; pro priore enim manifesto est $y>3ff$, et pro altero $y>9ff$; qui limites sunt nimis parvi; nimis magni autem hoc modo reperientur: Consideretur aequatio in genere

$$y^3=\alpha yy+\beta y+\gamma,$$

ubi α, β, γ sint positivi, ac primo erit $y>\alpha$; deinde cum sit $y=\alpha+\frac{\beta}{y}+\frac{\gamma}{yy}$, si in membro posteriori loco y scribatur α, hoc membrum fit nimis magnum, erit ergo $y<\alpha+\frac{\beta}{\alpha}+\frac{\gamma}{\alpha\alpha}$; ponatur hic limes $=\lambda$, sit $y<\lambda$, eritque vicissim $y=\alpha+\frac{\beta}{\lambda}+\frac{\gamma}{\lambda\lambda}$.

Exemplum. Sit pro casu priori $f=1$ et $a=1$, erit $b=2$ et $x=y$, hinc $y^3=3yy+9y+7$, unde statim $y>3$, hinc $y<7$, hinc $y>4\frac{3}{7}$, $y<5\frac{12}{31}$, radix ergo rationalis deberet esse 5, quae cum non sit divisor ultimi termini . . . . . .

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4. (*W. L. Krafft.*)

Problema. Invenire numeros x et y inter se primos, ut formula x^3+ny^3 fiat numerus quadratus.

Solutio. Si hi numeri non essent primi inter se, quaestio foret levissima; posito enim $x=pr$ et $y=qr$, formula nostra prodit $r^3(p^3+nq^3)$, quae aequetur quadrato r^4ss, ita ut hinc statim fiat

$$r=\frac{p^3+nq^3}{ss}, \quad \text{unde fit} \quad x=\frac{p^4+npq^3}{ss} \quad \text{et} \quad y=\frac{p^3q+nq^4}{ss}.$$

quod uti est facillimum, ita casus, quo x et y sint inter se primi, maxime est difficilis. Formulae istius factor simplex est $x+y\sqrt[3]{n}$, et si α denotet unam radicem cubicam unitatis, ita ut sit $\alpha^3=1$, quam constat $\frac{-1+\sqrt{-3}}{2}$, tum tertia radix cubica erit $\alpha\alpha$. Hinc formulae nostrae x^3+ny^3 alius factor simplex erit $x+\alpha y\sqrt[3]{n}$, ac tertius $x+\alpha\alpha y\sqrt[3]{n}$, ita ut formula nostra futura sit productum horum trium factorum

$$(x+y\sqrt[3]{n})(x+\alpha y\sqrt[3]{n})(x+\alpha\alpha y\sqrt[3]{n}),$$

qui singuli factores reddantur quadrata, hoc modo, quo statim patet, si unus fuerit quadratus, etiam reliquos fore quadratos.

Posito enim $x+y\sqrt[3]{n}=(p+q\sqrt[3]{n}+r\sqrt[3]{nn})^2$, per naturam rei fiet

$$x+\alpha y\sqrt[3]{n}=(p+\alpha q\sqrt[3]{n}+\alpha\alpha r\sqrt[3]{nn})^2 \quad \text{et} \quad x+\alpha\alpha y\sqrt[3]{n}=(p+\alpha\alpha q\sqrt[3]{n}+\alpha r\sqrt[3]{nn})^2.$$

Productum ergo, quod est x^3+ny^3, etiam erit quadratum, et quidem rationale, quippe cujus radix erit

$$p^3+nq^3+nnr^3-3npqr.$$

Tantum igitur opus est, primam illam positionem supra datam evolvi, ex qua consequimur

$$\begin{aligned} x+y\sqrt[3]{n} &= pp+2pq\sqrt[3]{n}+2pr\sqrt[3]{nn} \\ &+2nqr+nrr\sqrt[3]{n}+qq\sqrt[3]{nn}, \end{aligned}$$

unde statim sequitur fore $\quad x=pp+2nqr, \quad y=2pq+nrr.$

Praeterea vero esse oportet $2pr+qq=0$, unde $r=-\frac{qq}{2p}$. Sumatur ergo $p=2aa$ et $r=-bb$, fietque $q=2ab$, consequenter valores satisfacientes sunt

$$x=4a(a^3-nb^3) \quad \text{et} \quad y=b(8a^3-nb^3).$$

Aliter. Si ponatur $p=aa$ et $r=-2bb$, erit $q=2ab$ et $x=a(a^3-8nb^3)$ et $y=4b(a^3+nb^3)$, ubi a et b pro lubitu assumere licet.

Exemplum. Quaerantur duo cubi inter se primi x^3 et y^3, quorum summa fiat quadratum, cujusmodi quidem statim sunt obvii 1 et 8. Hic ob $n=1$, erit

$$x=a(a^3-8b^3) \quad \text{et} \quad y=4b(a^3+b^3).$$

Sit $a=3$, $b=1$, erit $x=57$, $y=112$, quorum cuborum summa fit quadratum, cujus radix $=1261$.

Sit $a=2$, $b=-1$, erit $x=32$, $y=-28$, sive $x=8$, $y=-7$.

In hac tamen solutione, etsi generalis videtur, casus quo $x=1$ et $y=2$ non continetur, cujus ratio sine dubio in eo est quaerenda, quod hoc casu numerus n ipse sit cubus, ideoque irrationalitas evanescat. Quod clarius patebit ex solutione magis directa, nam ut x^3+y^3 fiat quadratum, ponatur $x+y=p$ et $x-y=q$, ut sit $x=\frac{p+q}{2}$ et $y=\frac{p-q}{2}$, unde fit

$$x^3+y^3=\frac{p^3+3pqq}{4}=\frac{(pp+3qq)p}{4},$$

quae formula ut reddatur quadrata, debet esse $p(pp+3qq)$ quadratum, unde si hi duo factores sint inter se primi, uterque factor quadratum esse debet. Posterius vero tantum locum habet, si p divisibile sit per 3. Hinc duos casus evolvi convenit.

I. Sint hi factores inter se primi, atque ut $pp+3qq$ fiat quadratum, vidimus sumi debere $p=ff-3gg$ et $q=2fg$; at vero ut et p fiat quadratum, capiatur $f=hh+3kk$ et $g=2hk$. Ergo solutio hinc nata erit

$$x=\frac{h^4+4h^3k-6hhkk+12hk^3+9k^4}{2}, \quad y=\frac{h^4-4h^3k-6hhkk-12hk^3+9k^4}{2}.$$

II. Sit $p=3r$ et formula nostra erit $r(3rr+qq)$; fiat igitur $q=ff-3gg$ et $r=2fg$; fiet

$$3rr+qq=(ff+3gg)^2.$$

Jam ut et r fiat quadratum, sumatur $f=2hh$ et $g=kk$, ut fiat $r=4hhkk$, ideoque $p=12hhkk$ et $q=4h^4-3k^4$. At vero etiam alia solutio pro hoc casu locum habet, ponendo $q=\frac{3gg-ff}{2}$ et $r=fg$; tum vero etiam $f=hh$ et $g=kk$. Si $h=1=k$, erit $f=1$ et $g=1$, hinc $q=1$ et $r=1$, ergo $p=3$, $x=2$ et $y=1$, qui est casus cognitus.

In genere autem $q=\frac{3k^4-h^4}{2}$ et $r=hhkk$, $p=3hhkk$,

$$x=\frac{3k^4+6hhkk-h^4}{4}, \quad y=\frac{6hhkk-3k^4+h^4}{4}.$$

Supra observavimus, ut foret $a^3+b^3=c^3$, fore quoque $x^4-4xz^3=\square$ et vicissim. Cum ergo quadratum esse debeat $x(x^3-4z^3)$, unde uterque factor debet esse quadratum. Reddatur primo posterior x^3-4z^3 quadratum, pro quo casu est $n=-4$, unde colligitur $x=a(a^3+32b^3)$ et $y=4b(a^3-4b^3)$. Ut ergo et x fiat quadratum, debet esse $a(a^3+32b^3)=\square$, ergo uterque factor deberet esse $\square$, ideoque $a^3+32b^3=\square$. Loco $2b$ scribamus $-c$ et formula erit $a(a^3-4c^3)$; quocirca si in maximis numeris formula $x(x^3-4z^3)$ esset $\square$, hoc modo ad aliam similem formulam deveniretur $a(a^3-4c^3)$ etiam quadratum, ubi numeri a et c manifesto multo forent minores, quam illi x et y. Deinde ex his a et c simili modo deduceremur ad alios multo minores, puta d et e, ita ut similis forma $d(d^3-4e^3)$ esset $\square$ et ita porro; unde certe proditura esset in minimis numeris talis forma quadratum; quare cum in minimis numeris talis forma non detur, ne in maximis quidem talis existit. Casus autem obvius, quo $e=0$, hic nullam facit exceptionem; ad eum enim perveniri non potest, nisi jam in prima forma fuerit $z=0$, qui casus ne in quaestionem quidem cadit.

Problema. Reddere formulam x^3+ny^3 cubum.

Solutio. Statim manifestum est, ad hoc statui oportere

$$x+y\sqrt[3]{n}=(p+q\sqrt[3]{n}+r\sqrt[3]{nn})^3;$$

tum enim ipsius formulae x^3+ny^3 radix cubica erit $p^3+nq^3+nnr^3-3npqr$. Facta autem evolutione reperietur

$$x+y\sqrt[3]{n}=\begin{matrix}p^3\\+6npqr\\+nq^3\\+nnr^3\end{matrix}\;\left.\begin{matrix}+3ppq\\+3nprr\\+3nqqr\end{matrix}\right\}\sqrt[3]{n}\;\left.\begin{matrix}+3ppr\\+3pqq\\+3nqrr\end{matrix}\right\}\sqrt[3]{nn}$$

hinc

$$x=p^3+6npqr+nq^3+nnr^3$$
$$y=3ppq+3nprr+3nqqr$$
$$0=3ppr+3pqq+3nqrr,$$

ex qua aequatione fit

$$p=\frac{-qq\pm\sqrt{(q^4-4nqr^3)}}{2r},$$

unde quadratum esse deberet formula $q(q^3-4nr^3)$, ideoque uterque factor seorsim. Sit ergo $q=ss$ debetque esse $s^6-4nr^3=\square=tt$, sive $s^6-tt=4nr^3=4nf^3g^3$. Fiat ergo $s^3+t=2f^3$ et $s^3-t=2ng^3$, unde fit $s^3=f^3+ng^3$, quae formula similis est ipsi propositae, ubi litterae f et g sine dubio multo sunt minores quam x et y. Quare si in minimis numeris talis casus non datur, ne in maximis quidem dabitur.

p. 99—103.

5. (*Lexell.*)

Ad Theorema Fermatii supra memoratum, quo aequalitas $a^\lambda+b^\lambda=c^\lambda$ locum habere nequit praeter casus $\lambda=1$ et $\lambda=2$, reductio ibi tradita hoc modo facillime obtinetur: Si esset $c^\lambda=a^\lambda+b^\lambda$, foret

$$c^{2\lambda}-4a^\lambda b^\lambda=(a^\lambda-b^\lambda)^2=\square,$$

ideoque pro ab posito d, talis formula $c^{2\lambda}-4d^\lambda$ deberet esse quadratum, cujus igitur impossibilitatem ostendi oportet, praeter casus $\lambda=1$ et $\lambda=2$.

Corollarium. Simili modo ex formula $b^\lambda=c^\lambda-a^\lambda$ deducitur

$$b^{2\lambda}+4c^\lambda a^\lambda=(c^\lambda+a^\lambda)^2=\square,$$

quin etiam $a^{2\lambda}+4c^\lambda b^\lambda=\square$, quae ergo formulae etiam sunt impossibiles. Demonstratio pro casu saltem $\lambda=3$ ita tentetur: Cum sit $a^3+b^3=c^3$, erit $(a+b)(aa-ab+bb)=c^3$, quos factores ut primos inter se spectemus, cum casus, quo divisorem communem habent 3, nullam novam difficultatem implicet. Sit igitur uterque cubus $a+b=p^3$ et $aa-ab+bb=P^3$, fietque $c=Pp$; tum vero erit $p^6-P^3=3ab$, deinde ob $b^3=c^3-a^3=(c-a)(cc+ac+aa)$, fiat iterum $\quad c-a=q^3$ et $cc+ac+aa=Q^3$, fietque $b=Qq$ et $Q^3-q^6=3ac$,

denique ob $a^3=c^3-b^3=(c-b)(cc+bc+bb)$ sit $c-b=r^3$ et $cc+bc+bb=R^3$, unde $a=Rr$ et $R^3-r^6=3bc$. Introductis igitur litteris p, q, r et P, Q, R, ob $c=Pp$, $b=Qq$ et $a=Rr$, sequentes conditiones sunt adimplendae:

I. $p^3=Rr+Qq$, II. $q^3=Pp-Rr$, III. $r^3=Pp-Qq$,

IV. $P^3=RRrr-RrQq+QQqq$, V. $Q^3=PPpp+PpRr+RRrr$, VI. $R^3=PPpp+PpQq+QQqq$,

quibus praeterea adjungere licet

VII. $p^6-P^3=3QqRr$, VIII. $Q^3-q^6=3PpRr$, IX. $R^3-r^6=3PpQq$.

Denique etiam notasse juvabit $Q^3-P^3=(c-b)(a+c-b)=(Pp+Qq)(Rr+Pp-Qq)$. Totum ergo negotium huc redit, ut in his conditionibus contradictio detegatur.

p. 113.

6.

Theorema demonstrandum. Non dantur plures quantitates rationales veluti A, B, C, D, etc. quarum summa $A+B+C+D$ etc. per productum $ABCD$ etc. multiplicata producat unitatem. Sive si hoc signum $\boxminus$ denotet impossibilitatem aequalitatis, theorema hoc complectitur sequentes formas:

I. $AB(A+B)\boxminus 1$, II. $ABC(A+B+C)\boxminus 1$, III. $ABCD(A+B+C+D)\boxminus 1$, etc.

Hae formae etiam ita exhiberi possunt

I. $A+B\boxminus\frac{1}{AB}$, II. $A+B+C\boxminus\frac{1}{ABC}$, III. $A+B+C+D\boxminus\frac{1}{ABCD}$, etc.

Hinc si postremae formulae fractae referantur littera O, sequentes formae sunt notatu dignae:

I. $A+B\boxminus O$ existente $ABO=1$, II. $A+B+C\boxminus O$ existente $ABCO=1$,

III. $A+B+C+D\boxminus O$ existente $ABCDO=1$, etc.

Porro quia litterae A, B, C, O sunt fractiones, si ponamus

$$A=\frac{a}{b},\quad B=\frac{b}{c},\quad C=\frac{c}{d},\quad \text{etc.}$$

sequentes habebuntur relationes impossibiles:

I. $\frac{a}{b}+\frac{b}{c}\boxminus\frac{c}{a}$, II. $\frac{a}{b}+\frac{b}{c}+\frac{c}{d}\boxminus\frac{d}{a}$, III. $\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}\boxminus\frac{e}{a}$.

At si hujus theorematis demonstratio haberetur, inde facile sequentia theoremata demonstrari possent:

Theorema I. Summa duorum cuborum esse nequit cubus, sive $p^3+q^3\boxminus r^3$.

Demonstratio. Facta divisione per pqr, ut habeatur

$$\frac{pp}{qr}+\frac{qq}{pr}=\frac{rr}{pq},$$

ubi si faciamus $\frac{pp}{qr}=A$, $\frac{qq}{pr}=B$, erit $AB=\frac{pq}{rr}=\frac{1}{O}$, sive $A+B=\frac{1}{AB}$, quod cum impossibile sit, hujus theorematis veritas est evicta.

Theorema II. Summa trium biquadratorum biquadratum esse nequit, sive $p^4+q^4+r^4=s^4$.

Demonstratio. Facta divisione per $pqrs$ habebitur

$$\frac{p^3}{qrs}+\frac{q^3}{prs}+\frac{r^3}{pqs}=\frac{s^3}{pqr}$$

$$A+B+C=O$$

ubi manifesto est $ABCO=1$. Quod cum sit impossibile, etiam hoc theorema est demonstratum.

Theorema III. Non dantur quatuor potestates quintae, quarum summa sit potestas quinta, sive

$$p^5+q^5+r^5+s^5=t^5.$$

Demonstratio. Facta divisione per productum $pqrst$ et comparatione cum superioribus litteris A, B, C, D, O instituta, hoc modo

$$\frac{p^4}{qrst}+\frac{q^4}{prst}+\frac{r^4}{pqst}+\frac{s^4}{pqrt}=\frac{t^4}{pqrs}$$

$$A+B+C+D=O$$

hic statim apparet esse $ABCDO=1$. Sicque etiam hoc theorema est demonstratum.

Theorema generale. Existente n exponente potestatis, non dantur $n-1$ tales potestates, quarum summa esset similis potestas.

Corollarium 1. Hinc multo minus $n-2$, vel $n-3$, vel $n-4$, etc. tales potestates dantur, quarum summa esset similis potestas. Hoc ergo modo theorema illud Fermatii in multo majori extensione adeo esset demonstratum.

Corollarium 2. Quia potestates impares aeque negativae ac positivae esse possunt, litterae illae p, q, r, s, sive A, B, C, D utcunque ratione signorum variare poterunt, id quod hoc modo referri potest:

$$\text{I.}\quad \pm p^3\pm q^3\pm r^3=0, \qquad \text{II.}\quad \pm p^5\pm q^5\pm r^5\pm s^5\pm t^5=0 \quad \text{etc.}$$

Corollarium 3. Hoc autem nullo modo valet pro potestatibus paribus, quoniam $-p^4$ non est potestas quarta, unde hoc theorema non ad hanc formam debet extendi: $p^4+q^4-r^4=s^4$, quandoquidem statim in oculos incurrit casu $q=r$ hanc aequationem subsistere non posse, quemadmodum modo supra vidimus talem formam revera resolvi posse.

Huic fragmento manu J. A. Euleri *inscriptum:* Hujus autem falsitas infra fusius ostendetur.

pag. 115. 116.

7.

Ecce quatuor numeri, quorum tam summa quam productum unitati aequatur:

$$+\frac{4}{3},\quad +\frac{3}{2},\quad -\frac{1}{3},\quad -\frac{3}{2},$$

unde superior illa conjectura omni fundamento destituitur.

Problema. Invenire quotcunque numeros, quorum summa multiplicata per productum producat unitatem.

1. Si desiderentur duo tales numeri, ut sit $ab(a+b)=1$, ponatur $a=\alpha b$ eritque $\alpha b^3(\alpha+1)=1$, sive $b^3=\frac{1}{\alpha(\alpha+1)}$, sicque $\alpha(\alpha+1)$ debet esse cubus, quod fieri nequit.

2. Si tres desiderentur numeri $abc(a+b+c)=1$, ponatur $a=\alpha(b+c)$, ideoque

$$a+b+c=(1+\alpha)(b+c),\quad \text{ergo}\quad \alpha(\alpha+1)bc(b+c)^2=1.$$

Nunc ponatur $b=\beta c$ eritque

$$(b+c)^2=cc(1+\beta)^2,\quad \text{ideoque}\quad \alpha\beta(\alpha+1)(\beta+1)^2c^4=1,\quad \text{sive}\quad \frac{1}{c^4}=\alpha\beta(\alpha+1)(\beta+1)^2.$$

Sumatur $\alpha=\beta\beta+2\beta$, et debet esse

$$\frac{1}{c^4}=\beta\beta(\beta+2)(\beta+1)^4,\quad \text{sive}\quad \frac{1}{c^4(\beta+1)^4}=\beta\beta(\beta+2).$$

Sit $\beta=pp-2$, unde

$$\frac{1}{c^4(\beta+1)^4}=pp(pp-2)^2,\quad \text{ergo}\quad \frac{1}{cc(\beta+1)^2}=p(pp-2);$$

quod fit quadratum I. si sumatur $p=2$; tum enim erit $\frac{1}{c(\beta+1)}=2$; deinde $\beta=2$, $\alpha=8$, $c=\frac{1}{6}$, $b=\frac{1}{3}$, $a=4$. Consequenter tres numeri quaesiti 4, $\frac{1}{3}$, $\frac{1}{6}$, quorum summa est $\frac{9}{2}$ et productum $\frac{2}{9}$. Deinde $p(pp-2)$ fit quadratum sumendo $p=\frac{9}{4}$. Erit enim $p(pp-2)=\frac{9}{4}\cdot\frac{49}{16}$, hinc $\frac{1}{c(\beta+1)}=\frac{21}{8}$, porro $\beta=\frac{49}{16}$, $\alpha=\frac{49.81}{16^2}$, $c=\frac{8.16}{21.65}$, $b=\frac{7.8}{3.65}$, denique $a=\frac{49.81.8}{16^2.21}=\frac{7.27}{16.2}$, ergo tres numeri quaesiti $a=\frac{7.27}{32}$, $b=\frac{7.8}{3.65}$, $c=\frac{8.16}{21.65}$, quorum summa $\frac{65^2}{21.32}$, productum vero $\frac{21.32}{65^2}$.

Alia Solutio. Sumatur $\beta=pp-1$, ut fiat $\frac{1}{c^4p^4}=\alpha(\alpha+1)(pp-1)$, jam sumatur $\alpha=p-2$, unde $\frac{1}{c^4p^4}=(p-2)(p+1)(p-1)^2$, statuatur $(p-2)(p+1)=(p+1)^2qq$, unde $p-2=(p+1)qq$ et $p=\frac{2+qq}{1-qq}$, $p+1=\frac{3}{1-qq}$, $p-1=\frac{1+2qq}{1-qq}$, ideoque $\frac{1}{ccpp}=\frac{3q(1+2qq)}{(1-qq)^2}$. Superest ergo reddi quadratum $3q(1+2qq)$, quod manifesto fit sumto $q=\frac{1}{2}$; hinc $\frac{1}{cp}=2$, $p=3$, $\alpha=1$, $\beta=8$, $c=\frac{1}{6}$, ergo tres numeri sunt $a=\frac{3}{2}$, $b=\frac{4}{3}$, $c=\frac{1}{6}$, quorum summa est $=3$ et productum $=\frac{1}{3}$. Deinde solutionem praebet positio $q=\frac{1}{24}$.

Aliter, sumto statim $\alpha=1$, fit $\frac{1}{c^4p^4}=2\beta(\beta+1)^2$; sumatur $\beta=2pp$, fiet

$$\frac{1}{ccpp}=2p(2pp+1),\quad \text{cui satisfacit}\quad p=2.$$

3. Si desiderentur quatuor numeri, ut sit $abcd(a+b+c+d)=1$, ponatur

$$a=\alpha(b+c+d),\quad b=\beta(c+d)\quad \text{et}\quad c=\gamma d,$$

unde $b=\beta(\gamma+1)d$, $a=\alpha(\beta+1)(\gamma+1)d$ et summa omnium $(\alpha+1)(\beta+1)(\gamma+1)d$, productum vero $\alpha\beta\gamma(\beta+1)(\gamma+1)^2d^4$. Debet ergo esse

$$\alpha\beta\gamma(\alpha+1)(\beta+1)^2(\gamma+1)^3d^5=1.$$

Sumatur $\gamma=\beta$ eritque $\alpha\beta\beta(\alpha+1)(\beta+1)^5d^5=1$, ideoque $(\beta+1)^5d^5=\frac{1}{\alpha\beta\beta(\alpha+1)}$;

Sumatur porro $(\beta+1)d=\frac{kk}{\alpha(\alpha+1)}$ fietque $k^{10}=\frac{\alpha^4(\alpha+1)^4}{\beta\beta}$, $\beta=\frac{\alpha\alpha(\alpha+1)^2}{k^5}$,

ubi α et k pro arbitrio sumi possunt; tum autem habebitur $\beta=\frac{\alpha\alpha(\alpha+1)^2}{k^5}$; hinc

$$d=\frac{kk}{\alpha(\alpha+1)(\beta+1)},\quad \gamma=\frac{\alpha\alpha(\alpha+1)^2}{k^5};\quad \alpha=\text{arbitr.},\quad c=\gamma d,\quad b=\beta(c+d)=\beta(\gamma+1)d,$$
$$a=\alpha(\beta+1)(\gamma+1)d.$$

Exemplum. $\alpha=1$, $k=1$, erit $\beta=4=\gamma$, $d=\frac{1}{10}$, $c=\frac{2}{5}$, $b=2$, $a=\frac{5}{2}$. Consequenter quatuor numeri sunt: $\frac{5}{2}$, 2, $\frac{2}{5}$, $\frac{1}{10}$ quorum summa est $=5$, et productum $=\frac{1}{5}$.

4. Si desiderentur quinque numeri, ut sit $abcde(a+b+c+d+e)=1$, ponatur

$$d=\delta e,\quad c=\gamma(\delta+1)e,\quad b=\beta(\gamma+1)(\delta+1)e,\quad a=\alpha(\beta+1)(\gamma+1)(\delta+1)e.$$

Hinc summa omnium $=(\alpha+1)(\beta+1)(\gamma+1)(\delta+1)e$ et productum $\alpha\beta\gamma\delta(\beta+1)(\gamma+1)^2(\delta+1)^3e^5$,

$$\text{ergo}\quad \alpha\beta\gamma\delta(\alpha+1)(\beta+1)^2(\gamma+1)^3(\delta+1)^4e^6=1.$$

Sumatur $\qquad \delta=\beta$ et $\gamma=pp-1$, erit $\alpha\beta\beta(pp-1)(\alpha+1)p^6(\beta+1)^6e^6=1$.

Sit $p(\beta+1)e=\frac{1}{k}$ eritque $\alpha\beta\beta(pp-1)(\alpha+1)=k^6$. Fiat $\alpha(\alpha+1)(pp-1)=\alpha\alpha qq$, inde $\alpha=\frac{pp-1}{qq-pp+1}$, $\alpha\beta q=$ ergo $\beta=\frac{k^3}{\alpha q}$. Sit $p=2$ et $q=2$, hinc $\alpha=3$, $\beta=\frac{k^3}{6}$. Ponatur $k=2$, erit $\beta=\frac{4}{3}=\delta$, $e=\frac{3}{28}$, $d=$ $c=\frac{3}{4}$, $b=\frac{4}{3}$, $a=7$; consequenter quinque numeri 7, $\frac{4}{3}$, $\frac{3}{4}$, $\frac{1}{7}$, $\frac{3}{28}$, quorum summa est $\frac{28}{3}$ et productum omnium $\frac{3}{28}$.

Conjectura igitur supra proposita maxime fallit, ita ex casu ultimo, quo volebamus demonstrare non dari quinque potestates sextas, quarum summa sit potestas sexta, tum demum demonstratio haberetur, si ostendi posset, quinque illos numeros a, b, c, d, e nunquam ita definiri posse, ut eorum quilibet, per quemcunque reliquorum divisus, praebeat potestatem sextam. Si igitur demonstrari posset omnes has fractiones

$$\frac{a}{b},\quad \frac{a}{c},\quad \frac{a}{d},\quad \frac{a}{e}$$

non esse posse potestates sextas, tum simul demonstratum esset, non dari quinque potestates sextas, potestati sextae aequales.

8. (*J. A. Euler.*)

Ad casum superiorem secundum pro tribus numeris, quo formula

$$\alpha\beta(\alpha+1)(\beta+1)^2$$

debet esse biquadratum, sumatur $\beta=4\alpha(\alpha+1)$, eritque formula

$$4\alpha\alpha(\alpha+1)^2(2\alpha+1)^4=\frac{1}{c^4},\quad \text{ergo}\quad 2\alpha(\alpha+1)(2\alpha+1)^2=\frac{1}{cc},\quad \text{sive}\quad 2\alpha(\alpha+1)=\frac{1}{cc(2\alpha+1)^2}.$$

Debet ergo $2\alpha(\alpha+1)$ esse quadratum. Ponatur ergo $2\alpha(\alpha+1)=\alpha\alpha pp$, erit $\alpha=\frac{2}{pp-2}$, hincque fit

$$\alpha\alpha pp=\frac{4pp}{(pp-2)^2}=\frac{1}{cc(2\alpha+1)^2},\quad \text{ergo}\quad \frac{2p}{pp-2}=\frac{1}{c(2\alpha+1)}=\frac{pp-2}{c(pp+2)},\quad \text{hinc}\quad c=\frac{(pp-2)^2}{2p(pp+2)}.$$

Porro $\beta=\frac{8pp}{(pp-2)^2}$, unde tres numeri erunt

$$\text{I.}\quad a=\frac{pp+2}{p(pp-2)},\qquad \text{II.}\quad b=\frac{4p}{pp+2},\qquad \text{III.}\quad c=\frac{(pp-2)^2}{2p(pp+2)}.$$

Exemplum. Si $p=2$ fit $a=\frac{3}{2}$, $b=\frac{4}{3}$, $c=\frac{1}{6}$. Summa $=3$, productum $=\frac{1}{3}$.

Eodem redeunt sequentes solutiones

$$1.\quad \alpha=\frac{8kk}{(2kk-1)^2} \quad \text{et} \quad \beta=2kk, \quad \text{sive} \quad \beta=\frac{1}{2kk}.$$

$$2.\quad \alpha=\frac{1}{2kk-1} \quad \text{et} \quad \beta=\frac{(2kk-1)^2}{8kk}, \quad \text{sive} \quad \beta=\frac{8kk}{(2kk-1)^2}.$$

Nam si fuerit $\alpha(\alpha+1)\beta(\beta+1)^2=$ biquadrato, casu $\beta=b$, tum erit etiam casu $\beta=\frac{1}{b}$.

3. Si $kx(mxx+n)=\square$ casu $x=a$, tum etiam erit quadratum casu $x=\frac{n}{ma}$.

Problema. Invenire tres numeros p, q et r ita, ut formula $(pp-qq)(qq-rr)$ fiat biquadratum: veluti evenit I. si $p=51$, $q=3$, $r=1$; II. si $p=14$, $q=13$, $r=11$; III. si $p=29$, $q=25$, $r=23$.

Hic notasse juvabit ex casu quovis cognito facile erui alios, scilicet

$$p'=p+2q-r, \quad q'=p+r, \quad r'=p-2q-r,$$

sive etiam

$$p'=p+2q+r, \quad q'=p-r, \quad r'=p-2q+r.$$

Hoc problema facillime ex praecedente, quo numeri illi a, b, c sunt inventi, resolvitur. Sumatur enim

$$q=a+b \quad \text{et} \quad p=\sqrt{\left(qq+\frac{4}{ab}\right)} \quad \text{et} \quad r=a-b.$$

Exemplum. Sumto $a=\frac{3}{2}$, $b=\frac{4}{3}$, erit $a+b=q=\frac{17}{6}$, $r=\frac{1}{6}$ et $p=\sqrt{\left(\frac{289}{36}+2\right)}=\frac{19}{6}$; sive $p=19$, $q=17$ et $r=1$; hincque alii reperiuntur

$$p'=52, \quad q'=20, \quad r'=16, \quad \text{sive}$$
$$p'=13, \quad q'=5, \quad r'=4,$$

vel etiam

$$p'=54, \quad q'=18, \quad r'=14, \quad \text{sive}$$
$$p'=27, \quad q'=9, \quad r'=7.$$

Ex solutione generali sumatur $a=\frac{8kk}{2k(kk+2)}$ et $b=\frac{(kk-2)^2}{2k(kk+2)}$, fietque

$$q=\frac{(kk+2)^2}{2k(kk+2)}, \quad r=\frac{k^4-12kk+4}{2k(kk+2)} \quad \text{et} \quad p=\sqrt{\left(\frac{(kk+2)^2}{4kk}+\frac{2(kk+2)^2}{(kk-2)^2}\right)}=\frac{(kk+2)^2}{2k(kk-2)}, \quad \text{vel}$$

$$p=(kk+2)^3, \quad q=(kk-2)(kk+2)^2, \quad r=(kk-2)(k^4-12kk+4).$$

Analysis, qua haec solutio innititur, ita se habet:

Inventis ternis numeris a, b, c, ut supra, sumatur $q=a+b$ et $r=a-b$ et $p=a+b\pm 2c$; tum enim fiet

$$pp-qq=4cc+4c(a+b)=4c(a+b+c), \quad \text{at} \quad qq-rr=4ab,$$

ergo

$$(pp-qq)(qq-rr)=16abc(a+b+c)=16.$$

Hinc porro colligimus $p'=2a+4b+2c$, vel

$$p'=a+2b+c, \quad q'=a+c, \quad r'=a-c,$$

vel etiam

$$p'=2a+b+c, \quad q'=b+c, \quad r'=b-c.$$

Alia analysis. Loco a, b, c scribantur $\frac{x}{s}$, $\frac{y}{s}$ et $\frac{z}{s}$, ut debeat esse

$$xyz(x+y+z)=s^4.$$

Jam fiat primo $s^4=(x+y+z)^2pp$, eritque $xyz=(x+y+z)pp$, hinc

$$z=\frac{(x+y)pp}{xy-pp} \quad \text{et} \quad x+y+z=\frac{xy(x+y)}{xy-pp}, \quad \text{ergo} \quad ss=\frac{xyp(x+y)}{xy-pp}.$$

Ponatur porro $x = nqq$ et $y = nrr$, fietque

$$ss = \frac{n^3 qqrrp(qq+rr)}{nnqqrr - pp}, \quad \text{sive} \quad \frac{ss}{nnqqrr} = \frac{np(qq+rr)}{nnqqrr - pp}.$$

Ponatur nunc $nqr - p = k(qq+rr)$ eritque $p = nqr - k(qq+rr)$, ergo

$$\frac{ss}{nnqqrr} = \frac{nnqr - nk(qq+rr)}{k(2nqr - k(qq+rr))} = \frac{n(k(qq+rr) - nqr)}{k(k(qq+rr) - 2nqr)}.$$

Casus I. Sit $n = 2k$, erit $\frac{ss}{nnqqrr} = \frac{qq+rr-2qr}{qq+rr-4qr} = \frac{(q-r)^2}{qq+rr-4qr}.$

Sicque quadratum esse debet $qq+rr-4qr$, cujus radix ponatur $q + \frac{f}{g}r$, ita ut fiat

$$rr - 4qr = \frac{2f}{g}qr + \frac{ff}{gg}rr, \quad \text{vel} \quad ggr - 4ggq = 2fgq + ffr,$$

$$\text{vel} \quad (gg - ff)r = (4gg + 2fg)q, \quad \text{sive} \quad \frac{q}{r} = \frac{gg - ff}{4gg + 2fg}.$$

Sumatur ergo $q = gg - ff$ et $r = 4gg + 2fg$, eritque

$$\frac{ss}{nnqqrr} = \frac{(q-r)^2}{(q + \frac{f}{g}r)^2}, \quad \text{ideoque} \quad \frac{s}{nqr} = \frac{q-r}{q + \frac{f}{g}r} = \frac{3gg + ff + 2fg}{gg + ff + 4fg} \quad \text{et} \quad s = \frac{2k(gg-ff)(4gg+2fg)(3gg+ff+2fg)}{gg+ff+4fg}.$$

Quocirca erit $p = 2kqr - k(qq+rr) = -k(q-r)^2$, ideoque $x = 2kqq$, $y = 2krr$, $z = \frac{(x+y)pp}{xy - pp}$.

Casus II. Sumatur $n = k$, erit $\frac{ss}{kkqqrr} = \frac{qq+rr-qr}{(q-r)^2}$. Sit $\sqrt{(qq+rr-qr)} = q + \frac{f}{g}r$, erit

$$rr - qr = \frac{2f}{g}qr + \frac{ff}{gg}rr, \quad \text{vel} \quad ggr - ggq = 2fgq + ffr, \quad \text{hinc} \quad \frac{q}{r} = \frac{gg - ff}{gg + 2fg}.$$

Sumatur ergo $q = gg - ff$ et $r = gg + 2fg$, ita ut sit

$$\frac{s}{kqr} = \frac{q + \frac{f}{g}r}{q - r} = \frac{gg + fg + ff}{ff + 2fg}, \quad \text{hincque erit}$$

$$p = kqr - k(qq+rr), \quad \text{vel} \quad p = k(qr - qq - rr), \quad \text{indeque} \quad y = krr, \quad x = kqq, \quad z = \frac{(x+y)pp}{xy - pp}.$$

Aliter. Cum sit $ss = \frac{xyp(x+y)}{xy - pp}$, dividetur per xy, eritque $ss = \frac{p(x+y)}{1 - \frac{pp}{xy}}$. Sumatur $p = \frac{2xy}{x+y}$, fietque $ss = \frac{2xy(x+y)^2}{(x-y)^2}$. Superest ergo ut $2xy$ fiat quadratum. Sumatur $x = 2qq$ et $y = rr$ fietque

$$ss = \frac{4qqrr(2qq+rr)^2}{(2qq-rr)^2}, \quad \text{ideoque} \quad s = \frac{2qr(2qq+rr)}{2qq - rr}, \quad \text{hincque} \quad p = \frac{4qqrr}{2qq+rr},$$

$$\text{indeque} \quad x = 2qq, \quad q = rr, \quad z = \frac{pp(2qq+rr)}{2qqrr - pp}, \quad \text{vel} \quad z = \frac{8qqrr(2qq+rr)}{(2qq-rr)^2}.$$

Exemplum casus primi. Sit $g = 2$ et $f = 1$, erit $s = \frac{2.3.20.17}{13}k$, $q = 3$, $r = 20$, $p = 17^2 k$, $x = 2.9k$, $y = 2.20^2 k$, $z = \frac{818.17^4 k^3}{409.13^2 kk}$, vel $z = \frac{2.17^4 k}{13^2}$, hinc

$$a = \frac{x}{s} = \frac{3.13}{17.20}, \quad b = \frac{y}{s} = \frac{13.20}{3.17}, \quad c = \frac{z}{s} = \frac{17^3}{3.13.20},$$

quorum productum est $\frac{13.17}{3.20}$ et summa . . . . falsa!

p. 122 — 127.

9. (*N. Fuss I.*)

TENTAMEN DEMONSTRATIONIS THEOREMATIS FERMATIANI, quod esse nequeat $x^n + y^n = z^n$, statim ac n superat binarium.

Pro casu $n = 3$ res eo redit, ut demonstretur hanc formulam $ab(a+b)$ cubum esse non posse, ubi a et b sint primi inter se. Ponatur ergo $ab(a+b) = x^3$ eritque $4a^3b + 4aabb = 4ax^3$, sive $(aa+2ab)^2 = 4ax^3 + a^4$, unde $aa + 2ab = \sqrt{(4ax^3 + a^4)}$. Quoniam hic x et a non sunt numeri primi inter se, sit d maximus eorum communis divisor, ac ponatur $a = dp$ et $x = dz$, sicque p et z erunt primi inter se, et quia a et b etiam sunt primi inter se, erit quoque b primus ad d et p, tum igitur erit

$$2dbp + ddpp = dd\sqrt{(4pz^3 + p^4)}, \quad \text{ideoque} \quad \sqrt{(4pz^3 + p^4)} = \frac{2bp}{d} + pp.$$

Erit ergo $\frac{2bp}{d}$ numerus integer. Quia ergo b primus ad d, necesse est, ut $\frac{p}{d}$ sit integer; ponatur ergo $p = dq$, erit $\sqrt{(4dqz^3 + d^4q^4)} = 2bq + ddqq$. Unde quia radix factorem habet q, at z ad q primus, necesse ut d habeat factorem. Sit ergo $d = qr$ eritque $\sqrt{(4qqrz^3 + q^8r^4)} = 2bq + q^4rr$, seu $\sqrt{(4rz^3 + q^6r^4)} = 2b + q^3rr$. Sicque erit r factor quantitatis post signum, dum alter factor est $4z^3 + q^6r^3$, unde necesse est $r = \square$. Sit ergo $r = ss$, erit $\sqrt{(4z^3 + q^6s^6)} = \frac{2b}{s} + q^3s^3$. At vero $\frac{2b}{s}$ numerus integer esse non potest, unde patet aequationem nullo modo subsistere posse. Sicque impossibile erit, ut sit $ab(a+b) = x^3$, neque ergo unquam esse poterit $a^3 + b^3 = c^3$. Facile autem patet, hoc modo rem de altioribus potestatibus demonstrari posse. Verum haec conclusio maxime est incerta, cum fieri posset tam $s = 1$, quam $s = 2$. Ceterum theorema Fermatianum huc redit, ut demonstretur nunquam fieri posse, ut haec formula $1 + 4x^n$, vel etiam $1 - 4x^n$ unquam evadat quadratum, simul ac exponens n binarium superaverit; hic autem x omnes numeros rationales tam fractos, quam integros significare potest. Reducatur enim res ad numeros integros, ponendo $x = \frac{pq}{rr}$ et formula evadet

$$r^{2n} \pm 4p^nq^n = \square,$$

cujus radix statuatur $r^n + 2\nu$, ita ut ν primus ad r, erit

$$r^{2n} + 4p^nq^n = r^{2n} + 4\nu r^n + 4\nu\nu, \quad \text{unde erit} \quad r^n = \frac{4p^nq^n - 4\nu\nu}{4\nu}, \quad \text{sive} \quad p^nq^n = \nu(r^n + \nu),$$

qui duo factores sunt primi inter se, unde uterque debet esse potestas exponentis n. Capi ergo poterit $\nu = p^n$, tum autem erit $r^n + \nu = q^n$, ideoque $r^n + p^n = q^n$.

Quare si haec formula $1 \pm 4x^n$ fuerit impossibilis, etiam impossibile erit

$$\text{ut} \quad r^n + p^n = q^n.$$

A. m. T. II. p. 161.

10. (*J. A. Euler.*)

Ut fiat $x^3 + y^3 = \square$, sumatur $x + y = 3aabb$, $x - y = \frac{3a^4 - b^4}{2}$.

Summae vel differentiae duorum cuborum,

quae sint quadrata:

I. $2^3 + 1^3 = 3^2$, II. $8^3 - 7^3 = 13^2$, III. $65^3 + 56^3 = 671^2$, IV. $74^3 - 47^3 = 549^2$.

11. (*Lexell.*)

V. $37^3 + 11^3 = 228^2$, VI. $71^3 - 23^3 = 588^2$.

Proposito problemate, quo quaeruntur duo cubi inter se primi, quorum summa $x^3 + y^3$ sit quadratum, duo casus sunt perpendendi, alter, quo ambo numeri x et y sunt impares, alter vero, quo unus par, alter impar.

Pro *casu priori* erit $x=a+b$ et $y=a-b$, numerorum a et b altero existente pari, altero impari, hinc autem fit

$$x^3+y^3=2a^3+6abb=2a(aa+3bb),$$

ubi iterum duo casus occurrunt: *primo* vel $2a$ et $aa+3bb$ sunt primi inter se, quia $aa+3bb$ est impar, ergo uterque factor seorsim esse debet quadratum, unde patet a esse debere parem, b vero imparem; ponatur ergo $2a=4cc$, et quadratum insuper esse debet $aa+3bb=4c^4+3bb=\square$, quod facile fit; vel *secundo* $2a$ et $aa+3bb$ communem factorem habere possunt 3, quod fit si a sit divisibile per 3, existente a pari; sit ergo $a=6c$, fiet $12c(36cc+3bb)=\square$, hinc $4c(12cc+bb)=\square$. Sit ergo $c=dd$, fierique debet $12d^4+bb=\square$, quod facile fit. Pro *posteriori casu* poni debet $x=\frac{a+b}{2}$ et $y=\frac{a-b}{2}$, ubi uterque a et b impar; tum igitur quadratum esse debet $\frac{2a(aa+3bb)}{8}=\square$, sive $a(aa+3bb)=\square$. Hic iterum vel a non est divisibile per 3, vel divisibile per 3; illo casu sit $a=cc$, ideoque $c^4+3bb=\square$, hoc vero casu sit $a=3cc$, unde

$$3cc(9c^4+3bb)=\square, \quad \text{sive} \quad 3c^4+bb=\square.$$

A. m. T. I. p. 129.

12. (*N. Fuss I.*)

Si esse $4a^3=b^3+c^3$, foret $a^6-4b^3c^3=(b^3-c^3)^2$. Hinc ergo si demonstrari posset nunquam esse $a^6-4d^3=\square$, theorema foret demonstratum. Quoniam igitur haec forma a^6-4d^3 continetur in hac A^2-dB^2, etiam ejus radix quadrata similem formam habeat necesse est, quae ergo sit $pp-dqq$. Ergo fit $a^3=pp+pq^3=p(p+q^3)$. Hinc prior factor p debet esse cubus $=r^3$, et alter factor r^3+q^3 pariter cubus. Unde si foret $b^3+c^3=$ cubo, alius casus hinc deduceretur $r^3+q^3=$ cubo.

A. m. T. II. p. 211.

13.

Observatio circa theorema Fermatii, quo affirmat, hanc aequalitatem $a^n+b^n=c^n$ semper esse impossibilem, simul ac exponens n excedat binarium, cujus autem demonstrationem nemo adhuc invenire potuit.

Reduci potest ista forma ad formulas, quae quadrata fieri debent. Multiplicetur enim formula proposita per $4a^n$ et utrinque addatur b^{2n}, prodibit

$$(2a^n+b^n)^2=4a^nc^n+b^{2n}=\square=BB.$$

Simili modo erit $4b^nc^n+a^{2n}=AA$, item $c^{2n}-4a^nb^n=CC$. Totum negotium ergo eo redit, num impossibilitas harum formularum ostendi possit. Ceterum apparet sufficere, casus examinare, quibus n est numerus primus; nam si $a^n+b^n=c^n$, erit etiam $a^{\lambda n}+b^{\lambda n}=c^{\lambda n}$, sicque n spectari poterit ut numerus impar; tum autem formula a^n+b^n factorem habet $a+b$. Debet ergo etiam esse $a+b=p^n$, similique modo $c-a=q^n$ et $c-b=r^n$. Quod si ergo hae conditiones cum praecedentibus conjungantur, impossibilitas fortasse facilius ostendi poterit. Non solum igitur ostendi oportet hanc formulam $4a^nc^n+b^{2n}$ non esse posse quadratum ita, ut simul $a+b=p^n$.

Pro casu $a=1$ et $b=1$ fit illa formula $4c^n+1=\square$, quod in integris nunquam evenire posse ita ostendo, quod quidem manifestum est si n est par. Pro imparibus autem statim patet c non esse posse numerum imparem. Sit igitur par $=2d$, erit formula $2^{n+2}d^n+1$, cujus ergo radix esse debet $1+2^{n+1}s$,

$$2^{n+2}d^n+1=1+2^{n+2}s+2^{2n+2}ss, \quad \text{unde} \quad d^n=s+2^nss=s(2^ns+1),$$

qui factores cum sint primi inter se, debebit esse $s=t^n$, alter vero factor erit $2^nt^n+1=(2t)^n+1$, quod est impossibile.

A. m. T. III. p. 165-166.

14.

Theorema. Formula $1+2x^3$ nullo casu fit quadratum, neque in integris, neque in fractis, praeter casum $x=0$.

Demonstratio innititur huic fundamento, quod omnes cubi per 7 non divisibiles sint formae $7n\pm 1$. Hinc ergo omnes potestates sextae erunt formae $7n+1$. Deinde omnia quadrata sunt vel $7n$, vel $7n+1$, vel $7n+2$, vel $7n+4$, ita ut nulli numeri formae $7n+3$, $7n+5$, $7n+6$ sint quadrati. Jam forma $1+2x^3$ in integris quadratum esse non potest. Si enim x per 7 non sit divisibile, forma numeri $1+2x^3$ erit $7n+3$ et $7n+6$, quorum neuter quadratum esse potest. Sumto autem $x=7a$, erit $1+2.7^3.a^3=zz$. Foret ergo $2.7^3.a^3=zz-1=(z+1)(z-1)$, ergo factorum $z+1$ et $z-1$ alter debet esse cubus, alter duplex cubus. Sumamus $z-1=7^3.b^3$, ideoque $z=1+7^3.b^3$, unde $2a^3=2b^3+7^3.b^6$, unde patet esse debere $a=bc$, erit ergo $2c^3=2+7^3.b^3$.

At si x est numerus fractus, ejus denominator debet esse quadratum. Ponatur ergo $x=\frac{a}{bb}$, fieri debet $b^6+2a^3=\square$, ubi nisi $b=7n$, semper erit $b^6=7n+1$ et $a^3=7n\pm 1$ (si a non est $7n$), ergo

$$b^6+2a^3=7n+1\pm 2,$$

hoc est vel $7n+3$, vel $7n-1$, neutro casu quadratum. Sit $a=7c$ erit $b^6+2.7^3.c^3=zz$. Sit $z=b^3+2.7^3d^3$, hinc $c^3=2b^3d^3+2.7^3.d^6$ et sumto $c=dc$ erit $c^3=2b^3+2.7^3.d^3$, ergo c^3-2b^3 divisibile esset per 7, quod fieri nequit.

Verum rigida demonstratio postulat profundiores indagationes.

Theorema I. Si fuerit $2x^3+1=\square$, dari poterunt duo cubi, quorum summa vel differentia sit cubus quadruplus.

Demonstratio. Loco x scribamus $\frac{x}{yy}$ fietque $2x^3+y^6=zz$. Jam ponatur $x=ab$ fierique debet $zz-y^6=2a^3b^3$. Fiat ergo $z+y^3=2a^3$ et $z-y^3=b^3$, unde fit $2y^3=2a^3-b^3$, ergo $b^3=2(a^3-y^3)$. Fiat $b=2c$, erit $4c^3=a^3-y^3$.

Theorema II. Si dentur duo cubi, quorum summa vel differentia aequetur cubo quadruplo, dari poterit x, ut sit $2x^3+1=\square$.

Demonstratio. Sit $a^3+b^3=4c^3$, erit $4a^6+4a^3b^3+b^6=16a^3c^3+b^6=\square$. Jam sumatur $x=\frac{2ac}{bb}$ erit $\square=2x^3+1$.

Theorema III. Non dantur duo cubi, quorum summa vel differentia sit cubus quadruplus.

Demonstratio. Si enim fuerit $x^3+y^3=4z^3$, evidens est ambos numeros x et y esse debere impares, unde statui poterit $x=a+b$ et $y=a-b$, ita ut numerorum a et b alter sit par alter impar, unde fiet $2a^3+6abb=4z^3$, sive $a(aa+3bb)=2z^3$, ubi $aa+3bb$ erit numerus impar, unde patet a esse debere parem et b imparem. Hinc porro si ambo factores a et $aa+3bb$ fuerint primi inter se, debet esse $a=2p^3$ et $aa+3bb=q^3$. At vero si a sit $3c$, ambo factores communem habebunt divisorem 3, eritque

$$9c(3cc+bb)=2p^3q^3,$$

unde $9c$ debet esse duplus cubus veluti $2.27d^3$, ita ut $c=2.3d^3$, ideoque $a=2.9d^3$. Tum vero $bb+3cc$ debet esse cubus, unde casus duo sunt considerandi: prior, quo $a=2p^3$ et $aa+3bb=q^3$; alter, quo $a=2.9p^3$ et $aa+3bb=3q^3$, sive posito $a=3c$ debet esse $bb+3cc=r^3$. Quod autem uterque casus sit impossibilis, demonstrari potest ope sequentis lemmatis.

*

Lemma. Si fuerit $xx+3yy=$ cubo, certum est ejus radicem ejusdem fore formae, puta $pp+3qq$, ita ut $xx+3yy=(pp+3qq)^3$. Erit ergo $x+y\sqrt{-3}=(p+q\sqrt{-3})^3$, $x-y\sqrt{-3}=(p-q\sqrt{-3})^3$, hoc est

$$x+y\sqrt{-3}=p^3-9pqq+(3ppq-3q^3)\sqrt{-3},$$

unde fit $\quad x=p^3-9pqq \quad$ et $\quad y=3ppq-3q^3$.

Demonstratio casus prioris: Cum igitur $aa+3bb=$ cubo, per lemma erit $a=p^3-9pqq$ et $b=3ppq-3q^3$. Quam ob rem debet esse $a=p^3-9pqq=2$ cubis, unde hoc productum $p(p-3q)(p+3q)$ cubo duplo aequari debet, et cum numerorum p et q alter sit par, alter impar, erit p par, ideoque $p=2$ cubis. At vero $p+3q$ et $p-3q=$ cubo. Ponatur ergo $p+3q=r^3$ et $p-3q=s^3$, erit $2p=r^3+s^3=4t^3$, quod fieri non potest, quia si darentur tales numeri $a^3+b^3=4c^3$, nunc darentur multo minores $r^3+s^3=4t^3$.

Demonstratio alterius casus. Cum fieri debeat $bb+3cc=$ cubo, erit $b=p^3-9pqq$ et $c=3ppq-3q^3$. Cum igitur $a=3c$, erit $a=9(ppq-q^3)=2.9s^3$, sive $q(pp-qq)=2s^3$, ubi q erit par, ideoque ponatur

$$p+q=t^3 \quad \text{et} \quad p-q=u^3, \quad \text{erit} \quad 2q=t^3-u^3=4v^3.$$

Unde si magni darentur numeri, etiam in minoribus dari deberent, ut fiat $2x^3+1=\square$, quod autem cum in minimis non fiat, etiam in maximis non succedet.

A. m. T. III. p. 167—169.

15.

Problema. Invenire duos cubos, quorum summa aequetur dato multiplo cujuspiam cubi, sive ut sit

$$x^3+y^3=nz^3.$$

Solutio. Ponatur $n=\alpha\beta\gamma$ et fiat $x=a+b$ et $y=a-b$; tum vero $z=2\nu$, erit $a(aa+3bb)=4\alpha\beta\gamma\nu^3$. Fiat $aa+3bb=(pp+3qq)^3$ et vidimus fore $a=p(pp-9qq)$ et $b=3q(pp-qq)$, esseque oportebit $a=\frac{4\alpha\beta\gamma\nu^3}{(pp+3qq)^3}$. Sumatur $\nu=fgh(pp+3qq)$, ut prodeat $a=4\alpha\beta\gamma f^3g^3h^3$. Cum igitur sit $a=p(p+3q)(p-3q)$, fiat $p=\alpha f^3$, $p+3q=2\beta g^3$ et $p-3q=2\gamma h^3$. Hinc erit $p=\beta g^3+\gamma h^3$ et $3q=\beta g^3-\gamma h^3$. Hinc ergo debet esse

$$\alpha f^3=\beta g^3+\gamma h^3,$$

quod si ergo hoc fieri potest, etiam aequatio proposita erit confecta. Ita sumtis f, g, $h=\pm 1$, solutio locum habebit si fuerit $\alpha=\beta\pm\gamma$. Sumto $f=2$, $g=h=\pm 1$ solutio locum habet quoties fuerit $8\alpha=\beta\pm\gamma$. Tali autem casu, quo $\alpha f^3=\beta g^3+\gamma h^3$, invento, erit $p=\alpha f^3$ et $q=\frac{\beta g^3-\gamma h^3}{3}$, unde porro deducitur $a=p(pp-9qq)$ et $b=3q(pp-qq)$, ex quibus denique $x=a+b$ et $y=a-b$. Tandem autem erit $z=2\nu=2fgh(pp+3qq)$.

Exemplum 1. Sit $\alpha=3$, $\beta=2$, $\gamma=1$, ideoque $n=6$, fiet $3f^3=2g^3+h^3$, quod fit si $f=1$, $g=1$, $h=1$, tum autem erit $p=3$ et $q=\frac{1}{3}$, unde deducitur $a=24$ et $b=\frac{80}{9}$. Erit ergo $a:b=27:10$. Sit ergo $a=27$ et $b=10$ eritque $x=37$, $y=17$, $z=\frac{56}{3}$. Cum jam sit $x^3+y^3=(x+y)(xx-xy+yy)$, erit $x+y=54$ et ob $x-y=20$, ergo $xx-2xy+yy=400$ et $xx-xy+yy=1029$, ergo $x^3+y^3=54.1029=6.3^3.7^3$.

Exemplum 2. Sit $\alpha=5$, $\beta=3$, $\gamma=1$, ideoque $n=15$, fiet $p=5f^3=3g^3+h^3$, quod fit si $h=2$, $g=-1$ et $f=1$; tum erit $p=5$, $q=-\frac{11}{3}$, unde $a=5.96$ et $b=\frac{104.11}{9}$. Sumatur $a=540$, $b=143$, $x=683$, $y=397$ fietque $x^3+y^3=15z^3$.

Observatio maximi momenti. Arbitratus sum, si fuerit $xx-nyy=(pp-nqq)^3$, etiam fore

$$x+y\sqrt{n}=(p+q\sqrt{n})^3 \quad \text{et} \quad x-y\sqrt{n}=(p-q\sqrt{n})^3,$$

unde facta evolutione fiat $x=p^3+3npqq$ et $y=3ppq+nq^3$. At nunc se mihi casus obtulit maxime discrepans

$16^2-3.23^2=(1-3.2^2)^3$, unde deberet esse $16+23\sqrt{3}=(1+2\sqrt{3})^3$, quod autem neutiquam contingit. Similique modo deberet esse $16-23\sqrt{3}=(1-2\sqrt{3})^3$. Interim tamen productum priorum

$$16^2-3.23^2=(1-3.2^2)^3=37^2-3.30^2.$$

Revera igitur hoc remedium afferri debet: Si fuerit $xx-nyy=(pp-nqq)^3$, tum sumtis factoribus dabuntur numeri f et g, ut sit $x+y\sqrt{n}=(f+g\sqrt{n})(p+q\sqrt{n})^3$ atque $x-y\sqrt{n}=(f-g\sqrt{n})(p-q\sqrt{n})^3$, ubi necesse est, ut sit $ff-ngg=1$. Haec ergo applicemus ad casum observatum, ubi est $x=16$, $n=3$, $y=23$, deinde $p=1$, $q=2$, et facto calculo litterae f et g ita determinantur, ut sit $f=-\frac{1478}{1331}$ et $g=-\frac{371}{1331}$, unde revera fit $ff-3gg=1$. Unde patet hujusmodi coëfficientes nullo modo divinari posse.

Sequens autem consideratio me ad hunc casum deduxit: Quaesivi numeros x et y, ut $(x+y)(xx+yy)$ fiat cubus, et vidi esse debere $x+y=4A^3$ et $xx+yy=2B^3$. Posui ergo $xx+yy=2(aa+bb)^3$ et inveni $x=a(aa-3bb)$ et $y=b(3aa-bb)$. Hinc porro inveni hanc solutionem $y=-9$ et $x=13$, deinde ex hoc casu elicui $x=7.37.61$ et $y=9.13.229$. — At vero valores litterarum f et g multo simplicius exhiberi possunt, uti ex sequente problemate patebit:

Problema. Invenire numeros inter se primos x et y, ut sit $xx-3yy$ cubus.

Solutio. Primo haec conditio est adjicienda, ut numeri x et y sint primi inter se: si enim compositi admittantur, solutio esset facillima sumendo

$$x=a(aa-3bb) \quad \text{et} \quad y=b(aa-3bb); \text{ tum enim foret } xx-3yy=(aa-3bb)^3.$$

Ponatur igitur $xx-3yy=(pp-3qq)^3$ et sumtis factoribus fiat

$$x+y\sqrt{3}=(f+g\sqrt{3})(p+q\sqrt{3})^3$$

similique modo $$x-y\sqrt{3}=(f-g\sqrt{3})(p-q\sqrt{3})^3,$$

sic enim fiet $xx-3yy=(ff-3gg)(pp-3qq)^3$. Necesse igitur est, ut sit $ff-3gg=1$, quod infinitis modis fieri potest. Primo $f=1$ et $g=0$; secundo $f=2$ et $g=1$; tertio $f=7$ et $g=4$; quarto $f=26$ et $g=15$. et in genere

$$f+g\sqrt{3}=\frac{1}{2}(2+\sqrt{3})^n+\frac{1}{2}(2-\sqrt{3})^n$$

$$f-g\sqrt{3}=\frac{1}{2}(2+\sqrt{3})^n-\frac{1}{2}(2-\sqrt{3})^n.$$

His notatis cum sit $(p+q\sqrt{3})^3=p(pp+9qq)+3q(pp+qq)\sqrt{3}$. Ponatur brevitatis gr.

$p(pp+9qq)=P$ et $q(pp+qq)=Q$, ut fiat $(p+q\sqrt{3})^3=P+3Q\sqrt{3}$ et $(p-q\sqrt{3})^3=P-3Q\sqrt{3}$. Hinc ergo erit

$$x+y\sqrt{3}=fP+(gP+3fQ)\sqrt{3}+9gQ, \text{ unde fit } x=fP+9gQ \text{ et } y=gP+3fQ;$$

ubi notetur litteras f et g tam negative quam positive accipi posse.

Sit nunc $p=1$ et $q=2$, erit $P=37$ et $Q=10$, ergo $x=37f+90g$ et $y=37g+30f$; quare sumto $f=1$ et $g=0$, erit $x=37$ et $y=30$. At sumto $f=2$ et $g=1$ erit $x=164$, $y=97$. Sumto vero $f=2$ et $g=-1$ erit $x=16$, $y=23$, qui est ipse casus supra tam difficilis visus. Hoc ergo modo omnes casus possibiles pro x et y erui poterunt, ut $xx-3yy$ fiat cubus, dummodo litteris f et g omnes valores tam positivi quam negativi successive tribuantur. Eodem modo problema generalius solvi potest, ut fiat $xx-nyy$ cubus, qui sit $(pp-nqq)^3$ et sumto

$$ff-ngg=1 \text{ erit } x+y\sqrt{n}=(f+g\sqrt{n})(p+q\sqrt{n})^3,$$

ubi erit $$(p+q\sqrt{n})^3=p(pp+3nqq)[=P]+q(3pp+nqq)\sqrt{n}[=Q\sqrt{n}].$$

Hinc ergo erit

$$x + y\sqrt{n} = fP + gP\sqrt{n} + fQ\sqrt{n} + ngQ, \quad \text{ideoque} \quad x = fP + ngQ \quad \text{et} \quad y = gP + fQ,$$

quod ergo infinitis modis fieri potest, si modo fuerit $ff - ngg = 1$, id quod semper praestari potest quoties fuerit numerus positivus. At si n fuerit numerus negativus, evidens est formulam $ff + ngg$, saltem pro f et g integris, aliter unitati aequalem esse non posse, nisi sit $f = 1$ et $g = 0$.

p. 169. 171.

68.

(Lexell.)

Problema. Datis numeris m et n, item a et b, invenire x et y, ut fiat

$$maa - nbb = nxx - myy, \quad \text{sive} \quad m(aa + yy) = n(bb + xx).$$

Solutio. Ponatur $x = mpa + qb$ et $y = qa + npb$, unde fiet

$$maa - nbb = m(mnppaa - qqaa) + n(qqbb - mnppbb) = maa(mnpp - qq) - nbb(mnpp - qq).$$

Oportet ergo sit $mnpp - qq = 1$. Manifestum ergo est hoc problema solutionem non admittere, nisi numeri m et n sint summae duorum quadratorum. Quoties autem fuerint tales, ope problematis Pelliani semper invenire licet numeros p et q, ut fiat $mnpp = qq + 1$, sive $mnpp - 1 = \square$.

(J. A. Euler.)

Excipiuntur tamen casus, quibus vel mn est ipse numerus quadratus, vel in duo quadrata inter se prima resolvi nequit; cujusmodi sunt: 8, 18, 20, 32, 40, 45, etc. Sic si $mn = 13$, erit $p = 5$ et $q = 18$; nam $13.5^2 = 18^2 + 1$, et si $mn = 125$, erit $p = 61$ et $q = 682$, nam $125.61^2 = 682^2 + 1$ etsi $125 = 100 + 25$ quae non sint prima inter se, sed notandum est esse $125 = 11^2 + 2^2$, quae utique sunt prima.

A. m. T. I. p. 129.

69.

(N. Fuss I.)

Problema. Resolvere aequalitatem $ab(a + b)^2 = cd(c + d)^2$.

Solutio. Ponatur $m(a + b) = n(c + d)$ fierique debet $nnab = mmcd$. Porro sit

$$a = mmps, \quad c = nnqs, \quad b = qrs, \quad d = prs,$$

unde prior aequatio erit $m^3p + mqr = n^3q + npr$, unde fit $r = \frac{n^3q - m^3p}{mq - np}$. Ut ergo fractiones evitentur, sumatur $s = mq - np$ eritque in numeris integris

$$a = mmp(mq - np), \quad b = q(n^3q - m^3p), \quad c = nnq(mq - np), \quad d = p(n^3q - m^3p).$$

Hinc enim fit

$$a + b = n(nnqq - mmpp), \quad c + d = m(nnqq - mmpp),$$

quae solutio est generalis. Notetur autem, si litterae a, b, c, d sint quadrata, veluti $a = A^2$, $b = B^2$, $c = C^2$, $d = D^2$, tum aequalitatem propositam accipere hanc formam

$$AB(AA + BB) = CD(CC + DD),$$

ad quam igitur solvendam illae quatuor formulae quadrata fieri debent. Primo ergo quadratum erit

$$\frac{a}{c}=\frac{mmp}{nnq}, \quad \text{ideoque} \quad \frac{p}{q}=\square, \quad \text{sive} \quad pq=\square.$$

Praeterea vero quadratum esse debet

$$\frac{a}{b}=\frac{mmp\,(mq-np)}{q\,(n^3 q-m^3 p)}, \quad \text{sive} \quad \frac{mq-np}{n^3 q-m^3 p}=\square,$$

hocque modo omnes erunt quadrata, unde eadem solutio prodit, quae supra est data.

Resolutio succincta aequalitatis $(aa+bb)\,ab=(cc+dd)\,cd$.

Sumtis pro m et n numeris quibuscunque capiatur $\frac{f}{g}=\frac{mm+nn}{mm-nn}$; tum vero sumatur

$$p=4f^3+fgg-3g^3, \quad q=4f^3+fgg+3g^3 \quad \text{et} \quad s=4f^3-5fgg,$$

tum habebitur $a=mp$, $b=ns$, $c=ms$, $d=nq$. Veluti si sumatur $m=3$ et $n=1$, erit $\frac{f}{g}=\frac{5}{4}$, ideoque $f=5$, $g=4$, hinc $4ff+gg=116$, ergo $p=388$, $q=772$, $s=100$, seu $p=97$, $q=193$, $s=25$, unde fit $a=291$, $b=25$, $c=75$, $d=193$, hic scilicet numeros p, q, s per 4 deprimere licuit, quod semper evenit quando g numerus par.

Duo numeri a et b assignari possunt, ut fiat $10ab\,(aa+bb)=53$, quod utique in numeris integris fieri nequit. Hoc autem evenit sumendo $a=\frac{27}{10}$ et $b=\frac{4}{15}$, tum fit $ab=\frac{54}{75}$ et $10ab=\frac{540}{75}=\frac{36}{5}$; tum ob $a=\frac{81}{30}$ et $b=\frac{8}{30}$ erit $aa+bb=\frac{265}{30}$, ideoque $10ab\,(aa+bb)=53$.

Resolutio hujus formulae $ab\,(maa+nbb)=cd\,(mcc+ndd)$.

Posito $b=pc$ et $d=qa$, reperitur $\frac{aa}{cc}=\frac{mq-np^3}{mp-nq^3}$. Posito $p=q\,(1+z)$ et $\frac{a}{c}=1-5z$, porro $q=\frac{k}{h}$, ambo numeri h et k arbitrio relinquuntur. Tum sumatur $\frac{f}{g}=\frac{mhh+nkk}{mhh-nkk}$, eritque

$$a=h\,(4f^3-5fgg), \quad b=k\,(4f^3+fgg+3g^3), \quad c=h\,(4f^3+fgg-3g^3), \quad d=k\,(4f^3-5fgg).$$

Cum enim sit $\frac{mhh}{nkk}=\frac{f+g}{f-g}$, erit

$$maa+nbb=mn\,(2ff-gg)\,(4ff-3gg)\,(4f^3+fgg-3g^3)$$
$$mcc+ndd=mn\,(2ff-gg)\,(4ff-3gg)\,(4f^3+fgg+3g^3)$$

erit igitur
$$\frac{ab}{cd}=\frac{4f^3+fgg+3g^3}{4f^3+fgg-3g^3} \quad \text{et} \quad \frac{maa+nbb}{mcc+ndd}=\frac{4f^3+fgg-3g^3}{4f^3+fgg+3g^3}.$$

Ceterum hic patet, permutatis numeris h et k sumtoque g negativo, litteras a et b abire in d et c.

Alia Resolutio formulae $\frac{aa}{cc}=\frac{p^3-q}{q^3-p}$,

pro qua supra posuimus $q=p(1+z)$.

Nunc vero ponamus $q=nn(p-1)+p$, tum enim prodit

$$\frac{aa}{cc}=\frac{p^3-p-nn(p-1)}{n^6(p-1)^3+3n^4p(p-1)^2+3nnpp(p-1)+p^3-p},$$

ubi commode per $p-1$ dividitur prodibitque

$$\frac{aa}{cc}=\frac{pp+p-nn}{n^6(p-1)^2+3n^4p(p-1)+3nnpp+pp+p}.$$

Hoc modo habentur duae formulae ad quadratum reducendae scilicet

$$n^6pp+n^6p+n^8 \quad \text{et} \quad n^6pp+3n^4pp+3nn+pp-(2n^6+3n^4-1)p+n^6.$$

Necesse ergo est, ut sit $(nn+1)^3=\square$, ideoque etiam $nn+1$. Sit igitur $nn+1=mm$ eritque

$$\frac{aa}{cc}=\frac{pp+p-mm+1}{m^6pp-2mm(mm-1)^2p+(mm-1)^3};$$

hujusmodi autem binae formulae supra sunt resolutae. Ita si sumatur $m=2$, ut sit $q=4p-3$, hinc reperitur $p=\frac{8299}{64.3.5.7}$. Hae autem solutiones diversae erunt ab iis, quas prior solutio suppeditaverat. Ceterum hic statim ab initio scribi debuisset $mm-1$ loco nn.

Solutio generalior. Loco p et q scribatur $\frac{p}{r}$ et $\frac{q}{r}$ et formula resolvenda erit $\frac{p^3-qrr}{q^3-prr}$. Jam ponatur $p=1+\alpha z$, $q=1+\beta z$ et $r=1+\gamma z$ et habebimus

$$\frac{3\alpha-\beta-2\gamma+(3\alpha\alpha-2\beta\gamma-\gamma\gamma)z+(\alpha^3-\beta\gamma\gamma)zz}{3\beta-\alpha-2\gamma+(3\beta\beta-2\alpha\gamma-\gamma\gamma)z+(\beta^3-\alpha\gamma\gamma)zz}=\square.$$

Hic igitur tantum opus est, ut fiat

$$\frac{3\alpha-\beta-2\gamma}{3\beta-\alpha-2\gamma}=\frac{ff}{gg},\quad \text{unde}\quad 2\gamma=\frac{(3\alpha-\beta)gg-(3\beta-\alpha)ff}{gg-ff}.$$

Hoc modo prodit $\frac{p}{r}=1433$ et $\frac{q}{r}=473$. Veluti si $\alpha=2$ et $\beta=1$, fit $\gamma=\frac{5gg-ff}{gg-ff}=\frac{4gg}{gg-ff}+1$. Ergo si $g=1$ et $f=2$ erit $2\gamma=-\frac{1}{3}$, ideoque $\gamma=-\frac{1}{6}$, unde fit

$$\frac{3.64+433z+287zz}{3.16+132z+34zz}=\square.$$

Ceterum hic nil impedit, quominus sumatur vel $\alpha=0$, vel $\beta=0$, vel $\gamma=0$; tantum sumi non debet $\beta=\alpha$. Quovis autem casu simplicissima solutio ita reperitur: Cum fiat $\frac{A+Bz+Czz}{a+bz+czz}=\square$, in qua aequatione $\frac{A}{a}$ per hypothesin $=\square$, ponatur hoc $\square=\frac{A}{a}$, indeque prodit z. Sequens solutio imprimis est memorabilis, sumendo $p=(1+nn)z$, $q=1+z$, ac per artificium modo memoratum reperitur

$$z=\frac{(nn+1)(n^4-3nn+1)}{3n^4},\quad \text{unde fit}\quad p=\frac{n^6-2n^4+nn+1}{3nn}\quad \text{et}\quad q=\frac{n^6+n^4-2nn+1}{3n^4},$$

unde pro solutione formulae $ab(aa+bb)=cd(cc+dd)$ statim habetur $a=3n^5$, $b=n^6-2n^4+nn+1$, $c=3nn$, $d=n(n^6+n^4-2nn+1)$, quandoquidem posueramus $b=cp$, $d=aq$, hinc autem colligitur $\frac{aa}{cc}=n^6$, hincque $\frac{a}{c}=n^3$. Quodsi jam pro casu simplicissimo sumatur $n=2$, fit $a=96$, $b=37$, $c=12$, $d=146$ hincque erit

$$ab=2^5.3.37,\quad cd=2^3.3.73,\quad aa+bb=5.29.73,\quad cc+dd=4.5.29.37.$$

Theorema. Ex qualibet resolutione aequationis $ab(aa+bb)=cd(cc+dd)$ semper alia solutio deduci potest.

Demonstratio. Quia $ab(aa+bb)=cd(cc+dd)$, erit $(a+b)^4-(a-b)^4=(c+d)^4-(c-d)^4$ hinc $(a+b)^4-(c+d)^4=(a-b)^4-(c-d)^4$, seu

$$(a+b+c+d)(a+b-c-d)(\square+\square)=(a-b+c-d)(a-b-c+d)(\square+\square).$$

Quamobrem si ponamus $a'=a+b+c+d$ et $b'=a+b-c-d$; dein etiam

$$c'=a+b-c-d\quad \text{et}\quad d'=a-b-c+d$$

erit $a'b'(a'a'+b'b')=c'd'(c'c'+d'd')$. Quia igitur erat $a=291$, $b=25$, $c=75$, $d=193$, erit $a'=584$, $b'=48$, $c'=384$, $d'=148$, qui per 4 depressi dant

$$a'=146,\quad b'=12,\quad c'=96,\quad d'=37,$$

quae est solutio posterior minima.

Formulae sat concinnae

pro resolutione formulae $ab(aa+bb)=cd(cc+dd)$

Sumtis pro lubitu binis quadratis ff et gg, capiatur $\frac{\alpha}{\beta}=\frac{3ff+gg}{3gg+ff}$, erit

$$a=f(\alpha+\beta)(\alpha\alpha-3\alpha\beta+\beta\beta)$$
$$b=g(\beta^3-5\alpha\beta\beta+4\alpha\alpha\beta-2\alpha^3)$$
$$c=g(\alpha+\beta)(\alpha\alpha-3\alpha\beta+\beta\beta)$$
$$d=f(\alpha^3-5\alpha\alpha\beta+4\alpha\beta\beta-2\beta^3)$$

vel si ponatur $(\alpha+\beta)(\alpha\alpha-3\alpha\beta+\beta\beta)=\Delta$, erit

$$a=f\Delta,\quad b=g(\Delta-3\alpha(\alpha-\beta)^2),\quad c=g\Delta,\quad d=f(\Delta-3\beta(\alpha-\beta)^2).$$

Pro numeris α et β construatur haec tabula:

f	g	α	β
2	1	13	7
3	1	7	3
3	2	31	21
4	1	49	19
4	3	57	43
5	1	19	7
5	2	79	37
5	3	21	13
5	4	91	73

Hinc si $f=3$ et $g=1$, erit $\alpha=7$ et $\beta=3$, hincque $\Delta=-50$, unde $a=150$, $b=386$, $c=50$, $d=582$, sive per 2 deprimendo $a=75$, $b=193$, $c=25$, $d=291$. Sit $f=5$ et $g=1$, erit $\alpha=19$ et $\beta=7$, unde $\Delta=286$, $a=1430$, $b=7922$, $c=286$, $d=13690$, sive $a=715$, $b=3961$, $c=143$, $d=6845$. Hinc per theorema alii reperiuntur hoc modo:

$$a'=2966,\quad b'=578,\quad c'=2487,\quad d'=864.$$

Problema Diophanteum. Cognito uno casu, quo haec formula $\frac{A+Bz+Czz+Dz^3}{E+Fz}$ fit quadratum, ex eo alium casum derivare.

Solutio. Ponamus esse casu $z=e$, $\frac{A+Be+Cee+De^3}{E+Fe}=kk$, tum sumatur

$$s=\frac{B+2Ce+3Dee-Fkk}{2k(E+Fe)},\quad \text{quo facto alius casus erit}\quad z=\frac{C+2De-Ess-2Fks}{Fss-D};$$

tum enim erit
$$\frac{A+Bz+Czz+Dz^3}{E+Fz}=(k-es+sz)^2.$$

Hoc ergo modo ex unico casu innumerabiles alii successive deduci possunt.

Analysis. Ponatur $\frac{A+Bz+Czz+Dz^3}{E+Fz}=(k+s(z-e))^2$ et facta evolutione erit

$$A+Bz+Czz+Dz^3=Ekk+2kEs(z-e)+Ess(z-e)^2+Fkkz+2Fksz(z-e)+Fssz(z-e)^2,$$

hinc subtrahatur aequatio $A+Be+Cee+De^3=Ekk+Fekk$ et dividendo per $z-e$ prodit

$$B+C(z+e)+D(zz+ez+ee)=2Eks+Fkk+Ess(z-e)+2Fksz+Fssz(z-e).$$

Jam ponatur $z=e+\nu$, unde terminis ad eandem partem translatis erit

$$\left.\begin{aligned} & B + 2Ce + 3Dee - 2Eks - Fkk - 2Feks \\ & + C\nu + 3De\nu - Ess\nu - Fsse\nu - 2Fks\nu \\ & + D\nu\nu - Fss\nu\nu \end{aligned}\right\} = 0.$$

Nunc littera s ita determinetur, ut prima linea evanescat, hoc est ponendo

$$s = \frac{B + 2Ce + 3Dee - Fkk}{2k(E + Fe)},$$

tum dividendo per ν reperietur

$$\nu = \frac{C + 3De - Ess - Fsse - 2Fks}{Fss - D}, \quad \text{hincque} \quad z = \frac{C + 2De - Ess - 2Fks}{Fss - D},$$

tum igitur erit $\qquad \dfrac{A + Bz + Czz + Dz^3}{E + Fz} = (k - es + sz)^2.$

Aliud problema. Ex cognito casu, quo $\dfrac{A + Bz + Czz}{D + Ez + Fzz}$ fit quadratum, invenire alium casum, quo idem obtineatur.

Solutio. Sit $z = e$ casus ille cognitus, quo fiat $\dfrac{A + Be + Cee}{D + Ee + Fee} = kk$ et ponatur $\dfrac{A + Bz + Czz}{D + Ez + Fzz} = kk$, ut sit

$$A + Bz + Czz = Dkk + Ekkz + Fkkzz \quad \text{hinc subtrahatur aequatio}$$

$$A + Be + Cee = Dkk + Eekk + Feekk$$

et facta divisione per $z - e$ prodibit

$$B + C(z + e) = Ekk + Fkk(z + e), \quad \text{unde statim elicitur} \quad z = \frac{Ekk + Fkke - B - Ce}{C - Fkk}.$$

Verum hoc modo unicus alius valor reperitur, namque ex invento iterum pristinus valor prodiret.

A. m. T. II. p. 157—161.

Theorema. Hae duae formulae $ab(aa + bb)$ et $2cd(cc - dd)$ ita inter se conveniunt respectu factorum non quadratorum, ut altera in alteram transformari possit.

Prior enim in posteriorem transmutatur ponendo $a = 2cd$ et $b = cc - dd$. Vicissim autem posterior in priorem transmutatur ponendo $c = aa + bb$ et $d = 2ab$; tum enim fit

$$2cd(cc - dd) = 4ab(aa + bb)^3.$$

Tres numeri formae $xy(xx - yy)$ infinitis modis dari possunt, qui inter se prorsus sint aequales, scilicet

$$\text{I.} \quad x = ff + 3gg \quad \text{et} \quad y = 4fg; \qquad \text{II.} \quad x = ff + 3gg \quad \text{et} \quad y = 3gg - ff + 2fg;$$

$$\text{III.} \quad x = ff + 3gg \quad \text{et} \quad y = 3gg - ff - 2fg.$$

His numeris resolvitur problema, *quo quaeruntur tria triangula rectangula, quorum areae sint inter se aequales*.

Nam in triangulo ABC sumtis cathetis $AC = 2xy$ et $BC = xx - yy$, erit hypotenusa $AB = xx + yy$, area $= xy(xx - yy)$. Ex prima igitur forma fit area

$$4fg(ff + 3gg)(f - g)(f - 3g)(f + g)(f + 3g).$$

Ex secunda et tertia idem. Ratio investigationis haec est:

Ponatur $pr(p + r)(p - r) = qs(q + s)(q - s)$ et sumatur $q = p$, erit $r(pp - rr) = s(pp - ss)$, unde fit

$$pp = rr + rs + ss, \quad \text{sive} \quad pp - \frac{3}{4}ss = (r + \frac{1}{2}s)^2.$$

Erit ergo $2r + s = \sqrt{(4pp - 3ss)} = 2p - \frac{fs}{g}$, unde $\frac{p}{s} = \frac{ff + 3gg}{4fg}$.

Sumatur ergo $p = q = ff + 3gg$ et $s = 4fg$ eritque

$2r + s = 2r + 4fg = \pm 2(ff + 3gg) - 4ff$, hincque vel $r = 3gg - ff - 2fg$, vel $r = 3(gg - ff) - 2fg$.

Varii numeri formae $xy(xx - yy)$, qui eosdem factores non quadratos involvunt, sub littera F contentos, in hac tabella exhibuntur:

F	x	y
2.3	2	1
	25	24
2.3.5.7	5	2
	6	1
	8	7
2.3.5.11	6	5
	8	3
	27	22
2.7.11	9	2
	18	7
3.5.7.11	11	4
	16	5

Conjectura. Posito $xy(xx - yy) = A^2 F$, inter numeros F videntur omnes numeri primi, vel ipsi, vel eorum dupla, vel ambo interdum occurrere, excepto scilicet binario, uti ex hac tabula colligere licet:

F	x	y
2.3	2	1
5	5	4
7	16	9
2.7	9	7
2.11	50	49
13	325	36
2.17	9	8
2.19	1250	289
23	156^2	133^2
2.23	121	23
29	29.13^2	70^2
31	1600	81

Problema. Invenire numeros p, q, x, y, ut sit $pq(pp - qq) = nxy(xx - yy)\,\square$, pro quolibet numero dato n.

Solutio tantum particularis tradi potest, et calculis satis molestis expeditis inveni sequentes valores

$$p = s^4 - 20nsstt - 8nnt^4 \qquad p + q = 2(ss - ntt)(ss - 4ntt)$$
$$q = (ss + 2ntt)(ss + 8ntt) \qquad p - q = 6ntt(5ss + 4ntt)$$
$$x = s^4 - 20nsstt - 8nnt^4 \qquad x + y = (5ss + 4ntt)(ss - 4ntt)$$
$$y = 4(ss - ntt)(ss + 2ntt) \qquad x - y = 3ss(ss + 8ntt).$$

Hic enim rejectis factoribus quadratis formula $xy(xx - yy)$ omnes continet factores alterius $pq(pp - qq)$, ac praeterea factorem n haec posterior continebit. Notandum hic est numerum n tam positive quam negative accipi posse. Deinde etiam valores singularum harum litterarum semper positivi capi possunt, etiamsi prodeant negativi.

Ita sumto $s=1$ et $t=1$ pro casu $n=2$ erit $p=71$, $q=85$, $x=71$, $y=20$. Hic loco p et q eorum semisumma et semidifferentia sumi possunt fietque $p=2.3.13$ et $q=7$. Hinc enim fiet

$$pq(pp-qq)=2.3.5.7.13.17.71 \quad \text{et} \quad xy(xx-yy)=3.5.7.13.17.71.$$

Sin autem sumatur $n=-2$ manente $s=t=1$, prodit $p=9$, $q=5.9$, sive $p=1$ et $q=5$, sive etiam $p=3$ et $q=2$, tum vero erit $x=9$, $y=4.9$, sive $x=1$ et $y=4$, tum enim erit

$$pq(pp-qq)=2.3.5 \quad \text{et} \quad xy(xx-yy)=3.5.$$

A. m. T. III. p. 121—123.

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b) *Quaestiones ad resolutionem plurium aequationum ducentes.*

70.

(*W. L. Krafft.*)

Problema. Efficere, ut fiat $x+y+z=\square$ et $xyz=1$, ideoque $z=\frac{1}{xy}$.

Sequentes solutiones particulares prodierunt

$$\text{I.}\quad x=\frac{1}{2},\qquad y=\frac{9}{2},\qquad z=\frac{4}{9}$$

$$\text{II.}\quad x=\frac{2}{9},\qquad y=\frac{50}{9},\qquad z=\frac{81}{100}$$

$$\text{III.}\quad x=\frac{13}{392},\qquad y=\frac{117}{392},\qquad z=\frac{392^2}{9.13^2}$$

$$\text{IV.}\quad x=\frac{2}{5},\qquad y=\frac{18}{5},\qquad z=\frac{25}{36}$$

$$\text{V.}\quad x=-2,\qquad y=-\frac{2}{9},\qquad z=\frac{9}{4}$$

$$\text{VI.}\quad x=\frac{1}{6},\qquad y=\frac{1}{12},\qquad z=72.$$

Si ratio x ad y sumatur $a:b$, et ponatur $x=ma$ et $y=mb$, erit $z=\frac{1}{mmab}$, unde fieri debet

$$m(a+b)+\frac{1}{mmab}=\square,\quad \text{sive}\quad m^3(a+b)+\frac{1}{ab}=\square,$$

seu

$$m^3aabb(a+b)+ab=\square.$$

A. m. T. I. p. 121.

71.

(*Lexell.*)

Ut formulae $aa+Mbb$ et $aa+Nbb$ quadrata reddi queant, posito $M=m\mu$ capiatur $N=(\alpha pp+m)(\alpha qq+\mu)$ sicque pro dato numero M infiniti valores idonei pro N reperiuntur, veluti si $M=1$, ideoque $m=\mu=\pm 1$ erit $N=(\alpha pp\pm 1)(\alpha qq\pm 1)$. Si $p=1$, erit $N=(\alpha\pm 1)(\alpha qq\pm 1)$, cujusmodi formulae sunt

$$\begin{array}{ll} \text{pro } \alpha=1, & N=2(qq+1) \\ \alpha=2, & N=3(2qq+1),\quad 2qq-1 \\ \alpha=3, & N=4(3qq+1),\quad 2(3qq-1) \\ \alpha=4, & N=5(4qq+1),\quad 3(4qq-1). \end{array}$$

Ut formulae $aa+mmbb$ et $aa+nnbb$ reddi possint quadrata, numeros m et n ex talibus formis sumi oportet, (ubi quidem ratio inter m et n est definienda)

$$pq(rr-1),\quad pr(qq-1),\quad qr(pp-1),\quad p(qq-rr),\quad q(pp-rr),\quad r(pp-qq)$$
$$p(qqrr-1),\quad q(pprr-1),\quad r(ppqq-1),\quad ppqq-rr,\quad pprr-qq,\quad qqrr-pp,$$
$$ppqqrr-1,$$

quae formulae omnes ita sunt comparatae, ut si earum quadratis addatur idem quadratum $4ppqqrr$, proveniant quadrata.

A. m. T. I. p. 122.

Problema. Dato numero A, invenire conditiones numeri N, ut ambae istae formulae $xx+Ayy$, $xx+Nyy$ simul fieri possint quadrata.

Solutio. Ponatur $A=\mu\nu$, pro casu scilicet, quo habet factores, et priori formae satisfiet sumendo $x=\mu pp-\nu qq$ et $y=2pq$, tum enim erit $xx+Ayy=(\mu pp+\nu qq)^2$. Simul vero etiam altera formula evadet quadratum, si fuerit $N=mmppqq+m(\mu pp-\nu qq)$, tum enim erit

$$xx+Nyy=(\mu pp-\nu qq)^2+4mppqq(\mu pp-\nu qq)+4mmp^4q^4=(\mu pp-\nu qq+2mppqq)^2,$$

erit ergo $N=mmppqq+m(\mu pp-\nu qq)$, ubi m pro lubitu assumere licet. Quin etiam pro m fractiones assumere licet, ita ut pro N nihilominus prodeant numeri integri, sumto enim $m=n+\frac{\nu}{pp}$, tum enim fiet

$$N=\left(n+\frac{\nu}{pp}\right)(nppqq+\mu pp)=(npp+\nu)(nqq+\mu).$$

A. m. T. I. p. 128.

72.

(N. Fuss 1.)

Problema Diophanteum. Invenire numerum x ut his duabus conditionibus satisfiat

$$xx+2ax+mmc=\square \quad\text{et}\quad xx+2bx+nnc=\square,$$

cujus solutio particularis est

$$x=\frac{(nna-mmb)^2-mmnn(m-n)^2c}{2mn(m-n)(na-mb)}.$$

Ita si proponantur hae duae formulae: $xx+2ax+c=\square$ et $xx+2bx+c=\square$, sumatur $m=1$ et $n=-1$

eritque $x=\dfrac{(a-b)^2-4c}{4(a+b)}$.

A. m. T. II. p. 154.

73.

Problema. Resolvere has duas aequalitates:

$$(2+2a)xx+(2-2a)yy=4AA \quad\text{et}\quad (2+2b)xx+(2-2b)yy=4BB.$$

Solutio. Hinc ergo primo erit $4A^2+4B^2=(4+2(a+b))xx+(4-2(a+b))yy$; posito ergo

$$a+b=2c,\quad\text{erit}\quad A^2+B^2=(1+c)xx+(1-c)yy.$$

Deinde vero erit $4A^2-4B^2=2(a-b)(xx-yy)$, unde posito $a-b=2d$, erit $A^2-B^2=d(xx-yy)$. Statuatur ergo $A+B=x+y$ eritque $A-B=d(x-y)$. Addantur quadrata eritque

$$2A^2+2B^2=(x+y)^2+dd(x-y)^2=2(1+c)xx+2(1-c)yy,$$

quae evoluta fit $(1+dd)xx+(1-dd)2xy+(1+dd)yy=2(1+c)xx+2(1-c)yy$, sive

$$(dd-2c-1)xx+2(1-dd)xy+(dd+2c-1)yy=0,$$

sive $$(dd-1)(xx-2xy+yy)-2c(xx-yy)=0,$$

quae commode per $x-y$ divisa fit $(dd-1)(x-y)-2c(x+y)=0$, unde reperitur

$$\frac{x}{y}=\frac{dd+2c-1}{dd-2c-1}.$$

Sumatur ergo $x=dd+2c-1$ et $y=dd-2c-1$, fietque

$$A+B=2(dd-1) \quad \text{et} \quad A-B=4cd, \quad \text{ergo} \quad A=dd+2cd-1 \quad \text{et} \quad B=dd-2cd-1.$$

Haec autem solutio tantum est particularis.

PROBLEMA. Resolvere has duas aequalitates:

$$xx+2axy+nccyy=A^2 \quad \text{et} \quad xx+2bxy+nddyy=B^2.$$

SOLUTIO. Eliminetur littera n, quaerendo $A^2dd-B^2cc=xx(dd-cc)+2xy(add-bcc)$. Ponatur nunc $Ad+Bc=x(d+c)$, eritque $Ad-Bc=x(d-c)+2y\left(\frac{add-bcc}{d+c}\right)$, unde erit $2Ad=2dx+2y.\frac{add-bcc}{d+c}$, unde fit $A=x+y.\frac{add-bcc}{d(d+c)}$, quo valore substituto erit

$$2ax+nccy=2x.\frac{add-bcc}{d(d+c)}+y\left(\frac{add-bcc}{d(d+c)}\right)^2,$$

$$\text{unde fit} \quad \frac{x}{y}=\frac{(add-bcc)^2-nccdd(d+c)^2}{2cd(d+c)(ad+bc)}.$$

A. m. T. II. p. 157.

74.

RESOLUTIO aequationum

$$xx+2axy+cyy=A^2 \quad \text{et} \quad xx+2bxy+dyy=B^2.$$

Cum sit $A^2-B^2=2(a-b)xy+(c-d)yy$, sumatur $A-B=(a-b)y$, erit

$$A+B=2x+\frac{c-d}{a-b}y, \quad \text{hinc additis quadratis erit}$$

$$2(AA+BB)=4xx+4.\frac{c-d}{a-b}xy+\left(\frac{c-d}{a-b}\right)^2yy$$
$$+(a-b)^2yy.$$

Cum igitur $2(AA+BB)=4xx+4(a+b)xy+2(c+d)yy$, inde sequitur

$$4.\frac{c-d}{a-b}x-4(a+b)x+\left[\left(\frac{c-d}{a-b}\right)^2+(a-b)^2-2(c+d)\right]y=0, \quad \text{hincque fit}$$

$$\frac{x}{y}=\frac{(c-d)^2+(a-b)^4-2(c+d)(a-b)^2}{4(a-b)(aa-bb-c+d)}.$$

Sumi ergo poterit $x=(c-d)^2+(a-b)^4-2(c+d)(a-b)^2$ et $y=4(a-b)(aa-bb-c+d)$. Haec solutio differt ab ea, quae supra est tradita, ubi loco c et d habuimus ncc et ndd, quod mirum non est, cum utraque solutio tantum sit particularis.

Eodem etiam modo hae aequalitates resolvi possunt

$$(2+a)xx+(nn-4)xy+(2-a)yy=A^2 \quad \text{et} \quad (2+b)xx+(nn-4)xy+(2-b)yy=B^2.$$

Facta enim simili operatione, dividi poterit per $x-y$, ac reperietur inde

$$x=n^4-8nn+(a-b)^2+2nn(a+b) \quad \text{et} \quad y=n^4-8nn+(a-b)^2-2nn(a+b).$$

In his autem formulis continetur casus supra tractatus, quando $n=2$.

A. m. T. II. p. 158.

75.

Problema. Invenire quatuor numeros positivos x, y, z, ν, inter se primos, quorum tam summa quam summa quadratorum sit biquadratum.

Solutio. Positis $x = aa + bb + cc - dd$, $y = 2ad$, $z = 2bd$, $\nu = 2cd$, erit

$$xx + yy + zz + \nu\nu = (aa + bb + cc + dd)^2.$$

Ut vero fiat biquadratum, sumatur $a = pp + qq + rr - ss$, $b = 2ps$, $c = 2qs$, $d = 2rs$ eritque

$$aa + bb + cc + dd = (pp + qq + rr + ss)^2.$$

Ut vero etiam ipsa summa $x + y + z + \nu$ fiat quadratum, hoc fiet sumendo $p = s + \frac{3}{2}r - q$. Hoc enim modo erit $\sqrt{(x + y + z + \nu)} = 2qq - 3qr - 2qs + \frac{13}{4}rr + 5rs + 2ss$. Quae quantitas ut denuo fiat quadratum, posito $q = r + t$ reperitur $r = \frac{uu - 2tt + 2ts - 2ss}{t + 3s + 3u}$; hocque modo problemati satisfiet. Hinc sequens exemplum

$p = 10$	$a = 3$	$x = 409$	$x^2 = 167281$	$x + y + z + \nu = 625 = 5^4$
$q = 2$	$b = 20$	$y = 24$	$y^2 = 576$	$xx + yy + zz + \nu\nu = 21^4$.
$r = 2$	$c = 4$	$z = 160$	$z^2 = 25600$	
$s = 9$	$d = 4$	$\nu = 32$	$\nu^2 = 1024$	

Problema. Invenire quinque numeros positivos et inter se primos x, y, z, ν, u, quorum tam summa quam summa quadratorum sit biquadratum.

Solutio. Statuatur $x = aa + bb + cc + dd - ee$, $y = 2ae$, $z = 2be$, $\nu = 2ce$, $u = 2de$ eritque summa

$$x^2 + y^2 + z^2 + \nu^2 + u^2 = (a^2 + b^2 + c^2 + d^2 + e^2)^2.$$

Praeterea capiatur $a = pp + qq + rr + ss - tt$, $b = 2pt$, $c = 2qt$, $d = 2rt$, $e = 2st$; hocque modo fiet

$$x^2 + y^2 + z^2 + \nu^2 + u^2 = (p^2 + q^2 + r^2 + s^2 + t^2)^2.$$

Jam ut etiam ipsa summa fiat quadratum, sumi debet $p = -q - r + \frac{3}{2}s + t$ atque radix summae erit

$$pp + qq + rr + ss + tt + 2st,$$

quae denuo evadet quadratum posito $r = q + s + g$, si capiatur

$$s = \frac{6qq - 4qt + tt + 6gq - 2gt - 2ft + 2gg - ff}{3f - g},$$

ubi quatuor litterae q, t, f, g nostro arbitrio relinquuntur, quas facile ita accipere licet, ut numeri quaesiti fiant positivi.

A. m. T. III. p. 125. 126.

76.

Problema. Invenire duos numeros positivos et inter se primos, x et y, quorum summa sit quadratum, summa autem quadratorum cubus.

Tales numeri simpliciores sunt $x = 29601 = 9.11.13.23$, $y = 25624 = 8.3203$, unde $x + y = 235^2$, $xx + yy = 1153^3$.

Analysis. Cum debeat esse $xx + yy = p^3$, necesse est, ut sit $p = aa + bb$; tum vero evadit $x = a(aa - 3bb)$ et $y = b(3aa - bb)$. Hinc autem erit $x + y = a^3 + 3aab - 3abb - b^3 = (a - b)(aa + 4ab + bb) = \square$. Fiat igitur $a - b = cc$ eritque $x + y = cc(6bb + 6bcc + c^4) = \square$. Sit igitur $c^4 + 6bcc + 6bb = \left(cc + 3b\frac{f}{g}\right)^2$, unde haec relatio

$$\frac{b}{cc} = \frac{2g(f - g)}{2gg - 3ff} = \frac{2g(g - f)}{3ff - 2gg}.$$

Quia numeri x et y debent esse positivi, necesse est, ut sit $aa > 3bb$, sive $a > b\sqrt{3}$, ergo $b + cc > b\sqrt{3}$, sive $cc > b(\sqrt{3} - 1)$, ideoque $\frac{b}{cc} < \frac{1}{\sqrt{3}-1} < \frac{\sqrt{3}+1}{2}$. Ponatur igitur

$$b = 2g(g-f) \quad \text{et} \quad cc = 3ff - 2gg, \quad \text{ergo} \quad 3ff - 2gg = \square.$$

Huic satisfit sumendo $f = 11$ et $g = 1$, vel etiam sumto $f = -3$ et $g = 1$, unde fit $c = 5$, $b = 8$, $a = 33$ hinc $x = 29601$ et $y = 25624$.

Si quaerantur tres numeri x, y, z, positivi et primi inter se, quorum summa sit quadratum, quadratorum vero summa cubus, tales numeri sunt

$$35 + 9 + 5 = 7^2 \quad \text{et} \quad 35^2 + 9^2 + 5^2 = 11^3;$$

Simili modo etiam $\quad 67 + 9 + 5 = 9^2 \quad \text{et} \quad 67^2 + 9^2 + 5^2 = 19^3;$

at vero methodus tales numeros inveniendi adhuc latet.

A. m. T. III. p. 128.

77.

Problema. Has duas formulas $xx + abyy$ et $xx + cdyy$ ad quadratum reducere.

Solutio. Pro priore ponatur $x = \zeta(app - bqq)$, erit $y = 2\zeta pq$; pro altera ponatur $x = \eta(crr - dss)$ eritque $y = 2\eta rs$. Ponatur igitur $\zeta pq = \eta rs = \zeta\eta fghk$, fiatque $p = \eta fg$, erit $q = hk$ et $r = \zeta fh$ et $s = gk$. Qui valores pro x substituti praebent

$$\frac{gg}{hh} = \frac{\zeta}{\eta} \cdot \frac{\zeta\eta cff + bkk}{\zeta\eta aff + dkk}. \text{ Quadratum ergo esse debet } \zeta\eta(\zeta\eta cff + bkk)(\zeta\eta aff + dkk),$$

vel posito $\zeta\eta = \vartheta$, quadratum esse debet $\vartheta(\vartheta cff + bkk)(\vartheta aff + dkk)$, vel loco ϑ scribamus $\frac{\mu}{\nu}$, fieri debet $\mu\nu(\mu cff + \nu bkk)(\mu aff + \nu dkk)$, quod si reddi queat quadratum, tunc etiam ambae formulae propositae fiunt quadrata. Ubi scilicet litterae f, k, μ, ν pro lubitu accipi possunt.

A. m. T. III. p. 136.

Ut hae duae formulae $xx + nyy$ et $yy + nxx$ simul quadrata reddi queant, necesse est, ut numerus n in sequenti formula contineatur:

$$n = \frac{(s + xx - yy)(s - xx + yy)(s + xx + yy)(s - xx - yy)}{4ssxxyy}.$$

Quodsi fuerit $n = aa + 3$, tum ambae formulae quadrata reddi possunt; tum enim sumto $x = a + 1$ et $y = a - 1$ fiet $nxx + yy = (aa + a + 2)^2$ et $nyy + xx = (aa - a + 2)^2$. Hinc solus casus $a = 1$ excipitur; tum enim ob $n = 4$ foret $y = 0$. Praeterea vero innumeri alii dantur casus pro n, pro quibus inveniendis quaeratur numerus $c = \frac{xx + yy}{2d} \cdot \frac{dd - 1}{xy}$, tum enim semper erit $n = cc - dd + 1$.

Theorema. Hae duae formulae $xx + nyy$ et $yy + nxx$ simul quadrata reddi nequeunt, nisi $n + 1$ sit summa duorum quadratorum.

Demonstratio. Posito $xx + nyy = pp$ et $yy + nxx = qq$ erit $pp + qq = (n + 1)(xx + yy)$. Constat autem summam duorum quadratorum $pp + qq$ alios divisores non continere posse, nisi qui ipsi sint summae duorum quadratorum. Quoties ergo $n + 1$ non fuerit summa duorum quadratorum, quod fit his numeris pro n assumtis:

2, 5, 6, 8, 10, 11, 13, 14, 17, 18, 20, 21, 22, 23, 26, etc. ambae formulae simul quadrata fieri nequeunt.

Cum his formulis $7xx+yy=pp$ et $7yy+xx=qq$ satisfiat sumendo $x=3$ et $y=1$, unde fit $p=8$ et $q=4$, innumerabiles alii dabuntur valores pro x et y, ex quibus magno labore hos eruimus: $x=1121$ et $y=477$, unde fit $p=3004$ et $q=1688$, uti ex hoc schemate apparet:

$$\begin{array}{rlrl} xx = & 3256641 & 7xx = & 8796487 \\ 7yy = & \underline{1592703} & yy = & \underline{227529} \\ qq = & 2849344 & pp = & 9024016 \\ q = & 1688 & p = & 3004. \end{array}$$

A. m. T. III. p. 146.

77.

PROBLEMA. Invenire quatuor quadrata xx, yy, zz, $\nu\nu$, ut sit

$$xxyy-zz\nu\nu=A^2, \quad xxzz-yy\nu\nu=B^2, \quad yyzz-xx\nu\nu=C^2.$$

SOLUTIO. Quaeratur formula F, quae addita producat quadrata. Talis est

$$F=x^4+y^4+z^4-2xxyy-2xxzz-2yyzz+2xx\nu\nu+2yy\nu\nu+2zz\nu\nu+\nu^4,$$

cui si addatur $4xxyy-4zz\nu\nu$ prodit quadratum $(xx+yy-zz+\nu\nu)^2$.

Si addatur $4xxzz-4yy\nu\nu$ oritur $(xx+zz-yy+\nu\nu)^2$
et addito $4yyzz-4xx\nu\nu$ prodit $(yy+zz-xx+\nu\nu)^2$.

Hinc ergo facto $F=0$, erit

$$A=\frac{xx+yy-zz+\nu\nu}{2}, \quad B=\frac{xx+zz-yy+\nu\nu}{2}, \quad C=\frac{yy+zz-xx+\nu\nu}{2}.$$

Fiat igitur $F=0$, et per extractionem quaeratur $\nu\nu$, eritque

$$\nu\nu=-xx-yy-zz+2\sqrt{(xxyy+xxzz+yyzz)}.$$

Ponatur ergo $\sqrt{(xxyy+xxzz+yyzz)}=S$, ut fiat $\nu\nu=2S-xx-yy-zz$. Fingatur autem $S=xy+tz$, ita ut $SS=xxyy+2xytz+ttzz=xxyy+xxzz+yyzz$, unde fit

$$z=\frac{2txy}{xx+yy-tt}, \quad \text{hincque} \quad S=\frac{xy\,(xx+yy+tt)}{xx+yy-tt}.$$

Verum hi valores substituti pro $\nu\nu$ dant expressionem inextricabilem. Fieret enim sex dimensionum. Necesse ergo est, ut rem ad formulas simpliciores reducamus.

CASUS I. Sumatur $t=x-y$ fietque $z=x-y$ et $S=xx-xy+yy$, hincque porro $\nu\nu=0$, qui ergo casus prorsus est inutilis.

CASUS II. Sumatur $t=y$ fietque $z=\frac{2yy}{x}$ et $S=\frac{xxy+2y^3}{x}$.

ALIUS CONATUS. Ex aequatione $F=0$ quaeratur valor ipsius xx, qui est

$$xx=yy+zz-\nu\nu\pm2\sqrt{(yyzz-yy\nu\nu-zz\nu\nu)}=2S+yy+zz-\nu\nu.$$

Quia hic yy multiplicatur per $zz-\nu\nu$, pro z et ν tales valores sumantur, ut $zz-\nu\nu$ fiat quadratum, quod fit sumendo $z=5$ et $\nu=3$; tum erit $S=\sqrt{(16yy-225)}=4y-t$, unde $y=\frac{225+tt}{8t}$ et $S=\frac{225-tt}{2t}$. Tum igitur erit $xx=yy+16\pm2S$. Sumatur $t=5$, erit $y=\frac{25}{4}$ et $S=20$, ergo $xx=\frac{625}{16}+16\pm40=\frac{39^2}{4^2}$, ergo

$x=\frac{39}{4}$, $y=\frac{25}{4}$, $z=5$ et $v=3$, sive $x=39$, $y=25$, $z=20$, $v=12$. Tentemus etiam casum $t=3$, erit $y=\frac{39}{4}$ et $S=36$, ergo $xx=\frac{625}{16}$, ergo $x=\frac{25}{4}$, unde prodit casus praecedens.

Possemus etiam sumere $z=5$ et $v=4$: erit $S=\sqrt{(9yy-400)}=3y-t$, hinc

$$y=\frac{400+tt}{6t} \quad \text{et} \quad S=\frac{400-tt}{2t}, \quad \text{hinc} \quad xx=yy+9\pm 2S.$$

Sumatur $t=10$, erit $y=\frac{25}{3}$ et $S=15$, hinc $xx=$ incongruo.

Ex priore solutione $vv=2S-xx-yy-zz$ existente $z=\frac{2txy}{xx+yy-tt}$, $S=\frac{xy\,(x^2+y^2+t^2)}{x^2+y^2-t^2}$. Sumatur $x=5$ et $y=4$, fiet $z=\frac{40t}{41-tt}$ et $S=\frac{20\,(41+tt)}{41-tt}$. Porro si $t=\frac{13}{3}$, solutio supra data oritur, ex quo casu derivavi sequentem:

$$t=\frac{185}{153}, \quad \text{eritque} \quad z=\frac{5.185.153}{115693} \quad \text{et} \quad S=\frac{5.496997}{115693}.$$

A. m. T. III. p. 117. 118.

78.

Problème. Trouver trois nombres x, y, z tels, que le carré de chacun avec le produit des deux autres fasse un carré.

Solution. Qu'on pose, pour les deux premières conditions $xx+yz=pp$ et $yy+xz=qq$, et l'on aura $pp-qq=(x-y)(x+y-z)$. Soit donc $p-q=x-y$ et $p+q=x+y-z$, d'où l'on tire $p=x-\frac{1}{2}z$. Cette valeur substituée dans la première équation donne $z=4\,(x+y)$. Maintenant la troisième équation sera

$$16\,(x+y)^2+xy=\square.$$

Donc la racine sera plus grande que $4\,(x+y)$. Soit cette racine

$$4x+4y+s, \quad \text{et il y aura} \quad 8sx+8sy-xy=-ss.$$

Ajoutons de part et d'autre $-64ss$, pour avoir

$$(x-8s)(y-8s)=65ss=\frac{5ts}{u}\cdot\frac{13us}{t}.$$

Soit $x-8s=\frac{5ts}{u}$ et $y-8s=\frac{13us}{t}$, et pour ôter les fractions, supposons $s=tu$, et l'on aura $x=8tu+5tt$ et $y=8tu+13uu$. De là $z=4\,(x+y)=64tu+20tt+52uu$.

Exemple. Soit $t=1$ et $u=1$, et il y aura $x=13$, $y=21$, $z=136$, car alors

$$136^2+13.21=137^2, \quad 21^2+13.136=47^2, \quad 13^2+21.136=55^2.$$

Autre solution. Pour la première formule qu'on prenne $x=\frac{yz-ss}{2s}$, pour avoir $xx+yz=\left(\frac{yz+ss}{2s}\right)^2$. La seconde $yy+xz=\square$ donne $\frac{2syy+yzz-ssz}{2s}=\square=(y+pz)^2$, d'où l'on tire

$$y=\frac{2ppsz+ss}{z-4ps}, \quad \text{et de là} \quad x=\frac{p\,(pzz+2ss)}{z-4ps}.$$

Ces valeurs étant substituées dans la troisième équation $zz+xy=\square$, celle-ci deviendra

$$z^4+(2p^4-8p)\,sz^3+17ppsszz+4p^3s^3z+2ps^4=\square.$$

Pour rendre carré le dernier terme, je prends $p=2$, et la formule sera

$$z^4+16sz^3+68sszz+32s^3z+4s^4=\square.$$

Mais on s'aperçoit d'abord, qu'elle est carrée et que sa racine est $zz+8sz+2ss$; par conséquent z demeure arbitraire. Donc puisque $p=2$, nous aurons $y=\frac{8sz+ss}{z-8s}$ et $x=\frac{4zz+4ss}{z-8s}$, et pour ôter les fractions, mettons dans ces formules t au lieu de z, et ainsi on pourra multiplier tous ces nombres par $t-8s$ et l'on aura

$$x=4(tt+ss),\quad y=s(8t+s)\quad \text{et}\quad z=t(t-8s),$$

d'où il est clair que pour t il faut prendre une valeur $>8s$. En prenant $s=1$ et $t=9$, on aura $x=328$, $y=73$, $z=9$, plus grands que les précédents.

Cependant les deux solutions s'accordent; mais pour avoir le cas le plus simple, il faut prendre $t=13$ et $s=1$; car alors on aura

$$x=680,\quad y=105,\quad z=65,\quad \text{ou bien}\quad x=136,\quad y=21,\quad z=13.$$

A. m. T. III. p. 145.

79.

Ad PROBLEMA, quo quaeruntur tres numeri x, y, z, ut quadratum cujusque una cum producto reliquorum faciat quadratum, cujus solutio specialis facile invenitur haec $x=aa-8ab$, $y=bb+8ab$, $z=4aa+4bb$.

Generaliter statui potest $x=aa+2b$, $y=bb+2a$, $z=ab(ab-4)$, quibus satisfit duabus conditionibus $xx+yz=\square$, $yy+xz=\square$, et ut tertiae quoque $zz+xy=\square$ satisfiat, fieri debet

$$a^4b^4-(8a^3-2)b^3+17a^2b^2+4ab+2a^3=\square.$$

Hinc istos valores inveni: $x=33$, $y=185$, $z=608$, tum vero $x=297$, $y=377$, $z=320$.

A. m. T. III. p. 176.

80.

PROBLEMA. Invenire tria quadrata pp, qq, rr, ut semisumma binorum sit quadratum, scilicet

$$\frac{pp+qq}{2}=zz,\quad \frac{pp+rr}{2}=yy,\quad \frac{qq+rr}{2}=xx.$$

Hinc solutiones simpliciores hujus problematis erunt hae quinque

$$\begin{array}{rrrrr} p=\ 89, & 97, & 119, & 23, & 17\\ q=191, & 553, & 833, & 289, & 697\\ r=329, & 833, & 1081, & 527, & 1127. \end{array}$$

Directa autem hujus problematis solutio ita se habet:

$$p=(ff-2gg)(ss-2tt),\quad q=(ff+2gg+4fg)(2tt+ss+4st)-8fgst$$

Quia hic omnes litterae tam negative quam positive accipi possunt, haec formula plures admittit variationes, quarum una pro q, altera pro r accipi potest. Quo facto necesse est, ut $\frac{qq+rr}{2}$ reddatur quadratum. Praeterea vero notetur, quemlibet numerum formae $aa-2bb$ infinitis modis per similes formas exprimi posse. Ita formula $aa-2bb$ infinitis modis fieri potest $=\pm 1$, scilicet ponendo

$$\begin{array}{l} a=1,\ 3,\ 7,\ 17,\ 41,\ 99,\ \text{etc.}\\ b=1,\ 2,\ 5,\ 12,\ 29,\ 70,\ \text{etc.} \end{array}$$

Cum igitur numeri $ff-2gg$ et $ss-2tt$ infinitis modis per similes formas exprimi queant, formula pro q et r data infinities infinitis modis variari poterit. Si enim fuerit

$$aa-2bb=\pm 1,\quad \text{erit}\quad ff-2gg=(af\pm 2bg)^2-2(ag\pm bf)^2,$$

ita tamen, ut p eundem valorem retineat. At vero hoc modo quaelibet solutio particularis satis difficilem postulat evolutionem; unde praecedens solutio longissime praecellit, cum sumtis numeris c et d pro lubitu valores jam evolutos pro litteris p, q, r suppeditet. Hic etiam notasse juvabit infinitis modis fieri posse

$$aa - 2bb = 2cc - dd.$$

Cum enim fieri debeat $aa + dd = 2(bb + cc)$, hoc fiet sumendo $a = b + c$ et $d = b - c$. Erit ergo

$$c = a - b \quad \text{et} \quad d = 2b - a.$$

A. m. T. III. p. 178. 179.

81.

(Lexell.)

Problema de inveniendis quotcunque numeris p, q, r, s, t, etc., quorum quilibet ductus in summam reliquorum faciat quadratum, facile tentando sine analysi resolvi potest. Sit enim S summa omnium, et cum $p(S-p)$ debeat esse quadratum, ideoque $pS = pp + \square$, evidens est tam p quam S esse debere summam duorum quadratorum, ideoque pro S sumi conveniet talem numerum, qui pluribus modis in bina quadrata se resolvi patiatur, cujusmodi est $S = 130$, qui in binas partes secari debet, quarum productum sit quadratum; quandoquidem si una pars sit p, altera erit $S - p$, tales resolutiones hoc modo exhibemus:

$$S = 130, \qquad \begin{array}{r|cccccccc} p & 2, & 5, & 13, & 26, & 32, & 40, & 49, & 65 \\ S-p & 128, & 125, & 117, & 104, & 98, & 90, & 81, & 65 \end{array}$$

ubi notandum, valores ipsius p ex utraque columna sumi posse, ex his igitur excerpi oportebit vel ternos numeros, vel quaternos, vel quinos, vel senes etc., quorum summa faciat 130, ut sequitur: ternio 32, 49, 49. Sicque quinque habemus numeros 2, 5, 26, 32, 65, quorum quilibet in summam reliquorum ductus, producit quadratum. Alii quini: 2, 13, 26, 40, 49. Alii numeri idonei pro S assumendi, qui plurimas resolutiones admittunt, sunt 2210

$$S = 2210, \qquad \begin{array}{r|cccc} p & 1, & 5, & 13 & \ldots \\ S-p & 2209, & 2205, & 2197 & \ldots \end{array}$$

A. m. T. I. p. 112.

82.

Consideratio circa quadrata magica.

I. Quadrata magica facile eo reduci possunt, ut summae per columnas tam horizontales quam verticales evanescant, quod fit admittendo etiam numeros negativos; scilicet si quadratum fuerit impar, numeri inscribendi erunt 0, ± 1, ± 2, ± 3, ± 4, etc., sin autem quadratum fuerit par, numeri inscribendi erunt ± 1, ± 3, ± 5, ± 7, etc. Quodsi enim ad singulos addatur numerus quidam impar, et summae per 2 dividantur, orientur numeri naturales; ita si ad singulos illos numeros addatur 9, semisses dabunt hos numeros ordine 4, 5, 3, 6, 2, 7, 8, 1

II. In omni quadrato magico quaterna loca *connexa* voco, quando duo pro lubitu in columna quadam horizontali accipiuntur, quorum illud primum, alterum secundum voco, duo reliqua vero in alia columna horizontali ita accipiuntur, ut primum et tertium, itemque secundum et quartum in ea columna verticali existant, sive ut lineae 1...2 et 3....4 sint horizontales, rectae vero 1....3 et 2....4 verticales. Jam sumtis hujusmodi quaternis locis, si primo inscribatur numerus quicunque $+a$, secundo $-a$, tertio $-a$ et quarto $+a$, homodo nec summa horizontalium nec verticalium mutatur; hic enim nondum respicio ad eam conditionem, qua etiam summae per diagonales eaedem, hoc est nostro casu $=0$ esse debent.

III. Proposito igitur quadrato, in areolas quotcunque more solito diviso; omnes areolae hoc modo litteris inscribantur, ubi nihil impedit, quominus in eandem areolam quandoque duae, vel tres pluresve litterae hoc modo inscribantur: tot scilicet litteras hoc modo inscribi oportet, ut deinceps omnes numeri revera inscribendi obtineri queant, simulque proprietas diagonalium adimpleatur.

IV. Ita si proponatur quadratum novem areolarum, litterae inscribantur, ut in schemate adjecto videre est

$+a$	$+b$	$-a$ $-b$
$+c$	$-b$ $-c$	$+b$
$-c$ $-a$	$+c$	$+a$

pro diagonalibus autem praeterea esse debet $2a-b-c=0$, $2a+2b+2c=0$, unde sequitur $a=0$ et $b+c=0$, seu $c=-b$; unde quadratum ita se habebit

0	$+b$	$-b$
$-b$	0	$+b$
$+b$	$-b$	0

hinc autem numeri propositi obtineri nequeunt.

V. Commode autem in hoc quadrato areola media vacua relinquitur, scilicet cyphra implenda, ac tum inscriptio litterarum ita se habebit

$+a$ $+b$	$-b$	$-a$
$+c$	0	$-c$
$-c$ $-b$ $-a$	$+b$	$+c$ $+a$

ubi diagonales dant $2a+b+c=0$, ideoque $c=-2a-b$, et numeri inscripti erunt

I. $a+b$	II. $-b$	III. $-a$
IV. $-2a-b$	V. 0	VI. $2a+b$
VII. a	VIII. $+b$	IX. $-a-b$

Nunc quaerantur numeri pro a et b sumendi, ut prodeant numeri ± 1, ± 2, ± 3, ± 4. Sit igitur $a=1$ et $b=2$ et habebitur hoc quadratum

3	−2	−1
−4	0	+4
1	2	−3

ac si ad singulos numeros addatur 5, oritur quadratum solitum:

8	3	4
1	5	9
6	7	2

VI. Tentetur quadratum sedecim areolarum. Proprietas diagonalium hic dat

$2a+2b+2e+2f=0$, $2a+2b+2g+2h=0$, ergo $f=-(a+b+e)$ et $h=-(a+b+g)$

et numeri inscripti sunt

I. $a+e$	II. $c-e$	III. $-a-b-c-g$	IV. $b+g$
V. $d-e$	VI. $b+e$	VII. $a+g$	VIII. $-a-b-g-d$
IX. $-d+g$	X. $-b-g$	XI. $-a-e$	XII. $a+b+d+e$
XIII. $-a-g$	XIV. $-c+g$	XV. $a+b+c+e$	XVI. $-b-e$.

Notentur ii, quorum negativa non occurrunt: II. $c-e$, III. $-a-b-c-g$, V. $d-e$, VIII. $-a-b-g-d$, IX. $-d+g$, X. $-b-g$, XII. $a+b+d+e$, XIV. $-c+g$, XV. $a+b+c+e$. Ut horum cuique socius comparetur, statuatur $g=e$, et nunc bini socii junctim repraesententur:

I. $a+e$, II. $c-e$, III. $-a-b-c-e$, IV. $b+e$	XI. $-a-e$, XIV. $-c+e$, XV. $a+b+c+e$, X. $-b-e$
V. $d-e$, VI. $b+e$, VII. $a+e$	IX. $-d+e$, XVI. $-b-e$, XIII. $-a-e$
VIII. $-a-b-d-e$	XII. $a+b+d+e$.

Hic autem quidam bis occurrunt. Verum haec methodus accuratiorem evolutionem postulat.

A. m. T. I. p. 28—30.

83.

N. Fuss I.

Eine LEICHTE REGEL, alle magische Quadrate von ungeraden Zahlen, die sich nicht durch 3 theilen lassen, zu verfertigen, in welchen nicht nur alle Horizontal- und Vertikalreihen nebst den beiden Diagonalen, wie gewöhnlich erfordert wird, sondern auch die den Diagonalen parallel gezogenen Zahlenreihen, wenn man sie nämlich durch die gegenüberstehenden, gleich weit entfernten ergänzt, einerlei Summen geben, und wobei man zugleich nach Belieben in jedem Fache anfangen kann.

Dies geschieht vermittelst des bekannten Springerganges im Schachspiel, nach welchem man immer in die folgende Vertikalcolumne abwärts fortgeht, wobei zu merken, dass, wenn man unten an das Ende gekommen, man von da hinaufspringt, und ebenfalls, wenn man auf der rechten Seite ans Ende gekommen, wiederum in die erste Columne linker Hand einschlägt. Wo aber die Stelle schon besetzt ist, prallt man links nach eben dem Gang abwärts zurück, wie aus folgendem Schema zu ersehen:

1	14	22	10	18
25	8	16	4	12
19	2	15	23	6
13	21	9	17	5
7	20	3	11	24

Hievon wird die Ursache deutlicher werden, wenn man eben diese Operation auf eine allgemeine Art mit lateinischen Lettern a, b, c, d, e und griechischen α, β, γ, δ, ε anstellt, und dabei diese Ordnung beobachtet, dass man nach $a\alpha$, $a\beta$, $a\gamma$, $a\delta$, $a\varepsilon$ auf $b\alpha$, $b\beta$, $b\gamma$, u. s. w. fortgeht

$e\delta$	$b\beta$	$d\varepsilon$	$a\gamma$	$c\alpha$
$d\gamma$	$a\alpha$	$c\delta$	$e\beta$	$b\varepsilon$
$c\beta$	$e\varepsilon$	$b\gamma$	$d\alpha$	$a\delta$
$b\alpha$	$d\delta$	$a\beta$	$c\varepsilon$	$e\gamma$
$a\varepsilon$	$c\gamma$	$e\alpha$	$b\delta$	$d\beta$

A. m. T. II. p. 237. 238.

D. Miscellanea.

84.

(*J. A. Euler.*)

Wie blos aus den dreieckigten Zahlen alle vieleckigten Zahlen leicht gefunden werden können.

Wenn die m-eckigte Zahl für die Seite n gefunden werden soll, so suche man die dreieckigte Zahl für eben die Seite n und auch die vorhergehende dreieckigte Zahl, für die Seite $n-1$; diese multiplicire man mit $m-3$ und zum Product addire man jene, so hat man die verlangte vieleckigte Zahl.

Denn für die Seite n ist die dreieckigte Zahl $=\frac{nn+n}{2}$ und für die vorhergehende Seite $n-1$ ist die Dreieckzahl $=\frac{nn-n}{2}$; also diese mit $m-3$ multiplicirt gibt $(m-3)\left(\frac{nn-n}{2}\right)$, hiezu $\frac{nn+n}{2}$ addirt gibt

$$\frac{(m-2)\,nn-(m-4)\,n}{2}.$$

Also wenn die 365-eckigte Zahl von 12 verlangt wird, so ist $m-3=362$, die Dreieckszahl für 12 ist 78, die für 11 ist 66, also die gesuchte Zahl wird sein

$$362.66+78=23970.$$

A. m. T. I. p. 237.