XIV.

Enodatio insignis cujusdam paradoxi circa multiplicationem angulorum observati.

1. Singularis est proprietas formularum, quibus cosinus angulorum multiplorum per cosinum anguli simpli exprimuntur. Si enim anguli simpli φ cosinus ponatur $= x$, angulorum multiplorum cosinus ita se habent:

$$\cos 0\varphi = 1$$
$$\cos 1\varphi = x$$
$$\cos 2\varphi = 2xx - 1$$
$$\cos 3\varphi = 4x^3 - 3x$$
$$\cos 4\varphi = 8x^4 - 8xx + 1$$
$$\cos 5\varphi = 16x^5 - 20x^3 + 5x$$
$$\cos 6\varphi = 32x^6 - 48x^4 + 18xx - 1$$
$$\cos 7\varphi = 64x^7 - 112x^5 + 56x^3 - 7x$$
$$\cos 8\varphi = 128x^8 - 256x^6 + 160x^4 - 32xx + 1$$
etc.

unde concluditur fore in genere

$$\cos n\varphi = 2^{n-1} x^n - 2^{n-3} nx^{n-2} + 2^{n-5} \frac{n(n-3)}{1.2} x^{n-4} - 2^{n-7} \frac{n(n-4)(n-5)}{1.2.3} x^{n-6} + \text{etc.},$$

seu

$$\cos n\varphi = 2^{n-1} x^n \left(1 - \frac{n}{4} x^{-2} + \frac{n(n-3)}{4.8} x^{-4} - \frac{n(n-4)(n-5)}{4.8.12} x^{-6} + \frac{n(n-5)(n-6)(n-7)}{4.8.12.16} x^{-8} - \text{etc.}\right)$$

ubi ratio progressionis facile perspicitur.

2. Neque vero hinc concludere licet, hanc seriem eadem lege in infinitum continuatam cosinum anguli $n\varphi$ exprimere, ita ut istius seriei infinitae summa futura sit $= \cos n\varphi$; sed quoties n est numerus integer, seriem eousque tantum continuari oportet, donec ad exponentes negativos ipsius x perveniatur, quippe qui termini omnes sunt rejiciendi, iis solis ab initio seriei terminis retentis, qui constant potestatibus positivis ipsius x, et numero absoluto, qui si n sit numerus par, est vel $+1$, vel -1. Nisi haec cautela observetur, in errorem delabimur; quin etiam casu $n = 0$ expressio generalis veritati adversatur; prodit enim $2^{-1}x^0 = \frac{1}{2}$, cum tamen sit $\cos 0\varphi = 1$, quod certe insigne est paradoxon.

*

3. Quo clarius etiam in reliquis casibus falsitas formae generalis perspiciatur, ponamus $n=1$ et haec forma evadet

$$x\left(1-\frac{1}{4}x^{-2}-\frac{1.2}{4.8}x^{-4}-\frac{1.3.4}{4.8.12}x^{-6}-\frac{1.4.5.6}{4.8.12.16}x^{-8}-\frac{1.5.6.7.8}{4.8.12.16.20}x^{-10}-\text{etc.}\right),$$

quae cum sit $<x$, cum veritate certe consistere nequit. Ut autem hujus seriei valor verus exploretur, ea ad hanc formam reducta:

$$x\left(1-\frac{1}{4}x^{-2}-\frac{1.1}{4.4}x^{-4}-\frac{1.1.3}{4.4.6}x^{-6}-\frac{1.1.3.5}{4.4.6.8}x^{-8}-\frac{1.1.3.5.7}{4.4.6.8.10}x^{-10}-\text{etc.}\right)$$

ita exhiberi potest:

$$x-\frac{1}{2}x\left(\frac{1}{2}x^{-2}+\frac{1.1}{2.4}x^{-4}+\frac{1.1.3}{2.4.6}x^{-6}+\frac{1.1.3.5}{2.4.6.8}x^{-8}+\text{etc.}\right).$$

Cum jam sit

$$\left(1-x^{-2}\right)^{\frac{1}{2}}=1-\frac{1}{2}x^{-2}-\frac{1.1}{2.4}x^{-4}-\frac{1.1.3}{2.4.6}x^{-6}-\frac{1.1.3.5}{2.4.6.8}x^{-8}-\text{etc.}$$

nostra series hac finita forma continetur:

$$x-\frac{1}{2}x\left(1-(1-x^{-2})^{\frac{1}{2}}\right)=\frac{1}{2}x+\frac{1}{2}x\sqrt{\left(1-\frac{1}{xx}\right)}=\frac{1}{2}x+\frac{1}{2}\sqrt{(xx-1)};$$

ita ut casu $n=1$, seriei nostrae generalis summa futura sit $=\frac{x+\sqrt{(xx-1)}}{2}$, cum tamen sit $\cos 1\varphi=x$. Quin etiam, cum sit $x<1$, patet seriei in infinitum continuatae summam adeo fore imaginariam.

4. Idem etiam de quolibet alio valore ipsius n ostendi potest, unde eo magis mirandum est, expressionem nostram generalem, si justa limitatione adhibeatur, ut omnes termini exponentes negativos ipsius x habituri rejiciantur, veritati esse consentaneam, et valorem ipsius $\cos n\varphi$ praebere, cum tamen omni extensione sumta et in infinitum continuata longe aliam atque adeo imaginariam summam sortiatur: cujusmodi singulare phaenomenon nescio an in aliis analyseos partibus jam sit observatum. Praeterea vero etiam, quod haud minus est mirandum, notari convenit, limitatione quoque illa adhibita, ut potestates negativae ipsius x rejiciantur, veritatem non obtineri, nisi n sit numerus positivus integer; si enim n esset numerus negativus, ob omnes potestates ipsius x prodeuntes negativas, error foret manifestissimus, cum sit $\cos(-n\varphi)=\cos n\varphi$.

5. Sin autem pro n accipiatur numerus positivus quidem seu fractus, nullo modo inde veritatem elicere licet. Sit enim $n=\frac{1}{2}$, et expressio nostra generalis hanc induet formam:

$$\frac{\sqrt{x}}{\sqrt{2}}\left(1-\frac{1}{8}x^{-2}-\frac{1.5}{8.16}x^{-4}-\frac{1.7.9}{8.16.24}x^{-6}-\frac{1.9.11.13}{8.16.24.32}x^{-8}-\text{etc.}\right)$$

unde etiamsi termini negativas potestates ipsius x complexuri, omnes scilicet, praeter primum, expungantur, tamen neutiquam inde obtinetur $\cos\frac{1}{2}\varphi$, quippe cum sit $\cos\frac{1}{2}\varphi=\sqrt{\frac{1+x}{2}}$. Multo minus autem reliquis terminis admissis veritati consulitur, dum series prodit formulae $\sqrt{\frac{1+x}{2}}$ minime aequalis.

6. Hinc igitur abunde liquet, quid de forma illa canonica

$$\cos n\varphi = 2^{n-1} x^n \left(1 - \frac{n}{4} x^{-2} + \frac{n(n-3)}{4.8} x^{-4} - \frac{n(n-4)(n-5)}{4.8.12} x^{-6} + \frac{n(n-5)(n-6)(n-7)}{4.8.12.16} x^{-8} - \text{etc.}\right)$$

apud plurimos auctores mirifice laudata, sit judicandum. Ea scilicet veritati nunquam est consentanea, nisi hae restrictiones adhibeantur: primo, ut n sit numerus integer positivus, ubi quidem etiam cyphra est excludenda; deinde, ut termini, in quibus exponens potestatis x fit negativus, penitus extinguantur. Qui huic formulae plus tribuunt, eamque adeo ad casus, quibus n est numerus negativus vel fractus, extendere volunt, maxime decipiuntur et in gravissimos errores illabuntur. Quae cum sint adeo manifesta, mirandum videtur, quod istae tam necessariae cautelae, quantum equidem memini, a nemine sint animadversae.

7. Haec consideratio occasionem mihi praebet duplicem investigationem suscipiendi. Primo scilicet in veram summam nostrae expressionis generalis, siquidem in infinitum continuetur, sum inquisiturus, ut pateat, quantum ea quovis casu a valore $\cos n\varphi$ discrepet. Deinde similem expressionem generalem investigabo, quae revera valorem $\cos n\varphi$ exhibeat, et nulla restrictione adhibita veros cosinus angulorum multiplorum ipsius φ praebeat, ita ut singulis casibus, quibus n est numerus integer, formulae initio allatae prodeant, simulque veritas, quando n est numerus fractus vel negativus, obtineatur.

8. Quo utrique instituto facilius satisfaciam, considero hanc formulam

$$s = A\,(x + \sqrt{(xx-1)})^n$$

valorem ipsius s per seriem evoluturus, quae secundum potestates ipsius x procedat. Cum igitur sit

$$\sqrt{(xx-1)} = x - \frac{1}{2x} - \frac{1.1}{2.4x^3} - \text{etc.}, \quad \text{ob} \quad s = A\left(2x - \frac{1}{2x} - \frac{1.1}{2.4x^3} - \text{etc.}\right)^n,$$

observo terminum primum futurum esse $= 2^n A x^n$; in sequentibus autem exponentes potestatis x continuo binario decrescere, ita ut series hujusmodi habitura sit formam

$$s = \alpha x^n + \beta x^{n-2} + \gamma x^{n-4} + \delta x^{n-6} + \varepsilon x^{n-8} + \text{etc.}$$

ubi quidem est $\alpha = 2^n A$.

9. Ad hanc autem seriem commodissime eruendam, observo aequationem assumtam per differentiationem in aliam converti oportere, in qua tam potestas indefinita quam omnis irrationalitas absit, simulque quantitas s ubique plus una dimensione non sit habitura; hujusmodi enim aequatio facillime per seriem certa lege procedentem resolvitur. Hunc in finem primo logarithmis sumendis obtineo

$$ls = lA + nl\,(x + \sqrt{(xx-1)});$$

tum vero differentiando:

$$\frac{ds}{s} = \left(ndx + \frac{nxdx}{\sqrt{(xx-1)}}\right) : \left(x + \sqrt{(xx-1)}\right) = \frac{ndx}{\sqrt{(xx-1)}}.$$

Hic sumtis quadratis erit

$$\frac{ds^2}{ss} = \frac{nndx^2}{xx-1}, \quad \text{seu} \quad (xx-1)\,ds^2 = nnssdx^2,$$

quae aequatio denuo differentiata, sumto elemento dx constante, et per $2ds$ divisa dat

$$(xx-1)\,dds + xdxds = nnsdx^2,$$

quae jam formam habet desideratam, ita ut quantitas s nusquam plus una dimensione habeat, et quantitas x ab omni irrationalitate sit immunis.

10. Quia hic quantitas x in aliis terminis duas, in uno vero nullam tenet dimensionem, facta hujusmodi distinctione, ut sit

$$xxdds + xdxds - nnsdx^2 - dds = 0$$

ponamus $$s = \alpha x^m + \beta x^{m-2} + \gamma x^{m-4} \ldots\ldots + \mu x^{m-i} + \nu x^{m-i-2} + \text{etc.}$$

et facta substitutione, potestas x^{m-i-2} talem accipiet coëfficientem

$$\nu\left((m-i-2)(m-i-3) + m-i-2 - nn\right) - \mu\,(m-i)\,(m-i-1),$$

qui cum evanescere debeat, quantitas ν ex μ ita definitur, ut sit

$$\nu = \frac{(m-i)(m-i-1)}{(m-i-2)^2 - nn}\,\mu.$$

Statuatur jam pro initio $i = -2$, ut fiat $\nu = \alpha$ et $\mu = 0$, proditque $\alpha = \frac{(m+2)(m+1)}{mm-nn}\,0$, quae littera ut maneat indefinita, esse oportet $mm = nn$, ideoque vel $m = n$, vel $m = -n$.

11. Nostro autem casu est, ut supra vidimus, $m = n$, atque $\alpha = 2^n A$, quare posito

$$s = \alpha x^n + \beta x^{n-2} + \gamma x^{n-4} \ldots\ldots + \mu x^{n-i} + \nu x^{n-i-2} + \text{etc.}$$

erit $$\nu = \frac{(n-i)(n-i-1)}{(n-i-2)^2 - nn}\,\mu = -\frac{(n-i)(n-i-1)}{(i+2)(2n-i-2)}\,\mu,$$

unde sequentes prodeunt coëfficientium determinationes:

$$\beta = \frac{-n(n-1)}{4(n-1)}\,\alpha = \frac{-n}{4}\,\alpha$$

$$\gamma = \frac{-(n-2)(n-3)}{8(n-2)}\,\beta = \frac{+n(n-3)}{4.8}\,\alpha$$

$$\delta = \frac{-(n-4)(n-5)}{12(n-3)}\,\gamma = \frac{-n(n-4)(n-5)}{4.8.12}\,\alpha$$

$$\varepsilon = \frac{-(n-6)(n-7)}{16(n-4)}\,\delta = \frac{+n(n-5)(n-6)(n-7)}{4.8.12.16}\,\alpha.$$

etc.

12. Posito ergo $s = A\,(x + \sqrt{(xx-1)})^n$, ob $\alpha = 2^n A$, habebimus hanc seriem, qua quantitas s exprimitur:

$$s = 2^n A x^n \left(1 - \frac{n}{4}x^{-2} + \frac{n(n-3)}{4.8}x^{-4} - \frac{n(n-4)(n-5)}{4.8.12}x^{-6} + \frac{n(n-5)(n-6)(n-7)}{4.8.12.16}x^{-8} - \text{etc.}\right)$$

Quare si pro A capiatur $\frac{1}{2}$, orietur ipsa illa forma, quam initio pro $\cos n\varphi$ assignavimus, existente $x = \cos\varphi$, atque nunc quidem patet illius expressionis in infinitum continuatae verum valorem esse

$$\frac{1}{2}\left(x + \sqrt{(xx-1)}\right)^n;$$

sicque ratio aberrationis a valore $\cos n\varphi$ est manifesta, atque nunc quidem evidens est, cur sumto $n=0$, prodeat summa nostrae seriei $=\frac{1}{2}$; reliquis vero casibus summa fiat imaginaria, si quidem sit $x<1$. At si sumatur $x=1$, quicunque numerus pro n accipiatur, summa semper est $=\frac{1}{2}$, eritque propterea

$$1=2^n\left(1-\frac{n}{4}+\frac{n(n-3)}{4.8}-\frac{n(n-4)(n-5)}{4.8.12}+\frac{n(n-5)(n-6)(n-7)}{4.8.12.16}-\text{etc.}\right),$$

quod certe est theorema non inelegans.

13. Alio modo concinnius valor ipsius s exprimi potest; cum enim sit

$$x=\cos\varphi,\quad \text{erit}\quad \sqrt{(xx-1)}=\sqrt{-1}\,.\sin\varphi,$$

et ex notis sinuum proprietatibus

$$(\cos\varphi+\sqrt{-1}\,.\sin\varphi)^n=\cos n\varphi+\sqrt{-1}\,.\sin n\varphi.$$

Quare posito $\cos\varphi=x$, erit

$$\cos n\varphi+\sqrt{-1}\,.\sin n\varphi=2^n x^n\left(1-\frac{n}{4}x^{-2}+\frac{n(n-3)}{4.8}x^{-4}-\frac{n(n-4)(n-5)}{4.8.12}x^{-6}+\text{etc.}\right);$$

unde patet summam hujus seriei in infinitum continuatae esse imaginariam, nisi sit $x=1$, seu $\varphi=0$. Realis quidem semper erit dum sit $x>1$; sed his casibus non amplius ad sinus et cosinus referri potest. Veluti si $xx=2$, ob $s=A(1+\sqrt{2})^n$ erit

$$(\sqrt{2}+1)^n=2^{\frac{3n}{2}}\left(1-\frac{n}{8}+\frac{n(n-3)}{8.16}-\frac{n(n-4)(n-5)}{8.16.24}+\frac{n(n-5)(n-6)(n-7)}{8.16.24.32}-\text{etc.}\right).$$

At si ponamus $x+\sqrt{(xx-1)}=y$, fit $x=\frac{yy+1}{2y}$, unde obtinetur sequens summatio non contemnenda:

$$\left(\frac{yy}{yy+1}\right)^n=1-\frac{n}{1}\cdot\frac{yy}{(yy+1)^2}+\frac{n(n-3)}{1.2}\cdot\frac{y^4}{(yy+1)^4}-\frac{n(n-4)(n-5)}{1.2.3}\cdot\frac{y^6}{(yy+1)^6}+\text{etc.}$$

quae cum etiam vera sit sumto n negativo, erit

$$\left(\frac{yy+1}{yy}\right)^n=1+\frac{n}{1}\cdot\frac{yy}{(yy+1)^2}+\frac{n(n+3)}{1.2}\cdot\frac{y^4}{(yy+1)^4}+\frac{n(n+4)(n+5)}{1.2.3}\cdot\frac{y^6}{(yy+1)^6}+\text{etc.}$$

Sit porro $\frac{yy+1}{yy}=z$, et habebitur

$$z^n=1+\frac{n}{1}\cdot\frac{z-1}{zz}+\frac{n(n+3)}{1.2}\cdot\frac{(z-1)^2}{z^4}+\frac{n(n+4)(n+5)}{1.2.3}\cdot\frac{(z-1)^3}{z^6}+\text{etc.}$$

ubi pro n omnes numeros assumere licet.

14. Hinc etiam alteri requisito satisfacere poterimus, quo ejusmodi expressio infinita desideratur, quantitatem $\cos n\varphi$ sine ulla restrictione exhibens. Sumatur enim exponens n negative, et cum sit $\cos(-n\varphi)=\cos n\varphi$ et $\sin(-n\varphi)=-\sin n\varphi$, erit ex superiori forma

$$\cos n\varphi-\sqrt{-1}\,.\sin n\varphi=\frac{1}{2^n x^n}\left(1+\frac{n}{4}x^{-2}+\frac{n(n+3)}{4.8}x^{-4}+\frac{n(n+4)(n+5)}{4.8.12}x^{-6}+\text{etc.}\right)$$

addendis his formulis pars imaginaria tollitur, et summae semissis dabit

$$\cos n\varphi = +\, 2^{n-1} x^n \left(1 - \frac{n}{4} x^{-2} + \frac{n(n-3)}{4.8} x^{-4} - \frac{n(n-4)(n-5)}{4.8.12} x^{-6} + \text{etc.}\right)$$
$$+ \frac{1}{2^{n+1} x^n} \left(1 + \frac{n}{4} x^{-2} + \frac{n(n+3)}{4.8} x^{-4} + \frac{n(n+4)(n+5)}{4.8.12} x^{-6} + \text{etc.}\right).$$

Hae scilicet binae series conjunctae verum valorem ipsius $\cos n\varphi$, existente $\cos\varphi = x$, exprimunt, idque sine ulla restrictione, ita ut pro n omnes numeros tam negativos quam positivos, tam integros quam fractos assumere liceat. Ubi quidem per se est perspicuum, sive ipsi n tribuatur valor negativus quicunque, sive idem positivus, easdem binas series ordine mutato resultare.

15. Jam binae hae series conjunctae pro quovis numero integro n eosdem cosinus per formulas finitas exhibent, quos initio recensui. Pro casu quidem $n = 0$ res est manifesta, cum inde fiat $\cos 0\varphi = \frac{1}{2} + \frac{1}{2} = 1$. Reliquos igitur casus simpliciores evolvamus:

I. Sit $n = 1$ eritque

$$\cos\varphi = x\left(1 - \frac{1}{4} x^{-2} - \frac{1.2}{4.8} x^{-4} - \frac{1.3.4}{4.8.12} x^{-6} - \frac{1.4.5.6}{4.8.12.16} x^{-8} - \text{etc.}\right)$$
$$+ \frac{1}{4x}\left(1 + \frac{1}{4} x^{-2} + \frac{1.4}{4.8} x^{-4} + \frac{1.5.6}{4.8.12} x^{-6} + \frac{1.6.7.8}{4.8.12.16} x^{-8} + \text{etc.}\right),$$

ubi potestates negativae ipsius x sponte se destruunt, uti ex sequente repraesentatione fit perspicuum

$$\cos\varphi = x - \frac{1}{4} x^{-1} - \frac{1.2}{4.8} x^{-3} - \frac{1.3.4}{4.8.12} x^{-5} - \frac{1.4.5.6}{4.8.12.16} x^{-7} - \text{etc.}$$
$$+ \frac{1}{4} x^{-1} + \frac{1.1}{4.4} x^{-3} + \frac{1.1.4}{4.4.8} x^{-5} + \frac{1.1.5.6}{4.4.8.12} x^{-7} + \text{etc.}$$

ita ut sit $\cos\varphi = x$.

II. Sit $n = 2$ eritque

$$\cos 2\varphi = 2xx\left(1 - \frac{2}{4} x^{-2} - \frac{2.1}{4.8} x^{-4} - \frac{2.2.3}{4.8.12} x^{-6} - \frac{2.3.4.5}{4.8.12.16} x^{-8} - \text{etc.}\right)$$
$$+ \frac{1}{8xx}\left(1 + \frac{2}{4} x^{-2} + \frac{2.5}{4.8} x^{-4} + \frac{2.6.7}{4.8.12} x^{-6} + \frac{2.7.8.9}{4.8.12.16} x^{-8} + \text{etc.}\right),$$

quae binae series ita ordinate exhibeantur:

$$\cos 2\varphi = 2xx - 1 - 2\cdot\frac{2.1}{4.8} x^{-2} - 2\cdot\frac{2.2.3}{4.8.12} x^{-4} - 2\cdot\frac{2.3.4.5}{4.8.12.16} x^{-6} - \text{etc.}$$
$$+ \frac{1}{8} x^{-2} + \frac{1.2}{8.4} x^{-4} + \frac{1.2.5}{8.4.8} x^{-6} + \text{etc.},$$

ubi potestates negativae omnes tolluntur, ita ut prodeat $\cos 2\varphi = 2xx - 1$.

III. Sit $n = 3$ eritque

$$\cos 3\varphi = 4x^3\left(1 - \frac{3}{4} x^{-2} - 0x^{-4} - \frac{3.1.2}{4.8.12} x^{-6} - \frac{3.2.3.4}{4.8.12.16} x^{-8} - \text{etc.}\right)$$
$$+ \frac{1}{16x^5}\left(1 + \frac{3}{4} x^{-2} + \frac{3.6}{4.8} x^{-4} + \frac{3.7.8}{4.8.12} x^{-6} + \frac{3.8.9.10}{4.8.12.16} x^{-8} + \text{etc.}\right),$$

qui termini hoc modo in ordinem redigantur:

$$\cos 3\varphi = 4x^3 - 3x + 0x^{-1} - 4\cdot\frac{3.1.2}{4.8.12}x^{-3} - 4\cdot\frac{3.2.3.4}{4.8.12.16}x^{-5} - \text{etc.}$$
$$+ \quad \frac{1}{16}x^{-3} + \frac{1}{16}\cdot\frac{3}{4}x^{-5} + \text{etc.}$$

hincque $\cos 3\varphi = 4x^3 - 3x$.

16. His autem exemplis casu evenire videtur, ut potestates negativae se mutuo tollant, neque id pro terminis ulterioribus patet. Quamobrem, ne ullum dubium relinquatur, firma demonstratione evincendum est, singulas potestates negativas ipsius x in utraque serie paribus coëfficientibus signisque contrariis esse affectos, ita ut certum sit omnes se mutuo destruere. Hunc in finem utriusque seriei terminum generalem contemplemur, ac prioris quidem seriei ita repraesentatae

$$2^{n-1}x^n\left(1 - \frac{n}{4}x^{-2} - \frac{n(3-n)}{4.8}x^{-4} - \frac{n(4-n)(5-n)}{4.8.12}x^{-6} - \frac{n(5-n)(6-n)(7-n)}{4.8.12.16}x^{-8} - \text{etc.}\right)$$

terminus generalis colligitur fore:

$$-2^{n-1}x^{n-2\alpha}\cdot\frac{n(\alpha+1-n)(\alpha+2-n)(\alpha+3-n)\dots(2\alpha-1-n)}{4\,.\,8\,.\,12\,.\,16\,\dots\dots\,4\alpha}$$

ita, ut potestatis $x^{n-2\alpha}$ coëfficiens sit

$$-2^{n-1}\cdot\frac{n(\alpha+1-n)(\alpha+2-n)(\alpha+3-n)\dots(2\alpha-1-n)}{4\,.\,8\,.\,12\,.\,16\,\dots\dots\,4\alpha}.$$

Quando ergo haec potestas est negativa, seu $2\alpha > n$, patet hunc terminum evanescere his casibus: $2\alpha = n+1$, $2\alpha = n+2$, $2\alpha = n+3$, usque ad $2\alpha = 2n-2$, si quidem α fuerit numerus integer. Unde in priori serie omnium potestatum negativarum coëfficientes sponte evanescunt, nisi sit $2\alpha > 2n-2$, seu $\alpha > n-1$, quocirca docendum restat, si fuerit $\alpha > n-1$, istas potestates negativas per alteram seriem destrui, ita ut solae potestates positivae ipsius x relinquantur.

17. Alterius autem seriei, quae ita se habet

$$\frac{1}{2^{n+1}x^n}\left(1 + \frac{n}{4}x^{-2} + \frac{n(3+n)}{4.8}x^{-4} + \frac{n(4+n)(5+n)}{4.8.12}x^{-6} + \frac{n(5+n)(6+n)(7+n)}{4.8.12.16}x^{-8} + \text{etc.}\right)$$

terminus generalis colligitur

$$\frac{x^{-n-2\beta}}{2^{n+1}}\cdot\frac{n(\beta+1+n)(\beta+2+n)(\beta+3+n)\dots(2\beta-1+n)}{4.8.12.16\dots 4\beta}$$

unde potestatis negativae $x^{-n-2\beta}$ coëfficiens est

$$2^{-n-1}\cdot\frac{n(\beta+1+n)(\beta+2+n)(\beta+3+n)\dots(2\beta-1+n)}{4.8.12.16\dots 4\beta}.$$

Statuatur jam haec potestas praecedenti $x^{n-2\alpha}$ aequalis, seu $n-2\alpha = -n-2\beta$, fitque $\alpha = n+\beta$; sicque ipsae illae potestates negativae majores prodeunt, quarum coëfficientes in priori serie non sponte evanescunt. Ostendi ergo oportet, harum potestatum coëfficientes ex utraque serie ortos inter se esse aequales et se mutuo destruere, ubi quidem jam sponte patet alterum esse positivum, alterum negativum, ex quo utriusque aequalitas demonstrari debet.

18. Cum sit $\alpha = n+\beta$, erit $n = \alpha - \beta$, ideoque demonstrandum est fore

$$2^{\alpha-\beta-1}\cdot\frac{n(\beta+1)(\beta+2)(\beta+3)\ldots(\alpha+\beta-1)}{4.8.12.16\ldots4\alpha}=2^{\beta-\alpha-1}\cdot\frac{n(\alpha+1)(\alpha+2)(\alpha+3)\ldots(\alpha+\beta-1)}{4.8.12.16\ldots4\beta},$$

seu utrinque per $2^{\alpha+\beta+1}$ multiplicando

$$2^{2\alpha}\cdot\frac{n(\beta+1)(\beta+2)(\beta+3)\ldots(\alpha+\beta-1)}{4.8.12.16\ldots4\alpha}=2^{2\beta}\cdot\frac{n(\alpha+1)(\alpha+2)(\alpha+3)\ldots(\alpha+\beta-1)}{4.8.12.16\ldots4\beta}.$$

Cum jam in priori forma factorum denominatoris numerus sit $=\alpha$, singulique per quaternarium sint divisibiles, hos factores ita repraesentare licet

$$4^{\alpha}.1.2.3\ldots\alpha=2^{2\alpha}.1.2.3\ldots\alpha$$

simili modo denominator alterius formae ita exprimi poterit

$$4^{\beta}.1.2.3\ldots\beta=2^{2\beta}.1.2.3\ldots\beta$$

unde haec aequalitas ostendenda superest

$$\frac{n(\beta+1)(\beta+2)(\beta+3)\ldots(\alpha+\beta-1)}{1.2.3.4\ldots\alpha}=\frac{n(\alpha+1)(\alpha+2)(\alpha+3)\ldots(\alpha+\beta-1)}{1.2.3.4\ldots\beta},$$

quae per crucem multiplicata manifesto utrinque praebet idem productum

$$n.1.2.3.4\ldots(\alpha+\beta-1).$$

19. Paradoxon ergo initio propositum satis distincte explicatum videtur, simulque ratio patet, cur haec aequatio:

$$\cos n\varphi=2^{n-1}x^{n}\left(1-\frac{n}{4}x^{-2}+\frac{n(n-3)}{4.8}x^{-4}-\frac{n(n-4)(n-5)}{4.8.12}x^{-6}+\text{etc.}\right)$$

tum demum sit veritati consentanea, quando n denotat numerum integrum positivum, simulque omnes potestates ipsius x exponentes negativos habiturae expungantur, et cur his restrictionibus non observatis, haec expressio in errorem praecipitet.

20. Nunc autem pro casibus, quibus n est numerus fractus, veras series exhibere possumus, quae cosinus angulorum submultiplorum exprimant. Quod ut ostendam, sit primo $n=\frac{1}{2}$, eritque

$$\cos\frac{1}{2}\varphi=\frac{\sqrt{x}}{\sqrt{2}}\left(1-\frac{1}{8}x^{-2}-\frac{1.5}{8.16}x^{-4}-\frac{1.7.9}{8.16.24}x^{-6}-\frac{1.9.11.13}{8.16.24.32}x^{-8}-\text{etc.}\right)$$
$$+\frac{1}{2\sqrt{2x}}\left(1+\frac{1}{8}x^{-2}+\frac{1.7}{8.16}x^{-4}+\frac{1.9.11}{8.16.24}x^{-6}+\frac{1.11.13.15}{8.16.24.32}x^{-8}+\text{etc.}\right),$$

quae in ordinem secundum potestates redacta dat

$$\cos\frac{1}{2}\varphi=\frac{\sqrt{x}}{\sqrt{2}}\left(1+\frac{1}{2x}-\frac{1}{8xx}+\frac{1}{2.8x^3}-\frac{1.5}{8.16x^4}+\frac{1.7}{2.8.16x^5}-\frac{1.7.9}{8.16.24x^6}+\frac{1.9.11}{2.8.16.24x^7}-\text{etc.}\right)$$

ubi, si quilibet coëfficiens per praecedentem dividatur, haec resultat series:

$$\frac{1}{2},\quad-\frac{1}{4},\quad-\frac{1}{2},\quad-\frac{5}{8},\quad-\frac{7}{10},\quad-\frac{9}{12},\quad-\frac{11}{14},\quad\text{etc.}$$

unde fit $$\cos\frac{1}{2}\varphi=\frac{\sqrt{x}}{\sqrt{2}}\left(1+\frac{1}{2}x^{-1}-\frac{1.1}{2.4}x^{-2}+\frac{1.1.3}{2.4.6}x^{-3}-\frac{1.1.3.5}{2.4.6.8}x^{-4}+\text{etc.}\right)$$

ideoque manifesto habebitur $\cos\frac{1}{2}\varphi=\frac{\sqrt{x}}{\sqrt{2}}\left(1+\frac{1}{x}\right)^{\frac{1}{2}}=\sqrt{\frac{1+x}{2}}$, uti constat.

21. Evolvamus etiam casum $n=\frac{1}{3}$, ac reperimus

$$\cos\frac{1}{3}\varphi=\frac{\sqrt[3]{x}}{\sqrt[3]{4}}\left(1-\frac{1}{12}x^{-2}-\frac{1.8}{12.24}x^{-4}-\frac{1.11.14}{12.24.36}x^{-6}-\frac{1.14.17.20}{12.24.36.48}x^{-8}-\text{etc.}\right)$$
$$+\frac{1}{2\sqrt[3]{2x}}\left(1+\frac{1}{12}x^{-2}+\frac{1.10}{12.24}x^{-4}+\frac{1.13.16}{12.24.36}x^{-6}+\frac{1.16.19.22}{12.24.36.48}x^{-8}+\text{etc.}\right),$$

quae binae series ita conjungantur:

$$\cos\frac{1}{3}\varphi=\frac{1}{\sqrt[3]{4}}x^{\frac{1}{3}}+\frac{1}{\sqrt[3]{16}}x^{-\frac{1}{3}}-\frac{1}{12\sqrt[3]{4}}x^{-\frac{5}{3}}+\frac{1}{12\sqrt[3]{16}}x^{-\frac{7}{3}}-\frac{1.8}{12.24\sqrt[3]{4}}x^{-\frac{11}{3}}+\frac{1.10}{12.24\sqrt[3]{16}}x^{-\frac{13}{3}}-\text{etc.}$$

Jam ad irrationalitatem tollendam statuatur $x^{\frac{1}{3}}=y\sqrt[3]{4}$, seu $x=4y^3$, ac prodibit

$$\cos\frac{1}{3}\varphi=y+\frac{1}{4y}-\frac{1}{12.4^2y^5}+\frac{1}{12.4^3y^7}-\frac{1.8}{12.24.4^4y^{11}}+\frac{1.10}{12.24.4^5y^{13}}-\frac{1.11.14}{12.24.36.4^6y^{17}}+\text{etc.}$$

Sit porro $y=\frac{z}{2}$, erit

$$\cos\frac{1}{3}\varphi=\frac{z}{2}+\frac{1}{2z}-\frac{1}{6z^5}+\frac{1}{6z^7}-\frac{1.8}{2.3.6z^{11}}+\frac{1.10}{2.3.6z^{13}}-\frac{1.11.14}{2.3.6.9z^{17}}+\frac{1.13.16}{2.3.6.9z^{19}}-\text{etc.}$$

seu $$2\cos\frac{1}{3}\varphi=z+\frac{1}{z}-\frac{1}{3z^5}+\frac{1}{3z^7}-\frac{1.8}{3.6z^{11}}+\frac{1.10}{3.6z^{13}}-\frac{1.11.14}{3.6.9z^{17}}+\frac{1.13.16}{3.6.9z^{19}}-\text{etc.}$$

22. In genere autem casus $x=\frac{1}{2}$, unde fit $\varphi=60^0$, seu $\varphi=\frac{1}{3}\pi$, denotante π semicircumferentiam circuli, cujus radius $=1$, omni attentione dignus videtur. Nam ob $2x=1$, fit

$$\cos\frac{n}{3}\pi=\frac{1}{2}\left(1-\frac{n}{1}+\frac{n(n-3)}{1.2}-\frac{n(n-4)(n-5)}{1.2.3}+\frac{n(n-5)(n-6)(n-7)}{1.2.3.4}-\text{etc.}\right)$$
$$+\frac{1}{2}\left(1+\frac{n}{1}+\frac{n(n+3)}{1.2}+\frac{n(n+4)(n+5)}{1.2.3}+\frac{n(n+5)(n+6)(n+7)}{1.2.3.4}+\text{etc.}\right),$$

ubi notari convenit utriusque seriei summam seorsim sumtam esse imaginariam, et quia utraque est divergens, minime licet eas pro lubitu combinare. Veluti si termini ordinate conjungerentur, prodiret

$$\cos\frac{n}{3}\pi=1+\frac{nn}{1.2}+\frac{9nn}{1.2.3}+\frac{nn(nn+107)}{1.2.3.4}+\text{etc.}$$

unde sequeretur fore $\cos\frac{n}{3}\pi>1$, quod tamen est absurdum. Interim tamen binarum illarum prioris summa est $\frac{1}{2}\left(\cos\frac{n}{3}\pi+\sqrt{-1}.\sin\frac{n}{3}\pi\right)$, posterioris vero $\frac{1}{2}\left(\cos\frac{n}{3}\pi-\sqrt{-1}.\sin\frac{n}{3}\pi\right)$, sicque nullum est dubium, quin ambae conjunctim praebeant $\cos\frac{n}{3}\pi$, etiam si non pateat, quemadmodum hic valor ex conjunctione facta elici possit. Hinc ergo denuo insigne paradoxon resultat, cujus explicatio haud parum ardua videtur; sine dubio autem ex serierum divergentia est petenda, et series signis alternantibus ita scribenda, terminorum numero neque pari neque impari reputato:

$$2\cos\frac{n}{3}\pi=2-\frac{n}{1}+\frac{n}{1}-\frac{n(3-n)}{1.2}+\frac{n(3+n)}{1.2}-\frac{n(4-n)(5-n)}{1.2.3}+\frac{n(4+n)(5+n)}{1.2.3}-\text{etc.}$$

ita ut sit

$$\frac{2-2\cos\frac{n}{3}\pi}{n}=\frac{3-n}{1.2}-\frac{3-n}{1.2}+\frac{(4-n)(5-n)}{1.2.3}-\frac{(4+n)(5+n)}{1.2.3}+\frac{(5-n)(6-n)(7-n)}{1.2.3.4}-\text{etc.}$$

Incommoda autem effugere non licet, nisi quantitas x indefinita relinquatur, ac seriei termini secundum ejus potestates disponantur.

23. Verum etiam hoc modo haud leves difficultates relinquuntur; si enim numerum n sumamus infinite parvum, ut sit $\cos n\varphi = 1-\frac{1}{2}nn\varphi\varphi$, ob

$$2^n x^n = 1 + nl2x + \frac{1}{2}nn(l2x)^2 \quad \text{et} \quad \frac{1}{2^n x^n} = 1 - nl2x + \frac{1}{2}nn(l2x)^2$$

habebimus, in singulis terminis potestates ipsius n quadrato altiores negligendo,

$$\begin{aligned}2-nn\varphi\varphi = &+\left(1+nl2x+\frac{1}{2}nn(l2x)^2\right)\left(1-\frac{n}{4xx}-\frac{3n+nn}{4.8x^4}-\frac{20n+9nn}{4.8.12x^6}-\frac{210n+107nn}{4.8.12.16x^8}-\text{etc.}\right)\\ &+\left(1-nl2x+\frac{1}{2}nn(l2x)^2\right)\left(1+\frac{n}{4xx}+\frac{3n+nn}{4.8x^4}+\frac{20n+9nn}{4.8.12x^6}+\frac{210n+107nn}{4.8.12.16x^8}+\text{etc.}\right)\end{aligned}$$

Atque facta evolutione tam partes finitae quam infinite parvae ipso numero n affectae se mutuo destruunt, reliquae vero per nn divisae praebent

$$\begin{aligned}\varphi\varphi = &\,2l2x\left(\frac{1}{4xx}+\frac{3}{4.8x^4}+\frac{20}{4.8.12x^6}+\frac{210}{4.8.12.16x^8}+\text{etc.}\right)\\ &-2\left(\frac{1}{4.8x^4}+\frac{9}{4.8.12x^6}+\frac{107}{4.8.12.16x^8}+\text{etc.}\right)-(l2x)^2\end{aligned}$$

existente $x=\cos\varphi$. Ad legem hujus progressionis clarius percipiendam, ponamus $2x=y$, ut sit

$$\begin{aligned}&y=2\cos\varphi \quad \text{et} \quad \frac{3}{2}=A=\frac{3}{2}, \quad A\cdot\frac{1}{3} &&=\alpha=\frac{1}{2}\\ &\frac{4.5}{2.3}=B=\frac{10}{3}, \quad B\left(\frac{1}{4}+\frac{1}{5}\right) &&=\beta=\frac{3}{2}=A\\ &\frac{5.6.7}{2.3.4}=C=\frac{35}{4}, \quad C\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right) &&=\gamma=\frac{107}{24}=B+\frac{1}{2}AA\\ &\frac{6.7.8.9}{2.3.4.5}=D=\frac{126}{5}, \quad D\left(\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}\right) &&=\delta=\frac{55}{4}=C+AB\\ &\qquad\text{etc.} \qquad\qquad \text{etc.} && \varepsilon=D+AC+\frac{1}{2}BB\end{aligned}$$

eritque

$$\begin{aligned}\varphi\varphi = &\,2ly\left(\frac{1}{yy}+\frac{A}{y^4}+\frac{B}{y^6}+\frac{C}{y^8}+\frac{D}{y^{10}}+\text{etc.}\right)\\ &-2\left(\frac{\alpha}{y^4}+\frac{\beta}{y^6}+\frac{\gamma}{y^8}+\frac{\delta}{y^{10}}+\text{etc.}\right)-(ly)^2,\end{aligned}$$

ubi si brevitatis gratia statuamus

$$P=\frac{1}{yy}+\frac{A}{y^4}+\frac{B}{y^6}+\frac{C}{y^8}+\text{etc.}$$

fit $\varphi\varphi = 2Ply - PP - (ly)^2$, seu $\varphi\varphi = -(ly-P)^2$,

quod est absurdum.

24. Omnino autem notatu digna est relatio, quam hic inter numerorum A, B, C, D, etc. et numerorum α, β, γ, δ, etc. ordines observavi, et quae commodissime ita referri potest, ut sit

$$\alpha + \beta z + \gamma z^2 + \delta z^3 + \text{etc.} = \frac{1}{2}(1 + Az + Bz^2 + Cz^3 + Dz^4 + \text{etc.})^2,$$

cujus demonstratio haud parum ardua videtur. Operae igitur pretium est indolem horum numerorum accuratius contemplari:

$A = \frac{3}{2} = \frac{2.3}{2.2} \cdot 1$	$\alpha = A \cdot \frac{1}{3} = \frac{1}{2} \cdot 1^2$
$B = \frac{4.5}{2.3} = \frac{4.5}{3.3} A$	$\beta = B\left(\frac{1}{4} + \frac{1}{5}\right) = 1 \,.\, A$
$C = \frac{5.6.7}{2.3.4} = \frac{6.7}{4.4} B$	$\gamma = C\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) = 1 \,.\, B + \frac{1}{2} AA$
$D = \frac{6.7.8.9}{2.3.4.5} = \frac{8.9}{5.5} C$	$\delta = D\left(\frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9}\right) = 1 \,.\, C + AB$
$E = \frac{7.8.9.10.11}{2.3.4.5.6} = \frac{10.11}{6.6} D$	$\varepsilon = E\left(\frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11}\right) = 1 \,.\, D + AC + \frac{1}{2} BB$
etc.	etc.

25. Consideremus hanc proprietatem in solis numeris integris, ac formemus has binas progressiones:

$1 = 1$	
$\mathfrak{A} = 3$	$\mathfrak{a} = \mathfrak{A} \cdot \frac{1}{3}$
$\mathfrak{B} = 4 \,.\, 5$	$\mathfrak{b} = \mathfrak{B}\left(\frac{1}{4} + \frac{1}{5}\right)$
$\mathfrak{C} = 5 \,.\, 6 \,.\, 7$	$\mathfrak{c} = \mathfrak{C}\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right)$
$\mathfrak{D} = 6 \,.\, 7 \,.\, 8 \,.\, 9$	$\mathfrak{d} = \mathfrak{D}\left(\frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9}\right)$
$\mathfrak{E} = 7 \,.\, 8 \,.\, 9 \,.\, 10 \,.\, 11$	$\mathfrak{e} = \mathfrak{E}\left(\frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11}\right)$
$\mathfrak{F} = 8 \,.\, 9 \,.\, 10 \,.\, 11 \,.\, 12 \,.\, 13$	$\mathfrak{f} = \mathfrak{F}\left(\frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13}\right)$
etc.	etc.

eritque ut sequitur

$$\mathfrak{a} = 2 \,.\, \frac{1.1}{2}, \quad \mathfrak{b} = 3\mathfrak{A}, \quad \mathfrak{c} = 4\mathfrak{B} + 6\frac{\mathfrak{A}\mathfrak{A}}{2}, \quad \mathfrak{d} = 5\mathfrak{C} + 10\mathfrak{A}\mathfrak{B}, \quad \mathfrak{e} = 6\mathfrak{D} + 15\mathfrak{A}\mathfrak{C} + 20\frac{\mathfrak{B}\mathfrak{B}}{2},$$

$$\mathfrak{f} = 7\mathfrak{E} + 21\mathfrak{A}\mathfrak{D} + 35\mathfrak{B}\mathfrak{C},$$

$$\text{seu} \quad 2\mathfrak{f} = 7 \,.\, 1\mathfrak{E} + \frac{7.6}{1.2}\mathfrak{A}\mathfrak{D} + \frac{7.6.5}{1.2.3}\mathfrak{B}\mathfrak{C} + \frac{7.6.5.4}{1.2.3.4}\mathfrak{C}\mathfrak{B} + \frac{7.6.5.4.3}{1.2.3.4.5}\mathfrak{D}\mathfrak{A} + \frac{7.6.5.4.3.2}{1.2.3.4.5.6}\mathfrak{E} \,.\, 1,$$

unde lex progressionis est manifesta. Vel erit

$$\frac{\mathfrak{a}}{2} + \frac{\mathfrak{b}z}{2.3} + \frac{\mathfrak{c}zz}{2.3.4} + \frac{\mathfrak{b}z^3}{2.3.4.5} + \text{etc.} = \frac{1}{2}\left(1 + \frac{\mathfrak{A}z}{2} + \frac{\mathfrak{B}zz}{2.3} + \frac{\mathfrak{C}z^3}{2.3.4} + \frac{\mathfrak{D}z^4}{2.3.4.5} + \text{etc.}\right)^2.$$

26. Pro insigni autem hac proprietate sequentem inveni demonstrationem, qua simul indolem hujusmodi formularum penitius perspicietur. Ponamus brevitatis gratia $2x = \frac{1}{y}$, ut sit $\cos\varphi = \frac{1}{2y}$, atque ex superioribus habebimus

$$\cos n\varphi - \sqrt{-1}\,.\sin n\varphi = y^n\left(1 + \frac{n}{1}y^2 + \frac{n(n+3)}{1.2}y^4 + \frac{n(n+4)(n+5)}{1.2.3}y^6 + \text{etc.}\right).$$

Evolvatur haec series secundum potestates indicis n, fingaturque

$$\cos n\varphi - \sqrt{-1}\,.\sin n\varphi = y^n(1 + nP + nnQ + n^3R + n^4S + \text{etc.}),$$

quae forma quo facilius intelligi possit, novo signandi modo utamur, scilicet propositis quotcunque numeris α, β, γ, δ, etc.

haec scriptio	denotat
$(\alpha, \beta, \gamma, \delta, \text{etc.})^{(1)}$	summam singulorum $\alpha + \beta + \gamma + \delta + \text{etc.}$
$(\alpha, \beta, \gamma, \delta, \text{etc.})^{(2)}$	summam productorum ex binis
$(\alpha, \beta, \gamma, \delta, \text{etc.})^{(3)}$	summam productorum ex ternis
$(\alpha, \beta, \gamma, \delta, \text{etc.})^{(4)}$	summam productorum ex quaternis
etc.	

ubi observo si index suffixus aequalis sit multitudini numerorum, hac scriptione omnium productum exprimi, tum vero semper esse $(\alpha, \beta, \gamma, \delta, \text{etc.})^{(0)} = 1$. Hoc autem scribendi modo adhibito erit

$$P = \frac{1}{1}y^2 + \frac{(3)^{(1)}}{2}y^4 + \frac{(4.5)^{(2)}}{2.3}y^6 + \frac{(5.6.7)^{(3)}}{2.3.4}y^8 + \frac{(6.7.8.9)^{(4)}}{2.3.4.5}y^{10} + \text{etc.}$$

$$Q = \frac{(3)^{(0)}}{2}y^4 + \frac{(4.5)^{(1)}}{2.3}y^6 + \frac{(5.6.7)^{(2)}}{2.3.4}y^8 + \frac{(6.7.8.9)^{(3)}}{2.3.4.5}y^{10} + \text{etc.}$$

$$R = \frac{(4.5)^{(0)}}{2.3}y^6 + \frac{(5.6.7)^{(1)}}{2.3.4}y^8 + \frac{(6.7.8.9)^{(2)}}{2.3.4.5}y^{10} + \text{etc.}$$

$$S = \frac{(5.6.7)^{(0)}}{2.3.4}y^8 + \frac{(6.7.8.9)^{(1)}}{2.3.4.5}y^{10} + \text{etc.}$$

$$T = \frac{(6.7.8.9)^{(0)}}{2.3.4.5}y^{10} + \text{etc.}$$

etc.

27. Nunc autem observo fore simili modo

$$\cos\lambda n\varphi - \sqrt{-1}\,.\sin\lambda n\varphi = y^{\lambda n}(1 + \lambda nP + \lambda^2n^2Q + \lambda^3n^3R + \text{etc.})$$

Cum autem sit, uti constat

$$\cos\lambda n\varphi - \sqrt{-1}\,.\sin\lambda n\varphi = (\cos n\varphi - \sqrt{-1}\,.\sin n\varphi)^\lambda,$$

erit quoque $\cos\lambda n\varphi - \sqrt{-1}\,.\sin\lambda n\varphi = y^{\lambda n}(1 + nP + n^2Q + n^3R + \text{etc.})^\lambda$

ideoque $1 + \lambda nP + \lambda^2n^2Q + \lambda^3n^3R + \text{etc.} = (1 + nP + n^2Q + n^3R + \text{etc.})^\lambda$,

quae aequalitas subsistere nequit, nisi sit

$$Q = \frac{1}{2}P^2, \quad R = \frac{1}{2.3}P^3, \quad S = \frac{1}{2.3.4}P^4, \quad T = \frac{1}{2.3.4.5}P^5, \quad \text{etc.}$$

Atque hinc porro colligere licet, cum sit

$$\cos n\varphi - \sqrt{-1}\,.\,\sin n\varphi = e^{-n\varphi\sqrt{-1}}$$

nunc autem invenerimus

$$\cos n\varphi - \sqrt{-1}\,.\sin n\varphi = y^n e^{nP} = e^{n(P+ly)}$$

fore $-\varphi\sqrt{-1} = P + ly$, ideoque $\varphi\varphi = -(P+ly)^2$, quod cum videatur absurdum, ita resolvi oportet, quod P semper sit quantitas imaginaria; sicque explicatur paradoxon supra § 22 allatum.

28. Verum ut ad propositum revertar, cum sit $Q = \frac{1}{2}PP$, si brevitatis gratia valores § 24 explicatos introducamus, erit

$$P = yy + Ay^4 + By^6 + Cy^8 + Dy^{10} + \text{etc.}$$
$$\text{et}\quad Q = \alpha y^4 + \beta y^6 + \gamma y^8 + \delta y^{10} + \text{etc.}$$

unde valoribus Q et $\frac{1}{2}PP$ aequatis nanciscimur supra observatas relationes, scilicet

$$\alpha = \frac{1}{2},\quad \beta = A,\quad \gamma = B + \frac{1}{2}AA,\quad \delta = C + AB,\quad \varepsilon = D + AC + \frac{1}{2}BB$$

et ita porro. Hac igitur demonstratione confecta, aliarum similium formularum complicatarum resolutionem coronidis loco subjungam.

29. **Problema.** Hanc formulam $a(1+\sqrt{(1-x)})^n$ in seriem infinitam resolvere secundum potestates ipsius x progredientem.

Solutio. Statuatur $z = a(1+\sqrt{(1-x)})^n$ et posita serie, quae quaeritur,

$$z = A + Bx + Cxx + Dx^3 + Ex^4 + \text{etc.}$$

evidens est fore $A = 2^n a$, unde sequentes coëfficientes simili modo ut supra definire licebit. Sumtis logarithmis, habemus $lz = la + nl(1+\sqrt{(1-x)})$, et differentiando

$$\frac{dz}{z} = -\frac{ndx}{2\sqrt{(1-x)}} : (1+\sqrt{(1-x)}).$$

Multiplicetur numerator ac denominator per $1-\sqrt{(1-x)}$ prodibitque

$$\frac{dz}{z} = -\frac{ndx\,(1-\sqrt{(1-x)})}{2x\sqrt{(1-x)}} = \frac{ndx}{2x} - \frac{ndx}{2x\sqrt{(1-x)}}$$

et irrationalitatem tollendo

$$\left(\frac{ndx}{2x} - \frac{dz}{z}\right)^2 = \frac{nndx^2}{4xx(1-x)}.$$

Ponatur $z = x^{\frac{n}{2}}v$, ut sit $\frac{dz}{z} = \frac{dv}{v} + \frac{ndx}{2x}$, fietque $\frac{dv^2}{vv} = \frac{nndx^2}{4xx(1-x)}$, seu

$4xx(1-x)\,dv^2 = nnvv\,dx^2$, quae aequatio differentiata et per $2dv$ divisa praebet

$$4xx(1-x)\,ddv + 2x(2-3x)\,dxdv - nnv\,dx^2 = 0.$$

Cum nunc sit $\frac{dv}{v} = \frac{dz}{z} - \frac{ndx}{2x}$, erit differentiando

$$\frac{ddv}{v} - \frac{dv^2}{vv} = \frac{ddz}{z} - \frac{dz^2}{zz} + \frac{ndx^2}{2xx} \quad \text{at}$$

$$\frac{dv^2}{vv}=\frac{dz^2}{zz}-\frac{ndxdz}{xz}+\frac{nndx^2}{4xx},\qquad \text{ergo}$$

$$\frac{ddv}{v}=\frac{ddz}{z}-\frac{ndxdz}{xz}+\frac{n(n+2)dx^2}{4xx}$$

unde facta substitutione:

$$\left.\begin{aligned}4xx(1-x)\frac{ddz}{z}-4nx(1-x)\frac{dxdz}{z}+n(n+2)(1-x)dx^2\\ +2x(2-3x)\frac{dxdz}{z}-n(2-3x)dx^2\\ -nndx^2\end{aligned}\right\}=0,\qquad \text{seu}$$

$$4x(1-x)ddz-4(n-1)dxdz+2(2n-3)xdxdz-n(n-1)zdx^2=0.$$

Cum hic variabilis x partim unicam, partim duas dimensiones obtineat, distinguendo hos terminos

$$+4xddz-4(n-1)dxdz$$
$$-4xxddz+2(2n-3)xdxdz-n(n-1)zdx^2=0$$

statuamus $\qquad z=A+Bx+Cxx+Dx^3+\ldots+Mx^i+Nx^{i+1}+\text{etc.}$

et potestatis x^i coëfficiens erit

$$+N(4i(i+1)-4(n-1)(i+1))+M(-4i(i-1)+2(2n-3)i-n(n-1)),$$

qui cum evanescere debeat, habebitur

$$N=\frac{(2i-n)(2i-n+1)}{4(i+1)(i-n+1)}M.$$

Nunc autem novimus esse $A=2^n a$, quare sequentes coëfficientes erunt

$$B=-\frac{n(n-1)}{4(n-1)}A=-\frac{n}{4}A$$

$$C=-\frac{(n-2)(n-3)}{8(n-2)}B=+\frac{n(n-3)}{4.8}A$$

$$D=-\frac{(n-4)(n-5)}{12(n-3)}C=-\frac{n(n-4)(n-5)}{4.8.12}A$$

etc.

Sumatur $a=2^{-n}$, ut fiat $A=1$, eritque

$$\left(\frac{1+\sqrt{(1-x)}}{2}\right)^n=1-\frac{n}{4}x+\frac{n(n-3)}{4.8}xx-\frac{n(n-4)(n-5)}{4.8.12}x^3+\frac{n(n-5)(n-6)(n-7)}{4.8.12.16}x^4-\text{etc.},$$

quae est series quaesita.

30. **Coroll. 1.** Sumto x negativo, sequentis seriei

$$1+\frac{n}{4}x+\frac{n(n-3)}{4.8}xx+\frac{n(n-4)(n-5)}{4.8.12}x^3+\frac{n(n-5)(n-6)(n-7)}{4.8.12.16}x^4+\text{etc.}$$

summa erit $=\left(\frac{1+\sqrt{(1+x)}}{2}\right)^n$, ex cujus combinatione cum praecedente, alternis tantum terminis sumendis, summa assignari poterit.

31. **Coroll. 2.** Si exponens n negative capiatur, binae sequentes series ad summam revocabuntur

$$1 + \frac{n}{4}x + \frac{n(n+3)}{4.8}xx + \frac{n(n+4)(n+5)}{4.8.12}x^3 + \frac{n(n+5)(n+6)(n+7)}{4.8.12.16}x^4 + \text{etc.},$$

hujus seriei summa est $= \left(\frac{1+\sqrt{(1-x)}}{2}\right)^{-n} = 2^n\left(\frac{1-\sqrt{(1-x)}}{x}\right)^n$. Tum

$$1 - \frac{n}{4}x + \frac{n(n+3)}{4.8}xx - \frac{n(n+4)(n+5)}{4.8.12}x^3 + \frac{n(n+5)(n+6)(n+7)}{4.8.12.16}x^4 - \text{etc.},$$

cujus summa est $= 2^n\left(\frac{\sqrt{(1+x)}-1}{x}\right)^n$.

32. **Problema.** Hanc formulam $\left(\frac{\sqrt{(1+x)}+\sqrt{(1-x)}}{2}\right)^n$ in seriem infinitam resolvere, cujus termini secundum potestates ipsius x progredirentur.

Solutio. Posito $z = \left(\frac{\sqrt{(1+x)}+\sqrt{(1-x)}}{2}\right)^n$ erit quadratis sumendis $zz = \left(\frac{1+\sqrt{(1-xx)}}{2}\right)^n$, hincque

$$z = \left(\frac{1+\sqrt{(1-xx)}}{2}\right)^{\frac{n}{2}},$$

quae forma in priori continetur, simodo ibi loco x et n scribatur xx et $\frac{n}{2}$, quocirca colligitur statim series quaesita:

$$1 - \frac{n}{8}xx + \frac{n(n-6)}{8.16}x^4 - \frac{n(n-8)(n-10)}{8.16.24}x^6 + \frac{n(n-10)(n-12)(n-14)}{8.16.24.32}x^8 - \text{etc.}$$

quippe cujus summa est $= \left(\frac{\sqrt{(1+x)}+\sqrt{(1-x)}}{2}\right)^n$.

33. **Coroll.** Sumto n negativo, ut prodeat haec series

$$1 + \frac{n}{8}xx + \frac{n(n+6)}{8.16}x^4 + \frac{n(n+8)(n+10)}{8.16.24}x^6 + \frac{n(n+10)(n+12)(n+14)}{8.16.24.32}x^8 + \text{etc.}$$

hujus summa erit $= \left(\frac{\sqrt{(1+x)}+\sqrt{(1-x)}}{2}\right)^{-n}$, quae reducitur ad hanc formam:

$$\left(\frac{\sqrt{(1+x)}-\sqrt{(1-x)}}{x}\right)^n.$$

34. **Scholion.** Omnes series istas, quarum summam hic assignavi, in hac forma complecti licet:

$$s = 1 + \frac{n}{1}y + \frac{n(n+3)}{1.2}yy + \frac{n(n+4)(n+5)}{1.2.3}y^3 + \frac{n(n+5)(n+6)(n+7)}{1.2.3.4}y^4 + \text{etc.}$$

eritque $s = \left(\frac{1+\sqrt{(1-4y)}}{2}\right)^{-n}$; unde patet si fuerit $4y > 1$, seriei summam esse imaginariam; realem autem, si sit $4y < 1$. Casu autem $y = \frac{1}{4}$ erit, uti jam supra observavimus,

$$1 + \frac{n}{4} + \frac{n(n+3)}{4.8} + \frac{n(n+4)(n+5)}{4.8.12} + \frac{n(n+5)(n+6)(n+7)}{4.8.12.16} + \text{etc.} = 2^n.$$

Verum illa series pluribus modis transformari potest, ex quibus hunc solum casum affero, qui oritur differentialibus sumtis, erit scilicet

$$\frac{ds}{dy} = n + \frac{n(n+3)}{1}y + \frac{n(n+4)(n+5)}{1.2}yy + \frac{n(n+5)(n+6)(n+7)}{1.2.3}y^3 + \text{etc.}$$

At est
$$\frac{ds}{dy} = +n\left(\frac{1+\sqrt{(1-4y)}}{2}\right)^{-n-1} \cdot \frac{1}{\sqrt{(1-4y)}}.$$

Quare per n dividendo, hujus seriei

$$1 + \frac{n+3}{1}y + \frac{(n+4)(n+5)}{1.2}yy + \frac{(n+5)(n+6)(n+7)}{1.2.3}y^3 + \text{etc.}$$

summa est
$$= \frac{1}{\sqrt{(1-4y)}}\left(\frac{1+\sqrt{(1-4y)}}{2}\right)^{-n-1}.$$

Vel scribendo $n = m - 3$, hujus seriei

$$1 + \frac{m}{1}y + \frac{(m+1)(m+2)}{1.2}yy + \frac{(m+2)(m+3)(m+4)}{1.2.3}y^3 + \frac{(m+3)(m+4)(m+5)(m+6)}{1.2.3.4}y^4 + \text{etc.}$$

summa est
$$= \frac{1}{\sqrt{(1-4y)}}\left(\frac{1+\sqrt{(1-4y)}}{2}\right)^{-m+2}.$$