XIX.

Problematis ex theoria maximorum et minorum solutio.

Problema. (Fig. 48) *Super recta* AB *constituere triangulum* AOB, *ut si ex dato puncto* V *in* * *sublimi posito ducantur rectae* VA, VB *et* VO, *sit summa binorum triangulorum* AVO + BVO *minima.*

Solutio. Ex V in planum trianguli quaesiti demittatur perpendiculum VC, et ex C in rectas quaesitas AO et BO productas agantur perpendiculares CP et CQ: erunt rectae VP et VQ in easdem perpendiculares. Hinc colligitur area $\Delta\ AVO = \frac{1}{2} AO \cdot VP$ et $\Delta\ BVO = \frac{1}{2} BO \cdot VQ$, ideoque minimum effici oportet

$$AO\sqrt{(CV^2 + CP^2)} + BO\sqrt{(CV^2 + CQ^2)}.$$

Statuamus nunc rectas datas $CA = a$, $CB = b$, $AB = c$ et $CV = h$, itemque angulos datos $CAB = \alpha$ et $CBA = \beta$, hincque quaeramus binos angulos $BAO = \mu$, $ABO = \nu$, ideoque $AON = BOM = \mu + \nu$, unde colligimus

$$AO = \frac{c \sin \nu}{\sin (\mu + \nu)} \quad \text{et} \quad BO = \frac{c \sin \mu}{\sin (\mu + \nu)},$$

et ob angulos $CAP = \alpha - \mu$ et $CBQ = \beta - \nu$ fit

$$CP = a \sin (\alpha - \mu) \quad \text{et} \quad CQ = b \sin (\beta - \nu)$$

quare ob c constans minimum esse debet

$$\frac{\sin \nu \sqrt{(hh + aa \sin^2 (\alpha - \mu))}}{\sin (\mu + \nu)} + \frac{\sin \mu \sqrt{(hh + bb \sin^2 (\beta - \nu))}}{\sin (\mu + \nu)},$$

cujus ergo formulae differentiale, positis μ et ν variabilibus, nihilo est aequandum. Est vero

$$d \cdot \frac{\sin \nu}{\sin (\mu + \nu)} = \frac{d\nu \cos \nu}{\sin (\mu + \nu)} - \frac{(d\mu + d\nu) \sin \nu \cos (\mu + \nu)}{\sin^2 (\mu + \nu)} = \frac{d\nu \sin \mu - d\mu \sin \nu \cos (\mu + \nu)}{\sin^2 (\mu + \nu)},$$

$$d \cdot \frac{\sin \mu}{\sin (\mu + \nu)} = \frac{d\mu \cos \mu}{\sin (\mu + \nu)} - \frac{(d\mu + d\nu) \sin \mu \cos (\mu + \nu)}{\sin^2 (\mu + \nu)} = \frac{d\mu \sin \nu - d\nu \sin \mu \cos (\mu + \nu)}{\sin^2 (\mu + \nu)}.$$

Tum vero posito $\sqrt{(hh + aa \sin^2 (\alpha - \mu))} = P$ et $\sqrt{(hh + bb \sin^2 (\beta - \nu))} = Q$ erit differentiando

$$dP = \frac{- aa d\mu \sin (\alpha - \mu) \cos (\alpha - \mu)}{P} \quad \text{et} \quad dQ = \frac{- bb d\nu \sin (\beta - \nu) \cos (\beta - \nu)}{Q},$$

quibus valoribus substitutis prodit differentiale nihilo aequandum

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$$+\frac{P d\nu \sin\mu - P d\mu \sin\nu \cos(\mu+\nu)}{\sin^2(\mu+\nu)} - \frac{aa d\mu \sin\nu \sin(\alpha-\mu)\cos(\alpha-\mu)}{P\sin(\mu+\nu)},$$

$$+\frac{Q d\mu \sin\nu - Q d\nu \sin\mu \cos(\mu+\nu)}{\sin^2(\mu+\nu)} - \frac{bb d\nu \sin\mu \sin(\beta-\nu)\cos(\beta-\nu)}{Q\sin(\mu+\nu)} = 0,$$

ubi termini elementis $d\mu$ et $d\nu$ affecti seorsim evanescentes reddi debent, ita ut hae binae obtineantur aequationes finitae per $\sin^2(\mu+\nu)$ multiplicando

$$\text{I. } P\sin\mu - Q\sin\mu\cos(\mu+\nu) - \frac{bb\sin\mu\sin(\mu+\nu)\sin(\beta-\nu)\cos(\beta-\nu)}{Q} = 0,$$

$$\text{II. } Q\sin\nu - P\sin\nu\cos(\mu+\nu) - \frac{aa\sin\nu\sin(\mu+\nu)\sin(\alpha-\mu)\cos(\alpha-\mu)}{P} = 0.$$

Formetur hinc ista combinatio I. $\frac{Q}{\sin\mu}$ — II. $\frac{P}{\sin\nu}$, proditque

$$(PP-QQ)\cos(\mu+\nu) \begin{matrix} -bb\sin(\mu+\nu)\sin(\beta-\nu)\cos(\beta-\nu) \\ +aa\sin(\mu+\nu)\sin(\alpha-\mu)\cos(\alpha-\mu) \end{matrix} = 0,$$

at est $PP-QQ = aa\sin^2(\alpha-\mu) - bb\sin^2(\beta-\nu)$, ideoque

$$aa\sin(\alpha-\mu)\left(\cos(\mu+\nu)\sin(\alpha-\mu)+\sin(\mu+\nu)\cos(\alpha-\mu)\right) =$$
$$bb\sin(\beta-\nu)\left(\cos(\mu+\nu)\sin(\beta-\nu)+\sin(\mu+\nu)\cos(\beta-\nu)\right)$$

quae aequatio per reductionem sinuum abit in hanc

$$aa\sin(\alpha-\mu)\sin(\alpha+\nu) = bb\sin(\beta-\nu)\sin(\beta+\mu),$$

cujus vis quo distinctius perspiciatur, notetur in figura esse

$$\alpha-\mu = CAM,\ \alpha+\nu = CNB,\ \beta-\nu = CBN,\ \beta+\mu = CMA,$$

unde manifestum est fore

$$\frac{\sin(\alpha-\mu)}{\sin(\beta+\mu)} = \frac{\sin CAM}{\sin CMA} = \frac{CM}{CA} \quad \text{et} \quad \frac{\sin(\beta-\nu)}{\sin(\alpha+\nu)} = \frac{\sin CBN}{\sin CNB} = \frac{CN}{CB},$$

aequatio ergo nostra $\frac{aa\sin(\alpha-\mu)}{\sin(\beta+\mu)} = \frac{bb\sin(\beta-\nu)}{\sin(\alpha+\nu)}$ fit $CA.CM = CB.CN$ seu $CA:CB = CN:CM$, ita ut recta MN futura sit rectae AB parallela. Atque hinc porro concludere licet, si ex C per punctum O recta ducatur COJ, ab ea rectam AB bisectum iri, quod cum non sit adeo obvium, ita ostenditur.

Ob intersectionem rectarum AM et CJ in puncto O est

$$AJ:OJ = AB.CM:BM.CO,$$

similique modo ob rectarum BN et JC intersectionem in O

$$BJ:OJ = AB.CN:AN.CO,$$

unde alternando et multiplicando fit

$$AJ:BJ = CM.AN:CN.BM.$$

At ob rectam MN ipsi AB parallelam est

$$CM:CN = BM:AN \quad \text{seu} \quad CM.AN = CN.BM$$

ideoque $AJ = BJ$. Sicque unam conditionem jam elicuimus, qua novimus punctum quaesitum O in rectam OJ, qua AB bisecatur, cadere.

Solutionis pars altera. Restat ergo, ut conditio haec inventa in altera aequationum supra inventarum substituatur, indeque ambo anguli incogniti μ et ν, quorum jam quaedam relatio constat, determinentur: hoc autem modo in calculos nimis intricatos delaberemur, quam ut inde solutio commoda derivari posset. Expediet ergo novam resolutionem huic conditioni, quod punctum quaesitum O certo in recta CJ lineam datam AB bisecante reperitur, superstruere.

Fig. 49. In hac ergo recta CJ sit O punctum quaesitum. Ex A et B in eam demittantur perpendicula AF et BG, atque ob $AJ = BJ$ erit tam $AF = BG$ quam $JF = JG$. In calculum igitur introducamus has quantitates cognitas: $CJ = e$, $AF = BG = f$, $JF = JG = g$ et altitudinem $CV = h$. Tum vero sit intervallum quaesitum $JO = z$, erit $CO = e - z$. Hinc ob $OF = z + g$ et $OG = z - g$ habebitur *

$$AO = \sqrt{(ff + (z+g)^2)} \quad \text{et} \quad BO = \sqrt{(ff + (z-g)^2)}$$

simulque perpendicula ex C in rectas AO et BO demissa sic facile obtinentur

$$AO : AF = CO : CP \quad \text{et} \quad BO : BG = CO : CQ, \text{ ut sit}$$

$$CP = \frac{f(e-z)}{AO} \quad \text{et} \quad CQ = \frac{f(e-z)}{BO},$$

unde fit

$$AO \,.\, VP = \sqrt{(hh \,.\, AO^2 + ff(e-z)^2)} = \sqrt{(ffhh + hh(z+g)^2 + ff(e-z)^2)}$$
$$BO \,.\, VQ = \sqrt{(hh \,.\, BO^2 + ff(e-z)^2)} = \sqrt{(ffhh + hh(z-g)^2 + ff(e-z)^2)}$$

quorum productorum summa debet esse minima.

Ad calculum contrahendum statuamus

$$ffhh + gghh + eeff = E, \; ff + hh = F,$$
$$eff - ghh = G, \; eff + ghh = H,$$

ut haec expressio minima sit efficienda

$$\sqrt{(E - 2Gz + Fzz)} + \sqrt{(E - 2Hz + Fzz)},$$

unde differentiando colligimus

$$\frac{Fz-g}{\sqrt{(E-2Gz+Fzz)}} + \frac{Fz-H}{\sqrt{(E-2Hz+Fzz)}} = 0$$

et irrationalitate sublata

$$(G - Fz)^2 (E - 2Hz + Fzz) = (H - Fz)^2 (E - 2Gz + Fzz),$$

quae evoluta praebet

$$\left.\begin{array}{l} EGG - 2GGHz + FGGzz \\ \quad - 2EFGz + 4FGHzz - 2FFGz^3 \\ \quad\quad + EFFzz - 2FFHz^3 + F^3z^4 \end{array}\right\} = \begin{array}{l} EHH - 2GHHz + FHHzz \\ \quad - 2EFHz + 4FGHzz - 2FFHz^3 \\ \quad\quad + EFFzz - 2FFGz^3 + F^3z^4 \end{array}$$

et contrahitur in hanc formam

$$E(GG - HH) - 2(GGH + EFG - GHH - EFH)z + F(GG - HH)zz = 0.$$

Facta divisione per $G - H$ nanciscimur

$$E(G+H)-2(GH+EF)z+F(G+H)z^2=0$$

et radice extracta

$$z=\frac{GH+EF\pm\sqrt{(EF-GG)(EF-HH)}}{F(G+H)}.$$

Jam vero est

$$F=ff+hh,\qquad G+H=2eff,\qquad EF=eeff(ff+hh)+hh(ff+gg)(ff+hh)$$
$$GH=eef^4-ggh^4,\quad GG=eef^4-2effghh+ggh^4,\quad HH=eeff^4+2effghh+ggh^4$$

unde fit
$$GH+EF=ff(2eeff+eehh+ffhh+gghh+h^4)$$
$$EF-GG=ffhh((e+g)^2+ff+hh)$$
$$EF-HH=ffhh((e-g)^2+ff+hh)$$ sicque elicitur

$$z=\frac{2eeff+eehh+ffhh+gghh+h^4\pm hh\sqrt{(ff+hh+(e+g)^2)(ff+hh+(e-g)^2)}}{2e(ff+hh)},$$

hincque porro $CO=\frac{hh(ee-ff-gg-hh)\pm hh\sqrt{(ff+hh+(e+g)^2)(ff+hh+(e-g)^2)}}{2e(ff+hh)}.$

Transferamus has expressiones in figuram, huncque in finem ad CJ normaliter jungatur recta DE, in quam ex A et B demittantur perpendicula AD et BE junganturque VD et VE. Cum nunc sit $CV=h$, $CD=CE=f$, erit $DV=EV=\sqrt{(ff+hh)}$, $AD=e+g$, $BE=e-g$, hincque

$$AV=\sqrt{(ff+hh+(e+g)^2)},\quad BV=\sqrt{(ff+hh+(e-g)^2)}$$

et $AD.BE=ee-gg$. Ergo ob $CJ=e$ habebitur

$$CO=\frac{CV^2}{2CJ}\cdot\frac{AD.BE-DV.EV\pm AV.BV}{DV.EV},$$

ubi perspicuum est rationem triangulorum ADV et BEV praecipue teneri, quae ad D et E sunt rectangula. Quodsi ergo vocentur anguli $DAV=\delta$ et $EBV=\varepsilon$, erit

$$DV=AV\sin\delta,\ AD=AV\cos\delta\quad\text{atque}\quad EV=BV\sin\varepsilon,\ BE=BV\cos\varepsilon,$$

quibus introductis conficitur

$$CO=\frac{CV^2}{2CJ}\cdot\frac{\cos\delta\cos\varepsilon-\sin\delta\sin\varepsilon\pm1}{\sin\delta\sin\varepsilon}=\frac{CV^2}{2CJ}\cdot\frac{\cos(\delta+\varepsilon)\pm1}{\sin\delta\sin\varepsilon}.$$

Duplex igitur hinc nascitur solutio

$$\text{I.}\quad CO=\frac{CV^2}{CJ}\cdot\frac{\cos^2\left(\frac{\delta+\varepsilon}{2}\right)}{\sin\delta\sin\varepsilon},\qquad \text{II.}\quad CO=-\frac{CV^2}{CJ}\cdot\frac{\sin^2\left(\frac{\delta+\varepsilon}{2}\right)}{\sin\delta\sin\varepsilon},$$

quarum prior dat punctum O inter puncta C et J, uti problema postulat; posterior vero praebet punctum O in recta JC ultra C producta, cui quidem etiam minimum convenit, sed non tale, quale in quaestione desideratur, quia aequatio inventa etiam quaestionem resolvit, ubi differentia triangulorum AVO et BVO minima quaereretur. Quocirca sola solutio prior locum habere est censenda.

Coroll. 1. Si ergo ponamus $ff+hh=kk$ erit

$$z=e-\frac{hh}{2ekk}(ee-gg-kk+\sqrt{(kk+(e+g)^2)(kk+(e-g)^2)})$$

unde patet si altitudo $CV=h$ evanescat, fore $z=e$, seu punctum O in C cadere, quo casu utique ambo triangula AVO et BVO evanescunt.

Coroll. 2. Sin autem altitudo $CV = h$ fiat infinita, quo casu etiam $k = \infty$ et $\frac{hh}{kk} = 1$, tum formula irrationalis fit

$$\sqrt{(k^4 + 2(ee + gg)kk)} = kk + ee + gg$$

ideoque $z = e - \frac{hh}{2ekk} \cdot 2ee = 0$. Punctum scilicet O in J cadit; unde perspicuum est, quemcunque altitudo h valorem finitum sortiatur, punctum O inter C et J cadere.

Coroll. 3. Aequatio quadratica primum inventa praebet

$$E + Fzz = \frac{2(GH + EF)z}{G + H}.$$

Hinc fit

$$E - 2Gz + Fzz = \frac{2(EF - GG)z}{G + H} = \frac{hh}{e}(kk + (e + g)^2)z,$$

$$E - 2Hz + Fzz = \frac{2(EF - HH)z}{G + H} = \frac{hh}{e}(kk + (e - g)^2)z,$$

unde prodit quantitas minima facta

$$\frac{h\sqrt{z}}{\sqrt{e}}(\sqrt{(kk + (e + g)^2)} + \sqrt{(kk + (e - g)^2)})$$

quae aequatur duplae areae triangulorum AVO et BVO.

Coroll. 4. Sin autem intervallo JO alius quicunque valor $JO = x$ tribuatur, eorumdem triangulorum summa duplicata fit

$$fh\sqrt{(1 + \frac{(x + g)^2}{ff} + \frac{(e - x)^2}{hh})} + fh\sqrt{(1 + \frac{(x - g)^2}{ff} + \frac{(e - x)^2}{hh})}$$

qua superior semper est minor, nisi sit $x = z$. Hic autem sumto $x = 0$, fit ista quantitas

$$2fh\sqrt{(1 + \frac{gg}{ff} + \frac{ee}{hh})} = 2\sqrt{(ffhh + gghh + eeff)}$$

sin autem capiatur $x = e$, seu O in C capiatur, erit ea

$$h\sqrt{(ff + (e + g)^2)} + h\sqrt{(ff + (e - g)^2)}.$$