XXII.

De comparatione arcuum curvarum irrectificabilium.

Sectio prima

continens evolutionem hujus aequationis:

$$0=\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy.$$

I.

Si ex hac aequatione sigillatim utriusque variabilis x et y valor extrahatur, reperietur

$$y=\frac{-\beta-\delta x+\sqrt{(\beta\beta-\alpha\gamma+2\beta(\delta-\gamma)x+(\delta\delta-\gamma\gamma)xx)}}{\gamma},$$

$$x=\frac{-\beta-\delta y-\sqrt{(\beta\beta-\alpha\gamma+2\beta(\delta-\gamma)y+(\delta\delta-\gamma\gamma)yy)}}{\gamma}.$$

Ponatur brevitatis gratia $\beta\beta-\alpha\gamma=Ap$, $\beta(\delta-\gamma)=Bp$ et $\delta\delta-\gamma\gamma=Cp$, eritque

$$\beta+\gamma y+\delta x=+\sqrt{(A+2Bx+Cxx)p},$$

$$\beta+\gamma x+\delta y=-\sqrt{(A+2By+Cyy)p}.$$

II.

Litteris jam A, B, C pro lubitu assumtis, ex iis litterae α, β, γ, δ et p sequenti modo definientur: Primo ex aequalitate secunda fit $\delta-\gamma=\frac{Bp}{\beta}$, qui valor in tertia $\delta+\gamma=\frac{Cp}{\delta-\gamma}$ substitutus dat $\delta+\gamma=\frac{C\beta}{B}$; ita ut sit

$$\delta=\frac{C\beta}{2B}+\frac{Bp}{2\beta} \quad \text{et} \quad \gamma=\frac{C\beta}{2B}-\frac{Bp}{2\beta}.$$

Hinc autem aequalitas prima abit in hanc

$$\beta\beta-\frac{C\alpha\beta}{2B}+\frac{B\alpha p}{2\beta}=Ap$$

ex qua definietur $p=\frac{\beta\beta(2B\beta-C\alpha)}{B(2A\beta-B\alpha)}$, indeque porro

$$\delta=\frac{\beta(AC\beta+BB\beta-BC\alpha)}{B(2A\beta-B\alpha)} \quad \text{et} \quad \gamma=\frac{\beta\beta(AC-BB)}{B(2A\beta-B\alpha)}.$$

Sic ergo litterae α et β arbitrio nostro relinquuntur, quarum altera quidem unitate exprimi poterit, altera vero constantem arbitrariam, a coëfficientibus A, B, C non pendentem, exhibebit.

III.

Differentietur nunc aequatio proposita, ac prodibit

$$dx\,(\beta + \gamma x + \delta y) + dy\,(\beta + \gamma y + \delta x) = 0,$$

unde conficitur haec aequatio

$$\frac{dx}{\beta + \gamma y + \delta x} = \frac{-dy}{\beta + \gamma x + \delta y},$$

quae substitutis valoribus in articulo I inventis, abibit in hanc aequationem differentialem:

$$\frac{dx}{\sqrt{(A + 2Bx + Cxx)}} - \frac{dy}{\sqrt{(A + 2By + Cyy)}} = 0$$

cujus propterea integralis est ipsa aequatio assumta.

IV.

Proposita ergo vicissim hac aequatione differentiali

$$\frac{dx}{\sqrt{(A + 2Bx + Cxx)}} - \frac{dy}{\sqrt{(A + 2By + Cyy)}} = 0,$$

ejus integrale semper algebraice exhiberi poterit, quippe quod erit

$$0 = \alpha + 2\beta\,(x + y) + \frac{\beta\beta(AC - BB)\,(xx + yy) + 2\beta(AC\beta + BB\beta - BC\alpha)\,xy}{B\,(2A\beta - B\alpha)},$$

et quia hic continetur constans ab arbitrio nostro pendens, erit hoc integrale quoque completum aequationis differentialis propositae. Erit ergo retentis litteris graecis

$$\text{vel}\quad y = \frac{-\beta - \delta x + \sqrt{(A + 2Bx + Cxx)p}}{\gamma},$$

$$\text{vel}\quad x = \frac{-\beta - \delta y - \sqrt{(A + 2By + Cyy)p}}{\gamma}.$$

V.

Quemadmodum autem istarum formularum integralium differentia

$$\int\frac{dx}{\sqrt{(A + 2Bx + Cxx)}} - \int\frac{dy}{\sqrt{(A + 2By + Cyy)}},$$

est constans, siquidem inter x et y ea relatio subsistat, ut sit

$$0 = \alpha + 2\beta\,(x + y) + \gamma\,(xx + yy) + 2\delta xy,$$

ita etiam eadem manente relatione, differentia hujusmodi formularum

$$\int\frac{x^n dx}{\sqrt{(A + 2Bx + Cxx)}} - \int\frac{y^n dy}{\sqrt{(A + 2By + Cyy)}}$$

commode exprimi potest; quos valores indagasse operae pretium erit.

VI.

Posito ergo exponente $n = 1$, statuamus

$$\frac{x\,dx}{\sqrt{(A + 2Bx + Cxx)}} - \frac{y\,dy}{\sqrt{(A + 2By + Cyy)}} = dV,$$

eritque valoribus initio traditis pro his formulis irrationalibus substituendis

$$\frac{x\,dx\sqrt{p}}{\beta+\gamma y+\delta x}+\frac{y\,dy\sqrt{p}}{\beta+\gamma x+\delta y}=dV,$$

seu $x\,dx\,(\beta+\gamma x+\delta y)+y\,dy\,(\beta+\gamma y+\delta x)=\frac{dV}{\sqrt{p}}(\beta+\gamma y+\delta x)(\beta+\gamma x+\delta y)$; at est

$$(\beta+\gamma y+\delta x)(\beta+\gamma x+\delta y)=\beta\beta+\beta(\gamma+\delta)(x+y)+\gamma\delta(xx+yy)+(\gamma\gamma+\delta\delta)xy.$$

VII.

Quo hanc formulam facilius expediamus, ponamus $x+y=t$ et $xy=u$, erit

$$xx+yy=tt-2u \quad \text{et} \quad x^3+y^3=t^3-3tu,$$

sicque aequatio abit in hanc formam

$$\beta(x\,dx+y\,dy)+\gamma(xx\,dx+yy\,dy)+\delta xy(dx+dy)=\frac{dV}{\sqrt{p}}(\beta\beta+\beta(\gamma+\delta)t+\gamma\delta tt+(\gamma-\delta)^2u).$$

Ipsa autem aequatio assumta fit: $0=\alpha+2\beta t+\gamma tt+2(\delta-\gamma)u$, et penitus introductis litteris t et u habebimus

$$\beta(t\,dt-du)+\gamma(tt\,dt-t\,du-u\,dt)+\delta u\,dt=\frac{dV}{\sqrt{p}}(\beta\beta-\alpha\delta+\beta(\gamma-\delta)t+(\gamma\gamma-\delta\delta)u),$$

seu $dt(\beta t+\gamma tt-(\gamma-\delta)u)-du(\beta+\gamma t)=\frac{dV}{\sqrt{p}}(\beta\beta-\alpha\delta+\beta(\gamma-\delta)t+(\gamma\gamma-\delta\delta)u)$.

VIII.

Ex aequatione autem assumta si differentietur, fit $dt(\beta+\gamma t)=(\gamma-\delta)du$, unde aequationis ultimae prius membrum transformatur in

$$\frac{dt}{\gamma-\delta}(-\beta\beta-\beta(\gamma+\delta)t-\gamma\delta tt-(\gamma-\delta)^2u),$$

quod cum aequale esse debeat huic formulae

$$\frac{dV}{\sqrt{p}}(\beta\beta+\beta(\gamma+\delta)t+\gamma\delta tt+(\gamma-\delta)^2u),$$

commode inde oritur $\frac{dV}{\sqrt{p}}=\frac{-dt}{\gamma-\delta}$ et $V=\frac{-t\sqrt{p}}{\gamma-\delta}$.

IX.

Cum jam sit $t=x+y$, habebimus sequentem aequationem integratam

$$\int\frac{x\,dx}{\sqrt{(A+2Bx+Cxx)}}-\int\frac{y\,dy}{\sqrt{(A+2By+Cyy)}}=\text{Const.}-\frac{(x+y)\sqrt{p}}{\gamma-\delta},$$

existente $0=\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy$, siquidem relationes supra exhibitae inter litteras A, B, C et α, β, γ, δ ac p locum habeant. Hinc ergo eadem manente determinatione variabilium x et y erit generalius:

$$\int\frac{dx(\mathfrak{A}+\mathfrak{B}x)}{\sqrt{(A+2Bx+Cxx)}}-\int\frac{dy(\mathfrak{A}+\mathfrak{B}y)}{\sqrt{(A+2By+Cyy)}}=\text{Const.}-\frac{\mathfrak{B}(x+y)\sqrt{p}}{\gamma-\delta}.$$

X.

Progrediamur porro, ac statuamus

$$\frac{xx\,dx}{\sqrt{(A+2Bx+Cxx)}}-\frac{yy\,dy}{\sqrt{(A+2By+Cyy)}}=dV,$$

erit posito brevitatis ergo $\beta\beta + \beta(\gamma + \delta)t + \gamma\delta tt + (\gamma - \delta)^2 u = T$, si loco istarum formularum surdarum valores ante reperti substituantur

$xxdx(\beta + \gamma x + \delta y) + yydy(\beta + \gamma y + \delta x) = \frac{TdV}{\sqrt{p}}$, existente ut ante $t = x + y$ et $u = xy$.

XI.

Cum nunc sit $x^4 + y^4 = t^4 - 4ttu + 2uu$, erit eliminatis variabilibus x et y

$$\beta(ttdt - tdu - udt) + \gamma(t^3dt - ttdu - 2tudt + udu) + \delta u(tdt - du) = \frac{TdV}{\sqrt{p}},$$

sive
$$dt(\beta tt - \beta u + \gamma t^3 - 2\gamma tu + \delta tu) - du(\beta t + \gamma tt - \gamma u + \delta u) = \frac{TdV}{\sqrt{p}}.$$

Cum autem sit $du = \frac{dt(\beta + \gamma t)}{\gamma - \delta}$, erit hac facta substitutione

$$\frac{dt}{\gamma - \delta}(-\beta\beta t - \beta(\gamma + \delta)tt - \gamma\delta t^3 - (\gamma - \delta)^2 tu) = \frac{TdV}{\sqrt{p}} = \frac{-Ttdt}{\gamma - \delta},$$

sicque erit $\frac{dV}{\sqrt{p}} = \frac{-tdt}{\gamma - \delta}$ et $V = \frac{-tt\sqrt{p}}{2(\gamma - \delta)}$.

XII.

Hinc ergo adipiscimur sequentem aequationem integratam

$$\int\frac{xxdx}{\sqrt{(A + 2Bx + Cxx)}} - \int\frac{yydy}{\sqrt{(A + 2By + Cyy)}} = \text{Const.} - \frac{(x + y)^2\sqrt{p}}{2(\gamma - \delta)},$$

atque in genere concludimus fore

$$\int\frac{dx(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}xx)}{\sqrt{(A + 2Bx + Cxx)}} - \int\frac{dy(\mathfrak{A} + \mathfrak{B}y + \mathfrak{C}yy)}{\sqrt{(A + 2By + Cyy)}} = \text{Const.} - \frac{\mathfrak{B}(x + y)\sqrt{p}}{\gamma - \delta} - \frac{\mathfrak{C}(x + y)^2\sqrt{p}}{2(\gamma - \delta)},$$

siquidem fuerit $0 = \alpha + 2\beta(x + y) + \gamma(xx + yy) + 2\delta xy$. Erit autem ex relationibus supra assignatis $\frac{\sqrt{p}}{\gamma - \delta} = \frac{-\beta}{B\sqrt{p}}$ sive $\frac{\sqrt{p}}{\gamma - \delta} = -\sqrt{\frac{2A\beta - B\alpha}{B(2B\beta - C\alpha)}}$.

XIII.

Ponatur jam in genere

$$\frac{x^n dx}{\sqrt{(A + 2Bx + Cxx)}} - \frac{y^n dy}{\sqrt{(A + 2By + Cyy)}} = dV,$$

eritque ponendo $T = \beta\beta + \beta(\gamma + \delta)t + \gamma\delta tt + (\gamma - \delta)^2 u$,

$$x^n dx(\beta + \gamma x + \delta y) + y^n dy(\beta + \gamma y + \delta x) = \frac{TdV}{\sqrt{p}},$$

at ob $x + y = t$ et $xy = u$ habebimus $x = \frac{t + \sqrt{(tt - 4u)}}{2}$ et $y = \frac{t - \sqrt{(tt - 4u)}}{2}$, ideoque

$$\beta + \gamma x + \delta y = \frac{2\beta + (\gamma + \delta)t + (\gamma - \delta)\sqrt{(tt - 4u)}}{2},$$

$$\beta + \gamma y + \delta x = \frac{2\beta + (\gamma + \delta)t - (\gamma - \delta)\sqrt{(tt - 4u)}}{2}.$$

XIV.

Differentiando autem habebimus

$$dx = \frac{dt\sqrt{(tt - 4u)} + tdt - 2du}{2\sqrt{(tt - 4u)}} \quad \text{et} \quad dy = \frac{dt\sqrt{(tt - 4u)} - tdt + 2du}{2\sqrt{(tt - 4u)}},$$

at ante vidimus esse $du = \frac{dt(\beta + \gamma t)}{\gamma - \delta}$: quo valore substituto prodibit

$$dx = \frac{-dt(2\beta + (\gamma + \delta)t - (\gamma - \delta)\sqrt{(tt - 4u)})}{2(\gamma - \delta)\sqrt{(tt - 4u)}}$$

$$dy = \frac{dt(2\beta + (\gamma + \delta)t + (\gamma - \delta)\sqrt{(tt - 4u)})}{2(\gamma - \delta)\sqrt{(tt - 4u)}}.$$

Hisque valoribus substitutis

$$dx(\beta + \gamma x + \delta y) = \frac{-dt(4\beta\beta + 4\beta(\gamma + \delta)t + 4\gamma\delta tt + 4(\gamma - \delta)^2 u)}{4(\gamma - \delta)\sqrt{(tt - 4u)}} = \frac{-Tdt}{(\gamma - \delta)\sqrt{(tt - 4u)}},$$

$$\text{et} \quad dy(\beta + \gamma y + \delta x) = \frac{+Tdt}{(\gamma - \delta)\sqrt{(tt - 4u)}}.$$

XV.

Nostra ergo aequatione per T divisa habebimus

$$\frac{-dt(x^n - y^n)}{(\gamma - \delta)\sqrt{(tt - 4u)}} = \frac{dV}{\sqrt{p}} \quad \text{et} \quad V = \frac{-\sqrt{p}}{\gamma - \delta}\int\frac{dt(x^n - y^n)}{\sqrt{(tt - 4u)}},$$

existente $x = \frac{t + \sqrt{(tt - 4u)}}{2}$ et $y = \frac{t - \sqrt{(tt - 4u)}}{2}$ atque $u = \frac{\alpha + 2\beta t + \gamma tt}{2(\gamma - \delta)}$, unde

$$\sqrt{(tt - 4u)} = \sqrt{\frac{2\alpha + 4\beta t + (\gamma + \delta)tt}{\delta - \gamma}}.$$

Unde valores ipsius $\frac{x^n - y^n}{\sqrt{(tt - 4u)}}$ ex sequente progressione colligi poterunt:

$$\frac{x^0 - y^0}{\sqrt{(tt - 4u)}} = 0,$$

$$\frac{x^1 - y^1}{\sqrt{(tt - 4u)}} = 1,$$

$$\frac{x^2 - y^2}{\sqrt{(tt - 4u)}} = t,$$

$$\frac{x^3 - y^3}{\sqrt{(tt - 4u)}} = tt - u = \frac{(\gamma - 2\delta)tt - 2\beta t - \alpha}{2(\gamma - \delta)},$$

$$\frac{x^4 - y^4}{\sqrt{(tt - 4u)}} = t^3 - 2tu = \frac{-2\delta t^3 - 4\beta tt - 2\alpha t}{2(\gamma - \delta)},$$

$$\frac{x^5 - y^5}{\sqrt{(tt - 4u)}} = t^4 - 3ttu + uu = \frac{-(\gamma\gamma + 2\gamma\delta - 4\delta\delta)t^4 - 4\beta(2\gamma - 3\delta)t^3 + (4\beta\beta - 4\alpha\gamma + 6\alpha\delta)tt + 4\alpha\beta t + \alpha\alpha}{4(\gamma - \delta)^2}$$

etc. etc.

XVI.

Nanciscemur ergo formulas sequentes integratas

$$\int\frac{x^3 dx}{\sqrt{(A + 2Bx + Cxx)}} - \int\frac{y^3 dy}{\sqrt{(A + 2By + Cyy)}} = \text{Const.} - \frac{\sqrt{p}}{2(\gamma - \delta)^2}\left(\frac{1}{3}(\gamma - 2\delta)(x + y)^3 - \beta(x + y)^2 - \alpha(x + y)\right)$$

$$\int\frac{x^4 dx}{\sqrt{(A + 2Bx + Cxx)}} - \int\frac{y^4 dy}{\sqrt{(A + 2By + Cyy)}} = \text{Const.} + \frac{\sqrt{p}}{(\gamma - \delta)^2}\left(\frac{1}{4}\delta(x + y)^4 + \frac{2}{3}\beta(x + y)^3 + \frac{1}{2}\alpha(x + y)^2\right)$$

quae scilicet locum habent, si variabiles x et y ita a se invicem pendent, ut sit

$$0 = \alpha + 2\beta(x + y) + \gamma(xx + yy) + 2\delta xy,$$

atque hi coëfficientes pariter atque p secundum praescriptas formulas ex datis A, B, C determinentur.

XVII.

Hinc ergo infinitae formulae integrales exhiberi possunt, quae etsi ipsae non sint integrabiles, earum tamen differentia vel sit constans, vel geometrice seu algebraice assignari queat. Quae comparatio cum in analysi insignem habeat usum, tum imprimis in arcubus curvarum irrectificabilium inter se comparandis summam affert utilitatem, quam in aliquot exemplis ostendisse juvabit.

De comparatione arcuum Circuli.

1. Sit radius circuli $=1$, in eoque abscissa a centro sumta $=z$; erit arcus ei respondens $=\int\frac{dz}{\sqrt{(1-zz)}}$, cujus propterea sinus est $=z$. Ut igitur nostrae formulae hujusmodi arcus circuli exprimant, poni debet $A=1$, $B=0$, $C=-1$; quo facto habebimus

$$\beta\beta-\alpha\gamma=p,\quad \beta(\delta-\gamma)=0 \quad\text{et}\quad \delta\delta-\gamma\gamma=-p;$$

has enim determinationes ab ipsa origine peti oportet, quia ob $B=0$, valores inventi fiunt incongrui. Jam ex formula secunda sequitur vel $\delta-\gamma=0$, vel $\beta=0$, quorum ille valor $\delta=\gamma$ formulae tertiae adversatur. Erit ergo $\beta=0$, $\delta=\pm\sqrt{(\gamma\gamma-p)}$ et $\alpha=\frac{-p}{\gamma}$. Ambae ergo quantitates constantes γ et p arbitrio nostro relinquuntur.

2. Quo formulae nostrae fiant simpliciores, ponamus $\gamma=1$ et $p=cc$, eritque

$$\alpha=-cc,\quad \beta=0,\quad \gamma=1 \quad\text{et}\quad \delta=-\sqrt{(1-cc)},$$

ac nostra aequatio canonica, relationem variabilium x et y determinans, fiet

$$0=-cc+xx+yy-2xy\sqrt{(1-cc)},$$

ex qua colligitur

$$y=x\sqrt{(1-cc)}\pm c\sqrt{(1-xx)}.$$

3. Quodsi ergo iste valor ipsi y tribuatur, erit

$$\int\frac{dx}{\sqrt{(1-xx)}}-\int\frac{dy}{\sqrt{(1-yy)}}=\text{Const.}$$

Denotemus brevitatis gratia haec integralia ita

$$\int\frac{dx}{\sqrt{(1-xx)}}=\Pi.x \quad\text{et}\quad \int\frac{dx}{\sqrt{(1-yy)}}=\Pi.y$$

atque $\Pi.x$ et $\Pi.y$ indicabunt arcus circuli, abscissis seu sinibus x et y respondentes. Quocirca erit

$$\Pi.x-\Pi.(x\sqrt{(1-cc)}+c\sqrt{(1-xx)})=\text{Const.}$$

4. Ad constantem determinandam ponatur $x=0$, et ob $\Pi.0=0$, fiet Const. $=-\Pi.c$ sicque erit

$$\Pi.c+\Pi.x=\Pi.(x\sqrt{(1-cc)}+c\sqrt{(1-xx)});$$

unde arcus assignari poterit aequalis summae duorum arcuum quorumcunque. Ac si x capiatur negativum ob $\Pi.(-x)=-\Pi.x$, erit

$$\Pi.c-\Pi.x=\Pi.(c\sqrt{(1-xx)}-x\sqrt{(1-cc)}),$$

qua arcus, differentiae duorum arcuum aequalis, definitur.

5. Si in formula priori ponatur $x = c$, erit $2\Pi . c = \Pi . 2c\sqrt{(1-cc)}$. Ac si porro ponatur $x = 2c\sqrt{(1-cc)}$, ut sit $\Pi . x = 2\Pi . c$, erit ob $\sqrt{(1-xx)} = 1-2cc$,

$$3\Pi . c = \Pi . (3c - 4c^3).$$

Posito autem ultra $x = 3c - 4c^3$, erit

$$4\Pi . c = \Pi . (x\sqrt{(1-cc)} + c\sqrt{(1-xx)})$$

unde multiplicatio arcuum circularium est manifesta.

De comparatione arcuum Parabolae.

* 6. Existente (Fig. 55.) AB parabolae axe, sumentur abscissae AP in tangente verticis A, sitque parameter parabolae $= 2$; unde vocata abscissa quacunque $AP = z$, erit applicata $Pp = \frac{zz}{2}$, ideoque arcus $Ap = \int dz\sqrt{(1+zz)}$, quae expressio ut ad nostras formulas reducatur, in hanc abit $\int \frac{dz(1+zz)}{\sqrt{(1+zz)}}$. Quare fieri oportet $A = 1$, $B = 0$ et $C = 1$, unde ut ante habebimus

$$\beta = 0, \quad \alpha = \frac{-p}{\gamma} \quad \text{et} \quad \delta = \pm\sqrt{(\gamma\gamma + p)}.$$

Sit ergo $\gamma = 1$ et $p = cc$, atque aequatio relationem inter x et y exhibens erit

$$0 = -cc + xx + yy - 2xy\sqrt{(cc+1)}, \quad \text{seu} \quad y = x\sqrt{(1+cc)} + c\sqrt{(1+xx)}.$$

7. Deinde ob $\sqrt{p} = c$ et $\gamma - \delta = 1 + \sqrt{(1+cc)}$, facto $\mathfrak{A} = 1$, $\mathfrak{B} = 0$ et $\mathfrak{C} = 1$, erit ex formula XII data

$$\int \frac{dx\,(1+xx)}{\sqrt{(1+xx)}} - \int \frac{dy\,(1+yy)}{\sqrt{(1+yy)}} = \text{Const.} - \frac{c\,(x+y)^2}{2+2\sqrt{(1+cc)}}.$$

At est $x + y = x\,(1 + \sqrt{(1+cc)}) + c\sqrt{(1+xx)}$, ergo

$$(x+y)^2 = 2xx\,(1 + cc + \sqrt{(1+cc)}) + cc + 2cx\,(1+\sqrt{(1+cc)})\sqrt{(1+xx)}.$$

Quare formularum istarum integralium differentia erit

$$\text{Const.} - cxx\sqrt{(1+cc)} - ccx\sqrt{(1+xx)} = \text{Const.} - cxy.$$

8. Indicetur arcus parabolae abscissae cuicunque z respondens $\int dz\sqrt{(1+zz)}$ per $\Pi . z$, et nostra aequatio hanc induet formam:

$$\Pi . x - \Pi . (x\sqrt{(1+cc)} + c\sqrt{(1+xx)}) = -\Pi . c - cx\,(x\sqrt{(1+cc)} + c\sqrt{(1+xx)}),$$

sive $$\Pi . c + \Pi . x = \Pi . (x\sqrt{(1+cc)} + c\sqrt{(1+xx)}) - cx\,(x\sqrt{(1+cc)} + c\sqrt{(1+xx)}).$$

Datis ergo duobus arcubus quibuscunque, tertius arcus assignari potest, qui a summa illorum deficiat quantitate geometrice assignabili. Vel quo indoles hujus aequationis clarius perspiciatur, erit

$$\Pi . c + \Pi . x = \Pi . y - cxy$$

siquidem fuerit $$y = x\sqrt{(1+cc)} + c\sqrt{(1+xx)}.$$

9. Cum sit $y > x$, sint in figura abscissae $AE = c$, $AF = x$ et $AG = y$, erit arcus $Ae = \Pi . c$ et arcus $fg = \Pi . y - \Pi . x$; hinc ergo habebimus

$$\text{Arc} . Ae = \text{Arc} . fg - cxy, \quad \text{seu} \quad \text{Arc} . fg - \text{Arc} . Ae = cxy$$

existente $y = x\sqrt{(1+cc)} + c\sqrt{(1+xx)}$. Ex his igitur sequentia problemata circa parabolam resolvi poterunt.

10. **Problema 1.** Dato arcu parabolae Ae, in vertice A terminato a puncto quovis f, alium abscindere arcum fg, ita ut differentia horum arcuum $fg - Ae$ geometrice assignari queat.

Solutio. Ponatur arcus dati Ae abscissa $AE = e$, et abscissa, termino dato f arcus quaesiti fg respondens, $AF = f$; abscissa vero alteri termino g arcus quaesiti respondens, $AG = g$, quae ita accipiatur, ut sit $g = f\sqrt{(1+ee)} + e\sqrt{(1+ff)}$; eritque existente parabolae parametro $= 2$, uti constanter assumemus: $\text{Arc}.fg - \text{Arc}.Ae = efg$.

A puncto autem f quoque retrorsum arcus abscindi potest $f\gamma$, qui superet arcum Ae quantitate algebraica: ob signum radicale $\sqrt{(1+ff)}$ enim ambiguum, capiatur

$$AT = \gamma = f\sqrt{(1+ee)} - e\sqrt{(1+ff)}$$

eritque $\text{Arc}.f\gamma - \text{Arc}.Ae = ef\gamma$. Q. E. I.

11. **Coroll. 1.** Inventis ergo his duobus punctis g et γ, erit quoque arcuum fg et $f\gamma$ differentia geometrice assignabilis; erit enim

$$\text{Arc}.fg - \text{Arc}.f\gamma = ef(g-\gamma).$$

At est $g - \gamma = 2e\sqrt{(1+ff)}$; unde $e = \frac{g-\gamma}{2\sqrt{(1+ff)}}$. Tum vero habemus $g + \gamma = 2f\sqrt{(1+ee)}$, sive $\sqrt{(1+ee)} = \frac{g+\gamma}{2f}$; unde eliminanda e fit

$$1 = \frac{(g+\gamma)^2}{4ff} - \frac{(g-\gamma)^2}{4(1+ff)}, \quad \text{seu} \quad 4ff(1+ff) = (g+\gamma)^2 + 4ffg\gamma.$$

Fit ergo $$\gamma = -g(1+2ff) + 2f\sqrt{(1+ff)(1+gg)}.$$

12. **Coroll. 2.** Dato ergo arcu quocunque fg, existente $AF = f$ et $AG = g$, a puncto f retrorsum arcus $f\gamma$ abscindi potest, ita ut arcuum fg et $f\gamma$ differentia fiat geometrica. Capiatur scilicet $AT = \gamma = -g(1+2ff) + 2f\sqrt{(1+ff)(1+gg)}$ eritque

$$\text{Arc}.fg - \text{Arc}.f\gamma = 2f(g\sqrt{(1+ff)} - f\sqrt{(1+gg)})^2\sqrt{(1+ff)}.$$

Horum ergo arcuum differentia evanescere nequit, nisi sit vel $f = 0$, quo casu fit $\gamma = -g$, vel $g = f$, quo casu uterque arcus fg et $f\gamma$ evanescit.

13. **Coroll. 3.** Ut igitur positis $AE = e$, $AF = f$, $AG = g$ differentia arcuum fg et Ae fiat geometrice assignabilis scilicet $\text{Arc}.fg - \text{Arc}.Ae = efg$, oportet sit

$$g = f\sqrt{(1+ee)} + e\sqrt{(1+ff)},$$

seu ex trium quantitatum e, f, g binis datis tertia ita determinatur, ut sit

$$\text{vel} \quad g = f\sqrt{(1+ee)} + e\sqrt{(1+ff)},$$
$$\text{vel} \quad f = g\sqrt{(1+ee)} - e\sqrt{(1+gg)},$$
$$\text{vel} \quad e = g\sqrt{(1+ff)} - f\sqrt{(1+gg)}.$$

14. **Coroll. 4.** Cum sit $g = f\sqrt{(1+ee)} + e\sqrt{(1+ff)}$, erit

$$\sqrt{(1+gg)} = ef + \sqrt{(1+ee)(1+ff)},$$

unde colligitur $$g + \sqrt{(1+gg)} = (e + \sqrt{(1+ee)})(f + \sqrt{(1+ff)}).$$

Ergo ut arcus fg superet arcum Ae quantitate algebraica efg, oportet ut sit

$$\frac{g + \sqrt{(1+gg)}}{f + \sqrt{(1+ff)}} = e + \sqrt{(1+ee)}.$$

15. **Coroll. 5.** Haec ultima formula ideo est notatu digna, quod in ea quantitatum e, f et g functiones sint a se invicem separatae. Quod si ergo ponatur

$$e+\sqrt{(1+ee)}=E,\quad f+\sqrt{(1+ff)}=F,\quad g+\sqrt{(1+gg)}=G,$$

erit

$$e=\frac{EE-1}{2E},\qquad f=\frac{FF-1}{2F},\qquad g=\frac{GG-1}{2G}.$$

Quare si capiatur $\frac{G}{F}=E$, erit arcuum differentia

$$\text{Arc}.fg-\text{Arc}.Ae=efg=\frac{(EE-1)(FF-1)(GG-1)}{8EFG},$$

seu

$$\text{Arc}.fg-\text{Arc}.Ae=\frac{(FF-1)(GG-1)(GG-FF)}{8FFGG}=\frac{fg(GG-FF)}{2FG}.$$

16. **Problema 2.** Dato arcu parabolae quocunque fg, a puncto parabolae dato p alium abscindere arcum pq ita, ut differentia horum duorum arcuum fg et pq fiat geometrice assignabilis.

Solutio. Pro arcu dato fg ponantur abscissae $AF=f$, $AG=g$; pro arcu autem quaesito pq sint abscissae $AP=p$, $AQ=q$. Jam a vertice parabolae concipiatur arcus Ae respondens abscissae $AE=e$, cujus defectus ab utroque illorum arcuum sit geometrice assignabilis. Ad hoc autem vidimus (14) requiri, ut sit

$$\frac{g+\sqrt{(1+gg)}}{f+\sqrt{(1+ff)}}=e+\sqrt{(1+ee)}\ \text{ et }\ \frac{q+\sqrt{(1+qq)}}{p+\sqrt{(1+pp)}}=e+\sqrt{(1+ee)}.$$

Ponamus brevitatis gratia

$$f+\sqrt{(1+ff)}=F\qquad p+\sqrt{(1+pp)}=P$$
$$g+\sqrt{(1+gg)}=G\qquad q+\sqrt{(1+qq)}=Q$$

atque ut problemati satisfiat, necesse est sit $\frac{G}{F}=\frac{Q}{P}$. Porro autem cum sit ex (15)

$$\text{Arc}.fg-\text{Arc}.Ae=\frac{fg(GG-FF)}{2FG}\quad\text{similiterque}\quad\text{Arc}.pq-\text{Arc}.Ae=\frac{pq(QQ-PP)}{2PQ},$$

erit arcuum determinatorum differentia

$$\text{Arc}.pq-\text{Arc}.fg=\frac{pq(QQ-PP)}{2PQ}-\frac{fg(GG-FF)}{2FG}$$

ideoque geometrice assignabilis. Q. E. I.

17. **Coroll. 1.** Cum autem sit $\frac{G}{F}=\frac{Q}{P}$, erit $\frac{QQ-PP}{2PQ}=\frac{GG-FF}{2FG}$, unde differentia arcuum determinatorum prodit

$$\text{Arc}.pq-\text{Arc}.fg=\frac{(pq-fg)(GG-FF)}{2FG}.$$

Est autem $f=\frac{FF-1}{2F}$, $g=\frac{GG-1}{2G}$, $p=\frac{PP-1}{2P}$, $q=\frac{QQ-1}{2Q}$, ideoque ob $Q=\frac{GP}{F}$, erit

$$q=\frac{GGPP-FF}{2FGP}.$$

18. **Coroll. 2.** Erit ergo

$$pq=\frac{(PP-1)(GGPP-FF)}{4FGPP}\quad\text{et}\quad fg=\frac{(FF-1)(GG-1)}{4FG}\qquad\text{ideoque}$$

$$pq - fg = \frac{(PP - FF)(GGPP - 1)}{4FGPP}.$$

Hinc arcuum differentia prodit

$$\text{Arc}.\,pq - \text{Arc}.\,fg = \frac{(GG - FF)(PP - FF)(GGPP - 1)}{8FFGGPP}.$$

19. **Coroll. 3.** Ut igitur arcus pq arcui fg adeo fiat aequalis, esse oportet vel $GG - FF = 0$, vel $PP - FF = 0$, vel $GGPP - 1 = 0$. Primo autem casu arcus fg ideoque et pq evanescit; altero casu punctum p in f, ideoque et q in g cadit, arcusque ergo pq non prodit diversus ab arcu fg; tertius autem casus dat $P = \frac{1}{G}$, seu $p + \sqrt{(1 + pp)} = \frac{1}{g + \sqrt{(1 + gg)}} = \sqrt{(1 + gg)} - g$, unde fit $p = -g$ et $q = -f$, ita ut pq in alterum ramum parabolae cadat, arcuique fg similis et aequalis prodeat.

20. **Coroll. 4.** Hinc ergo sequitur, in parabola non exhiberi posse duos arcus dissimiles, qui sint inter se aequales. Interim proposito quocunque arcu fg, infinitis modis alius abscindi potest pq, qui illum quantitate algebraica superet, vel ab eo deficiat. Superabit scilicet, si fuerit $P > F$, seu $AP > AF$; deficiet autem, si $P < F$, seu $AP < AF$.

21. **Problema 3.** Dato parabolae arcu quocunque fg, a dato puncto p alium arcum abscindere pr, qui duplum arcus fg superet quantitate geometrice assignabili.

Solutio. Positis ut ante abscissis $AF = f$, $AG = g$, $AP = p$, $AQ = q$, sit $AR = r$ denotentque litterae majusculae F, G, P, Q, R istas functiones $f + \sqrt{(1 + ff)}$, $g + \sqrt{(1 + gg)}$ etc. minuscularum cognominum. Primum igitur si statuatur $\frac{Q}{P} = \frac{G}{F}$, erit

$$\text{Arc}.\,pq - \text{Arc}.\,fg = \frac{(pq - fg)(GG - FF)}{2FG}.$$

Simili autem modo si statuatur $\frac{R}{Q} = \frac{G}{F}$, erit

$$\text{Arc}.\,qr - \text{Arc}.\,fg = \frac{(qr - fg)(GG - FF)}{2FG}.$$

Addantur ergo invicem hae duae aequationes, erit

$$\text{Arc}.\,pr - 2\,\text{Arc}.\,fg = \frac{(pq + qr - 2fg)(GG - FF)}{2FG}.$$

Ut jam ex calculo eliminentur litterae q et Q, erit primo $\frac{R}{P} = \frac{GG}{FF}$; tum vero est $q = \frac{GGPP - FF}{2FGP}$, seu $q = \frac{F(PR - 1)}{2GP}$, et ob $p = \frac{PP - 1}{2P}$ et $r = \frac{G^4P^2 - F^4}{2F^2G^2P}$, erit

$$p + r = \frac{(FF + GG)(GGPP - FF)}{2FFGGP},$$

ideoque

$$pq + qr = \frac{(FF + GG)(GGPP - FF)^2}{4F^3G^3PP} \quad \text{et} \quad 2fg = \frac{2(FF - 1)(GG - 1)}{4FG}.$$

Sumto ergo $\frac{R}{P} = \frac{GG}{FF}$, arcus pr superabit duplum arcus fg quantitate algebraica. Q. E. I.

22. **Coroll. 1.** Punctum igitur p ita assumi poterit, ut excessus arcus pr supra duplum arcum $2fg$ sit datae magnitudinis; definietur enim P per aequationem algebraicam, ope extractionis radicis quadratae tantum.

23. **Coroll. 2.** Fieri igitur poterit, ut arcus pr praecise sit duplus arcus dati fg, quod evenit si P definiatur ex hac aequatione

$$(GGPP - FF)^2 = \frac{2(FF-1)(GG-1)FFGGPP}{FF+GG}$$

unde elicitur $$\frac{GGPP}{FF} = \frac{FFGG+1+\sqrt{(F^4-1)(G^4-1)}}{FF+GG}$$

et $$\frac{GP}{F} = \frac{\sqrt{\frac{1}{2}(FF+1)(GG+1)}+\sqrt{\frac{1}{2}(FF-1)(GG-1)}}{\sqrt{(FF+GG)}} = \frac{FR}{G}.$$

24. **Coroll. 3.** Haec autem determinatio arcus dupli pr maxime fit obvia, si arcus datus fg in vertice A incipiat; tum enim ob $F=1$ fit $GP=F$, seu $P=\frac{1}{G}=\sqrt{(1+gg)}-g$. Obtinetur ergo $p=-g$ et $R=G$, ideoque $r=g$. Hoc scilicet casu arcus pr in parabola circa verticem A utrinque aequaliter extendetur, sicque manifesto fit duplus arcus propositi.

25. **Coroll. 4.** Fieri quoque potest, ut arcus pr in ipso puncto g terminetur, sicque ambo arcus, simplus fg et duplus pr, evadant contigui. Hoc nempe evenit si $P=G$, quo casu haec habetur aequatio

$$F^6+F^4G^2-2F^4G^6+F^2G^8-2F^2G^4+G^{10}=0,$$

quae per $FF-GG$ divisa praebet

$$F^4-2FFG^6+2FFGG-G^8=0$$

unde elicitur

$$FF=GG(G^4-1)+GG\sqrt{(G^8-G^4+1)} \quad \text{ideoque} \quad F=G\sqrt{(G^4-1+\sqrt{(G^8-G^4+1)})}$$

$$\text{et} \quad R=\frac{G^3}{FF} \quad \text{seu} \quad R=\frac{\sqrt{(G^8-G^4+1)}-G^4+1}{G^3}.$$

26. **Coroll. 5.** Quantitas ergo G, seu parabolae punctum g pro lubitu assumi licet, in quo duo arcus terminabuntur, quorum alter alterius exacte erit duplus. Cum autem sumto g affirmativo, ideoque $G>1$, prodeat $F>G$, punctum f a vertice magis erit remotum quam punctum g; tum vero reperitur

$$r=\frac{RR-1}{2R}=\frac{-(GG-1)\sqrt{(G^8-G^4+1)}-G^6-G^4+GG+1}{2G^3},$$

cujus valor cum sit negativus, punctum r in alterum parabolae ramum incidit. Arcus ergo ita erunt * dispositi, ut habet figura 56, eritque

$$\text{Arcus } gr = 2\,\text{Arc}.fg.$$

27. **Coroll. 6.** Sit g valde parvum, erit $G=1+g+\frac{1}{2}gg$, hincque $G^2=1+2g+2gg$, $G^3=1+3g+\frac{9}{2}gg$, $G^4=1+4g+8gg$ et $G^8=1+8g+32gg$, unde

$$F=(1+g+\frac{1}{2}gg)(1+3g+\frac{9}{2}gg)=1+4g+8gg,$$

* ergo $f=\frac{FF-1}{2F}=4g$; porro $R=1-5g+\frac{25}{2}gg$, unde $r=-5g$. Quare (Fig. 56) si Ag valde parvum, erit proxime $AF=4AG$ et $AR=5AG$, ita ut sit quoque $GR=2GF$.

28. **Scholion.** Antequam ad ulteriorem arcuum parabolicorum multiplicationem progrediamur, etiamsi ea ex formulis datis non difficulter erui queat, tamen expediet differentiam algebraicam arcuum parabolicorum commodius exprimere. Cum igitur (Fig. 55) positis abscissis $AE=e$, $AF=f$, * $AG=g$ invenerimus (13) $\text{Arc}.Ag-\text{Arc}.Af-\text{Arc}.Ae=efg$, existente $e=g\sqrt{(1+ff)}-f\sqrt{(1+gg)}$, videndum est, num quantitas efg non possit transformari in terna membra, quae sint singula functiones certae ipsarum e, f et g, ita ut sit $efg=\text{funct}.g-\text{funct}.f-\text{funct}.e$; sic enim quaelibet harum functionum cum arcu cognomine comparari posset. Cum autem sit

$$efg=fgg\sqrt{(1+ff)}-ffg\sqrt{(1+gg)} \quad \text{et} \quad \sqrt{(1+ee)}=\sqrt{(1+ff)}(1+gg)-fg,$$

erit $$e\sqrt{(1+ee)}=g\sqrt{(1+gg)}+2ffg\sqrt{(1+gg)}-f\sqrt{(1+ff)}-2fgg\sqrt{(1+ff)},$$ hincque

$$fgg\sqrt{(1+ff)}-ffg\sqrt{(1+gg)}=efg=\tfrac{1}{2}g\sqrt{(1+gg)}-\tfrac{1}{2}f\sqrt{(1+ff)}-\tfrac{1}{2}e\sqrt{(1+ee)};$$

quae est expressio talis qualis desideratur. Quare si istas abscissarum e, f, g functiones brevitatis gratia ponamus $\frac{1}{2}e\sqrt{(1+ee)}=\mathfrak{E}$, $\frac{1}{2}f\sqrt{(1+ff)}=\mathfrak{F}$ et $\frac{1}{2}g\sqrt{(1+gg)}=\mathfrak{G}$, habebimus

$$\text{Arc}.Ag-\text{Arc}.Af-\text{Arc}.Ae=\mathfrak{G}-\mathfrak{F}-\mathfrak{E}=\text{Arc}.fg-\text{Arc}.Ae.$$

Si porro hae functiones cum illis, quibus ante usi sumus, comparemus, scilicet

$$e+\sqrt{(1+ee)}=E,\quad f+\sqrt{(1+ff)}=F,\quad g+\sqrt{(1+gg)}=G,$$

erit $$\mathfrak{E}=\frac{E^4-1}{8EE},\qquad \mathfrak{F}=\frac{F^4-1}{8FF},\qquad \mathfrak{G}=\frac{G^4-1}{8GG}$$

et ex natura horum arcuum est $\frac{G}{F}=E$. Si jam simili modo pro arcu pq procedamus, et ex abscissis $AP=p$ et $AQ=q$ has formemus functiones

$$p+\sqrt{(1+pp)}=P \qquad \tfrac{1}{2}p\sqrt{(1+pp)}=\mathfrak{P}$$
$$q+\sqrt{(1+qq)}=Q \qquad \tfrac{1}{2}q\sqrt{(1+qq)}=\mathfrak{Q},$$

erit simili modo $\text{Arc}.pq-\text{Arc}.Ae=\mathfrak{Q}-\mathfrak{P}-\mathfrak{E}$, existente $\frac{Q}{P}=E$. Hinc si illa aequatio ab hac subtrahatur, remanebit $\text{Arc}.pq-\text{Arc}.fg=(\mathfrak{Q}-\mathfrak{P})-(\mathfrak{G}-\mathfrak{F})$, si modo fuerit $\frac{Q}{P}=\frac{G}{F}$.

29. **Problema 4.** Dato arcu parabolae quocunque fg, abscindere arcum alium pz, qui ad arcum fg sit in data ratione $n:1$.

Solutio. Positis abscissis $AF=f$, $AG=g$, capiantur plures abscissae $AP=p$, $AQ=q$, $AR=r$, $AS=s$ et ultima $AZ=z$, ex quibus formentur geminae functiones, litteris majusculis cum latinis tum germanicis cognominibus denotandae, scilicet

$$f+\sqrt{(1+ff)}=F,\qquad g+\sqrt{(1+gg)}=G,\qquad p+\sqrt{(1+pp)}=P \text{ etc.}$$
$$\tfrac{1}{2}f\sqrt{(1+ff)}=\mathfrak{F},\qquad \tfrac{1}{2}g\sqrt{(1+gg)}=\mathfrak{G},\qquad \tfrac{1}{2}p\sqrt{(1+pp)}=\mathfrak{P} \text{ etc.}$$

sitque primo $\frac{Q}{P}=\frac{G}{F}$, erit

$$\text{Arc}.pq-\text{Arc}.fg=(\mathfrak{Q}-\mathfrak{P})-(\mathfrak{G}-\mathfrak{F}).$$

Deinde sit $\frac{R}{Q}=\frac{G}{F}$, seu $\frac{R}{P}=\frac{G^2}{F^2}$, erit

$$\text{Arc}.\,qr-\text{Arc}.\,fg=(\mathfrak{R}-\mathfrak{Q})-(\mathfrak{G}-\mathfrak{F})$$

qua aequatione ad priorem addita fit

$$\text{Arc}.\,pr-2\,\text{Arc}.\,fg=(\mathfrak{R}-\mathfrak{P})-2(\mathfrak{G}-\mathfrak{F}).$$

Sit porro $\frac{S}{R}=\frac{G}{F}$, seu $\frac{S}{P}=\frac{G^3}{F^3}$, erit

$$\text{Arc}.\,rs-\text{Arc}\,fg=(\mathfrak{S}-\mathfrak{R})-(\mathfrak{G}-\mathfrak{F}),$$

qua iterum ad praecedentem adjecta obtinebitur

$$\text{Arc}.\,ps-3\,\text{Arc}.\,fg=(\mathfrak{S}-\mathfrak{P})-3(\mathfrak{G}-\mathfrak{F}).$$

Simili modo si ulterius ponatur $\frac{T}{S}=\frac{G}{F}$, seu $\frac{T}{P}=\frac{G^4}{F^4}$, erit

$$\text{Arc}.\,pt-4\,\text{Arc}.\,fg=(\mathfrak{T}-\mathfrak{P})-4(\mathfrak{G}-\mathfrak{F}).$$

Unde perspicitur, si z sit ultimum punctum arcus pz qui quaeritur, et posita $AZ=z$ fit

$$Z=z+\sqrt{(1+zz)} \quad \text{et} \quad \mathfrak{Z}=\tfrac{1}{2}z\sqrt{(1+zz)},$$

poni debere $\frac{Z}{P}=\frac{G^n}{F^n}$, tumque fore

$$\text{Arc}.\,pz-n\,\text{Arc}.\,fg=(\mathfrak{Z}-\mathfrak{P})-n(\mathfrak{G}-\mathfrak{F}).$$

Nunc ut sit $\text{Arc}.\,pz=n\,\text{Arc}.\,fg$, reddi oportet $\mathfrak{Z}-\mathfrak{P}=n(\mathfrak{G}-\mathfrak{F})$. At est

$$\mathfrak{Z}=\frac{Z^4-1}{8ZZ},\quad \mathfrak{P}=\frac{P^4-1}{8PP},\quad \mathfrak{G}=\frac{G^4-1}{8GG} \quad \text{et} \quad \mathfrak{F}=\frac{F^4-1}{8FF}.$$

Verum ob $Z=\frac{G^nP}{F^n}$, erit $\mathfrak{Z}=\frac{G^{4n}P^4-F^{4n}}{8F^{2n}G^{2n}PP}$. Quibus valoribus substitutis sequens acquiretur aequatio resolvenda

$$\frac{G^{4n}P^4-F^{4n}}{F^{2n}G^{2n}PP}=\frac{P^4-1}{PP}+\frac{n(GG-FF)(1+FFGG)}{FFGG},$$

sive $$0=G^{2n}(G^{2n}-F^{2n})P^4+F^{2n}(G^{2n}-F^{2n})-nF^{2n-2}G^{2n-2}(G^2-F^2)(F^2G^2+1)PP,$$

seu $$P^4=\frac{+nF^{2n}(G^2-F^2)(F^2G^2+1)P^2}{F^2G^2(G^{2n}-F^{2n})}-\frac{F^{2n}}{G^{2n}}.$$

Quocunque ergo assumto multiplicationis indice n, sive numero integro, sive fracto, ex hac aequatione semper definiri potest P, unde arcus quaesiti pz alter terminus p innotescit. Quo invento pro altero termino z erit $Z=\frac{G^nP}{F^n}$, sicque obtinebitur arcus pz, ut sit $pz=n.fg$. Q. E. I.

30. **Coroll. 1.** Si loco P quaerere velimus Z, in ultima aequatione substitui oportet $\frac{F^nZ}{G^n}$, prodibitque

$$Z^4=\frac{nG^{2n}(G^2-F^2)(F^2G^2+1)ZZ}{F^2G^2(G^{2n}-F^{2n})}-\frac{G^{2n}}{F^{2n}},$$

ubi litterae F et G pariter uti P et Z sunt inter se commutatae.

31. **Coroll. 2.** Cum $G^{2n}-F^{2n}$ dividi queat per G^2-F^2, pro variis valoribus ipsius n formulae inventae ita se habebunt

si $n=1$, $P^4=\frac{(F^2G^2+1)P^2}{G^2}-\frac{F^2}{G^2}$ et $Z=\frac{GP}{F}$,

si $n=2$, $P^4=\frac{2F^2(F^2G^2+1)P^2}{G^2(G^2+F^2)}-\frac{F^4}{G^4}$ et $Z=\frac{G^2P}{F^2}$,

si $n=3$, $P^4=\frac{3F^4(F^2G^2+1)P^2}{G^2(G^4+F^2G^2+F^4)}-\frac{F^6}{G^6}$ et $Z=\frac{G^3P}{F^3}$,

si $n=4$, $P^4=\frac{4F^6(F^2G^2+1)P^2}{G^2(G^6+F^2G^4+F^4G^2+F^6)}-\frac{F^8}{G^8}$ et $Z=\frac{G^4P}{F^4}$,

etc. etc.

32. **Coroll. 3.** Ex solutione ceterum apparet pari modo pro arcu dato quocunque fg inveniri posse alium pz, qui illum arcum n vicibus sumtum data quantitate superet, vel ab eo deficiat; ut enim sit $\text{Arc}.pz - n\,\text{Arc}.fg = D$, resolvi oportebit hanc aequationem $\mathfrak{Z}-\mathfrak{P}=n(\mathfrak{G}-\mathfrak{F})+D$, quae non habet plus difficultatis, quam si esset $D=0$.

33. **Scholion.** Haec quidem, quae de circulo et parabola hic protuli, jam dudum satis sunt cognita, et quia utriusque rectificatio quasi in potestate est, (quae enim vel a quadratura circuli vel a logarithmis pendent, in ordinem quantitatum algebraicarum propemodum recipiuntur) nulli omnino difficultati sunt subjecta: ea tamen nihilominus aliquanto uberius hic exponere visum est, quod ex methodo prorsus singulari consequuntur. Quod autem imprimis notatu dignum est, haec methodus ad comparationem aliarum quoque curvarum manuducit, quarum rectificatio per calculum solitum nullo modo expediri potest; ita ut ex eodem quasi fonte plurimae eximiae affectiones tam cognitae quam incognitae hauriri queant, ex quo Analysi non contemnenda incrementa accedere censeri debebunt.

Sectio secunda

continens evolutionem hujus aequationis:

$$0=\alpha+\gamma(xx+yy)+2\delta xy+\zeta xxyy.$$

I.

Extrahatur ex hac aequatione sigillatim radix utriusque quantitatis variabilis x et y, ac reperietur

$$y=\frac{-\delta x+\sqrt{(\delta\delta xx-(\alpha+\gamma xx)(\gamma+\zeta xx))}}{\gamma+\zeta xx}$$

$$x=\frac{-\delta y-\sqrt{(\delta\delta yy-(\alpha+\gamma yy)(\gamma+\zeta yy))}}{\gamma+\zeta yy}$$

Ponatur brevitatis gratia $-\alpha\gamma=Ap$, $\delta\delta-\gamma\gamma-\alpha\zeta=Cp$ et $-\gamma\zeta=Ep$, eritque

$$\gamma y+\delta x+\zeta xxy=\sqrt{(A+Cxx+Ex^4)}\,p$$

$$\gamma x+\delta y+\zeta xyy=-\sqrt{(A+Cyy+Ey^4)}\,p.$$

II.

Si igitur coëfficientes A, C, E fuerint dati, ex iis litterarum graecarum valores facile definiuntur. Erit enim

$$\alpha=\frac{-Ap}{\gamma},\quad \zeta=\frac{-Ep}{\gamma}\quad\text{et}\quad \delta=\sqrt{(\gamma\gamma+Cp+\frac{AEpp}{\gamma\gamma})}.$$