absidum mutabilitatem inesse debere, quae tantum a figura Jovis non sphaerica proficiscatur; quod phaenomenon si per observationes confirmari posset, mirifice theoriae attractionis universalitatem confirmaret.

148. **Scholion 2.** Quanquam in hoc motu, quem hic definivimus, tam via a corpore descripta quam temporis ratio areis proportionalis maxime est transcendens, tamen calculum ita commode administrare licuit, ut determinatio motus vix difficilior, quam in casu ellipsis simplicis evaderet. Totum scilicet discrimen huc est perductum, ut linea absidum mobilis statueretur, dum reliqua omnia prorsus cum motu elliptico supra exposito conveniunt. Hoc compendio Astronomi jam pridem feliciter sunt usi, dum motus planetarum primariorum ita repraesentant, ac si in ellipsibus mobilibus circa solem revolverentur, in motu autem lunae insuper tam excentricitatem quam parametrum ellipsis variabilem statui oportere agnoverunt; quae idea eximium calculi alias intricatissimi compendium largitur. Atque non solum haec ita se habent, quando curva percursa sita est in eodem plano, sed etiam quando ejus planum est variabile; tum autem hujus variabilitatis rationem singulari modo ita ad quodpiam planum fixum referri convenit, ut ad quodvis tempus tam intersectio quam inclinatio planorum definiatur.

149. **Scholion 3.** Hoc modo approximationem institui conveniet, quando formulae analyticae, quibus motus determinatur, resolutionem non admittunt, quemadmodum in hoc capite usu venit, ubi hunc solum postremum casum, quo corpus M bina momenta inertiae respectu axium JA et JB aequalia habere, alterumque corpus N in ipso horum axium plano AJB moveri ponebatur, expedire licuit. Fundamentum autem hujus approximationis in hoc est situm, quod inter vires corpus N sollicitantes una prae ceteris eminet, quae ad punctum quasi fixum J dirigitur et quadratis distantiarum reciproce est proportionalis, reliquae autem vires prae hac sint valde exiguae. Tum enim motus corporis N non multum a ratione motus in sectione conica facti differet, cujus aberrationem tantum ab ista lege definivisse sufficiet. Quemadmodum ergo his casibus ope approximationis ad solutionem pervenire liceat, in sequente capite generatim explicabimus, in quo duplex investigatio erit instituenda, prout motus corporis N vel in eodem plano absolvetur, vel secus.

Caput V.

Determinatio motus corporis, quando inter vires, quibus sollicitatur, una ad punctum fixum tendens, quadrato distantiae ab eo est reciproce proportionalis, reliquae vero vires prae illa sunt valde parvae.

150. **Problema.** (Fig. 182.) Si corpus N circa punctum quasi fixum J in eodem plano moveatur, atque ad id trahatur vi quadrato distantiae reciproce proportionali, praeterea vero a viribus quibuscunque illius respectu valde parvis, corporis motum definire.

Solutio. Elapso tempore T sit distantia $JN = \varrho$ et angulus $AJN = \varphi$, ut sint

$$JX = x = \varrho \cos\varphi \quad \text{et} \quad XN = y = \varrho \sin\varphi.$$

Ponamus jam vim secundum directionem NJ esse $=\frac{LN}{vv}$, ac praeterea adesse vires valde parvas P et NQ, secundum directiones JX et XN agentes, et habebimus has aequationes:

$$ddx = -2g\,dt^2\left(\frac{Lx}{v^3} + P\right) \quad \text{et} \quad ddy = -2g\,dt^2\left(\frac{Ly}{v^3} + Q\right),$$

unde concludimus: $x\,ddy - y\,ddx = -2g\,dt^2\,(Qx - Py)$, hincque integrando

$$x\,dy - y\,dx = -2g\,dt \int dt\,(Qx - Py), \quad \text{seu} \quad vv\,d\varphi = -2g\,dt \int v\,dt\,(Q\cos\varphi - P\sin\varphi).$$

Statuamus nunc $v = \frac{p}{1 + q\cos s}$, ubi non solum angulus s, qui denotet anomaliam veram, sed etiam semiparameter p et excentricitas q sint quantitates variabiles, quarum variabilitas autem sit valde parva utpote a viribus P et Q proficiscens, quae si evanescerent, utique tam p quam q forent quantitates constantes. Ponamus brevitatis gratia $S = -2g \int v\,dt\,(Q\cos\varphi - P\sin\varphi)$, ut habeamus

$$d\varphi = \frac{S\,dt\,(1 + q\cos s)^2}{pp}.$$

Deinde ex primis aequationibus concludimus

$$x\,ddx + y\,ddy = -2g\,dt^2\left(\frac{L}{v} + Px + Qy\right);$$

at est $\quad x\,ddx + y\,ddy + dx^2 + dy^2 = d.v\,dv = v\,ddv + dv^2$ et $dx^2 + dy^2 = dv^2 + vv\,d\varphi^2$, hincque

$$x\,ddx + y\,ddy = v\,ddv - vv\,d\varphi^2,$$

$$\text{seu} \quad ddv - v\,d\varphi^2 = -2g\,dt^2\left(\frac{L}{vv} + P\cos\varphi + Q\sin\varphi\right),$$

ubi si pro $d\varphi$ valorem inventum substituamus, nanciscemur

$$\frac{ddv}{dt} = \frac{SS\,dt\,(1 + q\cos s)^3}{p^3} - \frac{2gL\,dt\,(1 + q\cos s)^2}{pp} - 2g\,dt\,(P\cos\varphi + Q\sin\varphi).$$

Hic primum observo si praeter P et Q etiam excentricitas q evanesceret, prodire debere $ddv = 0$ unde necesse est sit $SS = 2gLp$ et $S = \sqrt{2gLp}$. Quare habebimus

$$dS = \frac{dp}{2\sqrt{p}}\sqrt{2gL} = -2g\,v\,dt\,(Q\cos\varphi - P\sin\varphi),$$

ideoque $\quad dp = \frac{-4g\,dt\,(Q\cos\varphi - P\sin\varphi)\,p\sqrt{p}}{(1 + q\cos s)\sqrt{2gL}} \quad$ et $\quad d\varphi = \frac{dt\,(1 + q\cos s)^2\sqrt{2gL}}{p\sqrt{p}}.$

Tum vero nostra aequatio adhuc resolvenda erit

$$\frac{ddv}{dt} = \frac{2gLq\,dt\cos s}{pp}(1 + q\cos s)^2 - 2g\,dt\,(P\cos\varphi + Q\sin\varphi).$$

Jam quia per hypothesin dum fit $\sin s = 0$, etiam $\frac{dv}{dt}$ evanescere debet, statuamus $\frac{dv}{dt} = \sqrt{q}\sin s$ eritque

$$\sqrt{q}\,dt\sin s = dv = \frac{-4g\,dt\,(Q\cos\varphi - P\sin\varphi)\,p\sqrt{p}}{(1 + q\cos s)^2\sqrt{2gL}} - \frac{p\,d.q\cos s}{(1 + q\cos s)^2},$$

ita ut sit

$$d.q\cos s = \frac{-Vqdt\sin s\,(1+q\cos s)^2}{p} - \frac{4gdt\,(Q\cos\varphi - P\sin\varphi)\sqrt{p}}{\sqrt{2gL}}.$$

Porro autem ob $\frac{dv}{dt} = qdV\sin s + Vd.q\sin s$, erit

$$d.q\sin s = \frac{-qdV\sin s}{V} + \frac{2gLqdt\cos s}{Vpp}(1+q\cos s)^2 - \frac{2gdt\,(P\cos\varphi + Q\sin\varphi)}{V},$$

ex quibus duabus aequationibus concluditur

$$dq = \frac{-Vqdt\sin s\cos s\,(1+q\cos s)^2}{p} - \frac{4gdt\cos s\,(Q\cos\varphi - P\sin\varphi)\sqrt{p}}{\sqrt{2gL}}$$
$$- \frac{qdV\sin^2 s}{V} + \frac{2gLqdt\sin s\cos s}{Vpp}(1+q\cos s)^2 - \frac{2gdt\sin s\,(P\cos\varphi + Q\sin\varphi)}{V},$$

quae expressio evanescere debet casu $P=0$ et $Q=0$, ubi simul V fieret constans, ex qua conditione prodit

$$VV = \frac{2gL}{p} \quad \text{et} \quad V = \frac{\sqrt{2gL}}{\sqrt{p}} \quad \text{et} \quad dv = qdt\sin s\sqrt{\frac{2gL}{p}}.$$

Praeterea ob

$$\frac{dV}{V} = \frac{-dp}{2p} = \frac{2gdt\,(Q\cos\varphi - P\sin\varphi)\sqrt{p}}{(1+q\cos s)\sqrt{2gL}},$$

$$d.q\cos s = \frac{-qdt\sin s\,(1+q\cos s)^2\sqrt{2gL}}{p\sqrt{p}} - \frac{4gdt\,(Q\cos\varphi - P\sin\varphi)\sqrt{p}}{\sqrt{2gL}}$$

$$d.q\sin s = \frac{qdt\cos s\,(1+q\cos s)^2\sqrt{2gL}}{p\sqrt{p}} - \frac{2gdt\,(P\cos\varphi + Q\sin\varphi)\sqrt{p}}{\sqrt{2gL}}$$
$$- \frac{2gqdt\sin s\,(Q\cos\varphi - P\sin\varphi)\sqrt{p}}{(1+q\cos s)\sqrt{2gL}},$$

unde colligimus

$$dq = \frac{2gdt\sqrt{p}}{\sqrt{2gL}}\left(2(Q\cos\varphi - P\sin\varphi)\cos s + (P\cos\varphi + Q\sin\varphi)\sin s + \frac{q(Q\cos\varphi - P\sin\varphi)\sin^2 s}{1+q\cos s}\right),$$

$$qds = \frac{qdt(1+q\cos s)^2\sqrt{2gL}}{p\sqrt{p}} + \frac{2gdt\sqrt{p}}{\sqrt{2gL}}\left(2(Q\cos\varphi - P\sin\varphi)\sin s - (P\cos\varphi + Q\sin\varphi)\cos s - \frac{q(Q\cos\varphi - P\sin\varphi)\sin s\cos s}{1+q\cos s}\right),$$

ita ut hinc sit

$$ds = \frac{dt\,(1+q\cos s)^2\sqrt{2gL}}{p\sqrt{p}} + \frac{2gdt\sqrt{p}}{\sqrt{2gL}}\left(\frac{2(Q\cos\varphi - P\sin\varphi)\sin s}{q} - \frac{(P\cos\varphi + Q\sin\varphi)\cos s}{q} - \frac{(Q\cos\varphi - P\sin\varphi)\sin s\cos s}{1+q\cos s}\right).$$

Inde autem variatio excentricitatis q definitur, aeque ac semiparametri p, quibus inventis pro ipso motu erit

$$v = \frac{p}{1+q\cos s} \quad \text{et} \quad d\varphi = \frac{dt\,(1+q\cos s)^2\sqrt{2gL}}{p\sqrt{p}}.$$

Cum deinde $\varphi - s$ designet longitudinem absidis imae, et haec erit variabilis, habebiturque

$$d(\varphi - s) = \frac{2gdt\sqrt{p}}{q\sqrt{2gL}}\left((P\cos\varphi + Q\sin\varphi)\cos s - 2(Q\cos\varphi - P\sin\varphi)\sin s + \frac{q(Q\cos\varphi - P\sin\varphi)\sin s\cos s}{1+q\cos s}\right)$$

sicque omnia, quae ad motus determinationem attinent, sunt determinata.

151. **Coroll. 1.** Si ponamus $Q\cos\varphi - P\sin\varphi = T$ et $Q\sin\varphi + P\cos\varphi = U$, ut aequationes resolvendae sint:

$$vdd\varphi + 2dvd\varphi = -2gTdt^2 \quad \text{et} \quad ddv - vd\varphi^2 = \frac{-2gLdt^2}{vv} - 2gUdt^2,$$

eae posito $v = \frac{p}{1+q\cos s}$ ita resolventur, ut sit

$$1.\ d\varphi = \frac{dt\,(1+q\cos s)^2}{p\sqrt{p}}\sqrt{2gL}, \qquad 2.\ d\varphi - ds = \frac{2gdt\sqrt{p}}{q\sqrt{2gL}}\left(U\cos s - 2T\sin s + \frac{qT\sin s\cos s}{1+q\cos s}\right),$$

$$3.\ dp = \frac{-4gTpdt\sqrt{p}}{(1+q\cos s)\sqrt{2gL}}, \qquad 4.\ dq = \frac{-2gdt\sqrt{p}}{\sqrt{2gL}}\left(2T\cos s + U\sin s + \frac{qT\sin^2 s}{1+q\cos s}\right).$$

152. **Coroll. 2.** Si ex formulis N° 2. 3. 4. quantitates T et U elidantur, pervenietur ad hanc aequationem:

$$\frac{dp}{p} = \frac{dq\cos s + q\,(d\varphi - ds)\sin s}{1+q\cos s},$$

quae integrata quatenus licet dat

$$l\,\frac{p}{1+q\cos s} = \int\frac{qd\varphi\sin s}{1+q\cos s} = \int\frac{qdt\,(1+q\cos s)\sin s}{p\sqrt{p}}\sqrt{2gL}.$$

153. **Coroll. 3.** Cum quantitates P et Q sint per hypothesin valde parvae, erunt quantitates p et q fere constantes et $d\varphi = ds$, unde fit

$$dt = \frac{pds\sqrt{p}}{(1+q\cos s)^2\sqrt{2gL}},$$

cujus integrale spectatis p et q ut constantibus exhiberi poterit, quod cum sit prope verum, sufficiet deinceps hunc valorem pro dt in formulis 2, 3, 4 posuisse, ex iisque sumta sola s pro variabili valores proxime veros pro $\varphi - s$, p et q elicuisse.

154. **Coroll. 4.** Hoc autem pro dt valore inducto, aequationes nostrae evolvendae erunt:

$$2\quad d\varphi - ds = \frac{ppds}{Lq\,(1+q\cos s)^2}\left(U\cos s - 2T\sin s + \frac{qT\sin s\cos s}{1+q\cos s}\right),$$

$$3.\quad dp = \frac{-2Tp^3ds}{L\,(1+q\cos s)^3},$$

$$4.\quad dq = \frac{-ppds}{L\,(1+q\cos s)^2}\left(2T\cos s + U\sin s + \frac{qT\sin^2 s}{1+q\cos s}\right).$$

Revera autem in his formulis pro ds scribi oporteret $d\varphi$, sed quia saltem proxime est $d\varphi = ds$, ... in appropinquatione uti licebit.

155. **Scholion I.** Hoc modo solutio problematis ad determinationem motus in ellipsi variabili perducitur, ita ut ratio motus similis sit illi, quam supra pro casu duorum corporum sphaericorum assignavimus, praeterquam quod hic elementa ellipsis omnia variabilia statuantur. Primo enim tam semiparameter ellipsis p quam excentricitas q est variabilis, tum vero etiam ipsa linea absidum mobilis assumitur, denotante angulo s anomaliam veram, secundum eandem ideam, quam supra constituimus. Atque haec reductio eo magis est notatu digna, quod quaedam operationes prorsus ...

nostro sint institutae, ex quo apparet infinitis aliis modis etiam posito $\varrho = \frac{p}{1+q\cos s}$ relationem inter differentialia dp, dq, ds et $d\varphi$ ita constitui posse, ut motus rationi satisfiat. Loco enim determinationum $SS = 2gLp$ et $VV = \frac{2gL}{p}$, eosdem valores quantitatibus quibusdam exiguis per vires T et V definiendis augere liceret, quo pacto conditiones propositae aeque impleri possent, ut scilicet tam $q = 0$ quam $\sin s = 0$, evanescat $\frac{d\varrho}{dt}$, insuperque casu $T = 0$, $V = 0$ et $q = 0$ prodeat ... Cum enim loco unius variabilis ϱ tres novae p, q et s introducantur, mirum non est ... determinationem arbitrio nostro relinqui, quam ita constitui convenit, ut calculus commodissimus reddatur, in quo quidem negotio saepenumero maxima difficultas deprehenditur. Atque in solutione quidem, qua hic sumus usi, parum congruere videtur, quod expressio pro $d\varphi - ds$ inventa per excentricitatem q sit divisa, qua conditione determinatio motus lineae absidum lubrica redditur, praecipue quando excentricitas q est valde parva. Siquidem calculum perfecte expedire liceret, nullum incommodum hinc esset metuendum, quoniam perpetuo absides ibi existunt, ubi distantia ϱ est vel maxima vel minima, ita ut hic nulli incertitudini locus relinquatur. At cum approximatione contenti esse debeamus, ob hanc causam haud levia impedimenta occurrere possunt.

156. **Scholion 2.** Solutioni igitur summam extensionem tribuamus, et cum aequationes propositae sint:

$$\varrho dd\varphi + 2d\varrho d\varphi = -2gTdt^2, \qquad dd\varrho - \varrho d\varphi^2 = \frac{-2gLdt^2}{\varrho\varrho} - 2gVdt^2,$$

posito $\varrho = \frac{p}{1+q\cos s}$, statuamus $-2g\int T\varrho dt = \sqrt{2gp}\,(L+X) = \frac{\varrho\varrho d\varphi}{dt}$, eritque

$$\frac{-2gTpdt}{1+q\cos s} = \frac{dp}{2\sqrt{p}}\sqrt{2g}\,(L+X) + \frac{dX\sqrt{2gp}}{2\sqrt{(L+X)}}, \text{ hincque}$$

$$dp = \frac{-2Tpdt\sqrt{2gp}}{(1+q\cos s)\sqrt{(L+X)}} - \frac{pdX}{L+X} \quad \text{et} \quad d\varphi = \frac{dt\,(1+q\cos s)^2\sqrt{2gp}\,(L+X)}{pp}.$$

Porro statuatur $\frac{d\varrho}{dt} = q\sin s\sqrt{\frac{2g\,(L+Y)}{p}}$, eritque primo

$$qdt\sin s\sqrt{\frac{2g\,(L+Y)}{p}} = \frac{-2Tpdt\sqrt{2gp}}{(1+q\cos s)^2\sqrt{(L+X)}} - \frac{pdX}{(1+q\cos s)(L+X)} - \frac{pd.q\cos s}{(1+q\cos s)^2},$$

unde colligimus

$$d.q\cos s = \frac{-qdt\sin s\,(1+q\cos s)^2\sqrt{2g\,(L+Y)}}{p\sqrt{p}} - \frac{2Tdt\sqrt{2gp}}{\sqrt{(L+X)}} - \frac{(1+q\cos s)\,dX}{L+X}.$$

Deinde ex forma $\frac{d\varrho}{dt}$ assumta deducimus

$$\frac{dd\varrho}{dt} = \frac{\sqrt{2g\,(L+Y)}}{\sqrt{p}}\,d.q\sin s - \frac{qdp\sin s\sqrt{2g\,(L+Y)}}{2p\sqrt{p}} + \frac{qdY\sin s\sqrt{2g}}{2\sqrt{p\,(L+Y)}}, \text{ seu}$$

$$\frac{dd\varrho}{dt} = \frac{\sqrt{2g\,(L+Y)}}{\sqrt{p}}\,d.q\sin s + \frac{2gTqdt\sin s\sqrt{(L+Y)}}{(1+q\cos s)\sqrt{(L+X)}} + \frac{qdX\sin s\sqrt{2g\,(L+Y)}}{2\,(L+X)\sqrt{p}} + \frac{qdY\sin s\sqrt{2g}}{2\sqrt{p\,(L+Y)}}.$$

Ex aequatione proposita est

$$\frac{ddv}{dt} = \frac{2gdt\,(1+q\cos s)^3\,(L+X)}{pp} - \frac{2gLdt\,(1+q\cos s)^2}{pp} - 2gVdt, \quad \text{seu}$$

$$\frac{ddv}{dt} = \frac{2gLqdt\cos s\,(1+q\cos s)^2}{pp} + \frac{2gXdt\,(1+q\cos s)^3}{pp} - 2gVdt,$$

qua expressione cum praecedente collata fit

$$d.\,q\sin s = \frac{2gLqdt\cos s\,(1+q\cos s)^2}{p\sqrt{2gp}\,(L+Y)} + \frac{2gXdt\,(1+q\cos s)^3}{p\sqrt{2gp}\,(L+Y)} - \frac{2gVdt\sqrt{p}}{\sqrt{2g}\,(L+Y)} - \frac{Tqdt\sin s\sqrt{2gp}}{(1+q\cos s)\sqrt{(L+X)}}$$

$$- \frac{qdX\sin s}{2(L+X)} - \frac{qdY\sin s}{2(L+Y)}.$$

Hinc concludimus fore

$$dq = \frac{2gXdt\sin s\,(1+q\cos s)^3}{p\sqrt{2gp}\,(L+Y)} - \frac{2gVdt\sin s\sqrt{p}}{\sqrt{2g}\,(L+Y)} - \frac{dX\cos s\,(1+q\cos s)}{L+X} - \frac{qdX\sin^2 s}{2(L+X)}$$

$$\frac{-2gYqdt\sin s\cos s(1+q\cos s)^2}{p\sqrt{2gp}\,(L+Y)} - \frac{2Tdt\cos s\sqrt{2gp}}{\sqrt{(L+X)}} - \frac{Tqdt\sin^2 s\sqrt{2gp}}{(1+q\cos s)\sqrt{(L+X)}} - \frac{qdY\sin^2 s}{2(L+Y)};$$

$$-qds = \frac{2gLqdt\,(1+q\cos s)^2}{p\sqrt{2gp}\,(L+Y)} + \frac{2gYqdt\sin^2 s\,(1+q\cos s)^3}{p\sqrt{2gp}\,(L+Y)} + \frac{2Tdt\sin s\sqrt{2gp}}{\sqrt{(L+X)}} + \frac{dX\sin s\,(1+q\cos s)}{L+X}$$

$$+ \frac{2gXdt\cos s\,(1+q\cos s)^3}{p\sqrt{2gp}\,(L+Y)} - \frac{Vdt\cos s\sqrt{2gp}}{\sqrt{(L+Y)}} - \frac{qdX\sin s\cos s}{2(L+X)}$$

$$- \frac{Tqdt\sin s\cos s\sqrt{2gp}}{(1+q\cos s)\sqrt{(L+X)}} - \frac{qdY\sin s\cos s}{2(L+Y)}.$$

Si jam quantitates arbitrariae X et Y ita accipi possent, ut haec postrema expressio per q fieret divisibilis, incommodum supra memoratum tolleretur, id quod eveniret, si fieret

$$\frac{2gXdt\cos s}{p\sqrt{2gp}\,(L+Y)} + \frac{2Tdt\sin s\sqrt{2gp}}{\sqrt{(L+X)}} - \frac{Vdt\cos s\sqrt{2gp}}{\sqrt{(L+Y)}} + \frac{dX\sin s}{L+X} = 0,$$

vel formulae per q multiplicatae.

En ergo has determinationes, quae ob binas arbitrarias X et Y, maxime generales sunt habendae

1. $$d\varphi = \frac{dt\,(1+q\cos s)^2\sqrt{2g\,(L+X)}}{p\sqrt{p}} \quad \text{existente} \quad v = \frac{p}{1+q\cos s},$$

2. $$d\varphi - ds = \frac{dt\,(1+q\cos s)^2\sqrt{2g}}{p\sqrt{p}\,(L+Y)}\Big(\sqrt{(L+X)(L+Y)} - L - Y\sin^2 s - \frac{1}{q}X\cos s\,(1+q\cos s)\Big)$$

$$- \frac{2Tdt\sin s\sqrt{2gp}}{q\sqrt{(L+X)}} + \frac{Vdt\cos s\sqrt{2gp}}{q\sqrt{(L+Y)}} + \frac{Tdt\sin s\cos s\sqrt{2gp}}{(1+q\cos s)\sqrt{(L+X)}}$$

$$+ \frac{dX\sin s\cos s}{2(L+X)} + \frac{dY\sin s\cos s}{2(L+Y)} - \frac{dX\sin s\,(1+q\cos s)}{q(L+X)},$$

3. $$dp = \frac{-2Tpdt\sqrt{2gp}}{(1+q\cos s)\sqrt{(L+X)}} - \frac{pdX}{L+X},$$

4. $$dq = \frac{dt\sin s\,(1+q\cos s)^2\sqrt{2g}}{p\sqrt{p}\,(L+Y)}\Big(X(1+q\cos s) - Yq\cos s\Big)$$

$$- \frac{2Tdt\cos s\sqrt{2gp}}{\sqrt{(L+X)}} - \frac{Vdt\sin s\sqrt{2gp}}{\sqrt{(L+Y)}} - \frac{Tqdt\sin^2 s\sqrt{2gp}}{(1+q\cos s)\sqrt{(L+X)}}$$

$$- \frac{qdX\sin^2 s}{2(L+X)} - \frac{qdY\sin^2 s}{2(L+Y)} - \frac{dX\cos s\,(1+q\cos s)}{L+X}.$$

...autem est quantitates X et Y valde parvas capi debere, easque quatenus a p et q pendent pro constantibus esse habendas; sin autem insuper angulum φ vel s involvant, in earum differentiatione loco $d\varphi$ vel ds scribi posse

$$\frac{dt\,(1+q\cos s)^2\sqrt{2g\,(L+X)}}{p\sqrt{p}}.$$

Denique meminisse juvabit esse $dv = qdt\sin s\sqrt{\frac{2g\,(L+Y)}{p}}$.

Scholion 3. Ut pro litteris X et Y quovis casu commodissimi valores eligantur, id efficiendum videtur, ut quantitatum p et q variabilitas tam exigua reddatur quam fieri potest. Quodsi enim fieri queat, ut hae duae quantitates p et q evadant constantes, nullum est dubium, quin tum motus simplicissimo modo repraesentetur. Semper quidem has litteras X et Y ita definire liceret, ut fieret tam $dp = 0$ quam $dq = 0$, verum tum plerumque reliquae formulae nimis prodirent complicatae, quam ut hinc ullum commodum consequeremur; quare in hoc negotio ita versari conveniet, ut si non commode formulae pro dp et dq inventae ad nihilum redigi queant, eae saltem tam parvae efficiantur, quam fieri poterit, neque tamen ad hoc valores nimis perplexi pro X et Y adhibeantur: id enim imprimis cavendum est, ne hi valores unquam limites quantitatum prae L valde exiguarum superent. Quo igitur hoc judicium ratione formulae dq facilius instituatur, plerumque conveniet eam ita transformari, ut quantitas v cum suo differentiali dv, ponendo

$$1+q\cos s = \frac{p}{v} \quad \text{et} \quad qdt\sin s = \frac{dv\sqrt{p}}{\sqrt{2g\,(L+Y)}},$$

introducatur. Hoc modo obtinebimus

$$dq = \frac{pdv}{qvv\,(L+Y)}\left(\frac{pX}{v} - \frac{pY}{v} + Y\right) - \frac{2\,Tdt\cos s\sqrt{2gp}}{\sqrt{(L+X)}} - \frac{pdX\cos s}{v\,(L+X)} - \frac{Vpdv}{q\,(L+Y)} - \frac{Tvdv\sin s}{\sqrt{(L+X)}\,(L+Y)}$$
$$- \frac{qdX\sin^2 s}{2\,(L+X)} - \frac{qdY\sin^2 s}{2\,(L+Y)}.$$

Cum autem sit

$$dp = \frac{2\,Tvdt\sqrt{2gp}}{\sqrt{(L+X)}} - \frac{pdX}{L+X},$$

si hinc jam valorem idoneum pro X elegerimus, habebimus

$$dq = \frac{pdv}{qv^3\,(L+Y)}\,(pX - pY + vY) + \frac{dp\cos s}{v} - \frac{Vpdv}{q\,(L+Y)} - \frac{Tvdv\sin s}{\sqrt{(L+X)}\,(L+Y)} + \frac{qdp\sin^2 s}{2p}$$
$$+ \frac{Tqvdt\sin^2 s\sqrt{2gp}}{p\sqrt{(L+X)}} - \frac{qdY\sin^2 s}{2\,(L+Y)},$$

ubi termini littera T affecti se mutuo destruunt. Multiplicemus per q, et ob $q\cos s = \frac{p}{v} - 1$ et $qq\sin^2 s = qq - \left(\frac{p}{v} - 1\right)^2$, habebimus

$$qdq = \frac{pdv}{v^3\,(L+Y)}\,(pX - pY + vY) + \frac{dp\,(p-v)}{vv} - \frac{Vpdv}{L+Y} + \frac{qq\,dp}{2p} - \frac{dp\,(p-v)^2}{2pvv} - \frac{dY\,(qqvv - (p-v)^2)}{2vv\,(L+Y)},$$

quae reducitur ad hanc formam commodiorem

*

$$qdq = \frac{pdv\,(pX - pY + vY - Yv^3)}{v^3(L + Y)} - \frac{dp\,(1 - qq)}{2p} - \frac{dY\,(qqvv - (p - v)^2)}{2vv\,(L + Y)},$$

unde quovis casu haud difficulter maxime idoneus valor pro Y assumendus colligitur. Velimus $\frac{p}{1 - qq} = r$, fiet

$$\frac{dr}{r} = \frac{2rdv\,(pX - pY + vY - Yv^3) + vdY\,(pr - 2rv + vv)}{v^3\,(L + Y)},$$

quae formula si ad nihilum redigi possit, commodissimam solutionem suppeditabit. Videamus ergo quantum fructum hinc colligere queamus pro casu praecedentis capitis, ubi corpus N circa alterum M in plano AJB movetur.

158. **Problema.** Si corpus sphaericum N circa corpus M, figura quacunque praeditum, quod omni motu gyratorio destitutum ponitur, ita moveatur, ut perpetuo in plano binorum axium principalium AJB maneat, ejus motum definire.

Solutio. Maneant omnia ut in problemate § **128**, ac tantum opus est, ut hic ponamus $z = 0$, unde fiet $x = v \cos \varphi$ et $y = v \sin \varphi$. Quod si jam illas formulas ad has, quibus hic utimur, accommodemus, habebimus $L = M + N$ et

$$P = \frac{3L \cos\varphi}{2v^4}\,(3aa + bb + cc - 5aa \cos^2\varphi - 5bb \sin^2\varphi),$$

$$Q = \frac{3L \sin\varphi}{2v^4}\,(aa + 3bb + cc - 5aa \cos^2\varphi - 5bb \sin^2\varphi),$$

unde deducimus

$$T = \frac{3L\,(bb - aa) \sin\varphi \cos\varphi}{v^4} = \frac{3L\,(bb - aa) \sin 2\varphi}{2v^4},$$

$$V = \frac{3L}{4v^4}\,(2cc - aa - bb + 3\,(bb - aa) \cos 2\varphi).$$

Statuamus brevitatis gratia $bb - aa = n$ et $2cc - aa - bb = 2m$, eritque

$$T = \frac{3nL \sin 2\varphi}{2v^4} \quad \text{et} \quad V = \frac{3mL}{2v^4} + \frac{9nL \cos 2\varphi}{4v^4}.$$

Ponatur nunc $v = \frac{p}{1 + q \cos s}$, et cum invenerimus

$$dp = \frac{-3nLdt \sin 2\varphi \sqrt{2gp}}{v^3 \sqrt{(L + X)}} - \frac{pdX}{L + X},$$

notetur esse $d\varphi = \frac{dt \sqrt{2gp\,(L + X)}}{vv}$, unde fit

$$dp = \frac{-3nLd\varphi \sin 2\varphi}{v\,(L + X)} - \frac{pdX}{L + X},$$

ad quem valorem diminuendum ponamus

$$X = \frac{3nL}{2pv}\,(\alpha + \cos 2\varphi) + \beta, \quad \text{fietque} \quad dp = \frac{3nL(\alpha + \cos 2\varphi)}{2\,(L + X)}\left(\frac{dp}{pv} + \frac{dv}{vv}\right),$$

ubi dp est quam minimum, et dv involvit excentricitatem q tanquam factorem. Nunc pro expressione dq diminuenda habebimus

$$[illegible](L+Y)=2rdv\left(\frac{3nL(\alpha+\cos 2\varphi)}{2v}+\beta p-pY+vY-\frac{3mL}{2v}-\frac{9nL\cos 2\varphi}{4v}\right)+vdY(pr-2rv+vv),$$

ubi est $r=\frac{p}{1-qq}$. Statuamus $Y=\zeta+\frac{\eta}{v}$, fietque haec expressio

$$2rdv\left(\frac{3\alpha nL}{2v}-\frac{3nL\cos 2\varphi}{4v}+\beta p-\frac{3mL}{2v}-p\zeta-\frac{p\eta}{v}+v\zeta+\eta\right)$$

$$-\eta dv\left(\frac{pr}{v}-2r+v\right)+(vd\zeta+d\eta)(pr-2rv+vv),$$

ut jam termini vdv destruantur, sit $\eta=2r\zeta$; pro terminis autem dv prodit

$$2r\eta-2pr\zeta+2r\eta+2\beta pr=0 \quad \text{seu} \quad 2\eta-p\zeta+\beta p=0,$$

hincque $\beta=\zeta\left(1-\frac{4r}{p}\right)$. Tum vero termini $\frac{dv}{v}$ tollentur sumendo

$$3\alpha nLr-\tfrac{3}{2}nLr\cos 2\varphi-3mLr-3pr\eta=0, \quad \text{hincque}$$

$$\zeta=\frac{\alpha nL}{2pr}-\frac{nL\cos 2\varphi}{4pr}-\frac{mL}{2pr}.$$

Verum ne variabilitas anguli φ in differentiatione novum momentum introducat, omittamus hic potius terminum $\cos 2\varphi$, ponamusque $\alpha=0$, ut sit

$$X=\frac{3nL\cos 2\varphi}{2pv}+\frac{mL(4r-p)}{2ppr} \quad \text{et} \quad Y=\frac{-mL(2r+v)}{2prv}.$$

Vel eodem res redibit, si ponamus $\zeta=0$, $\eta=0$, $\beta=0$, ut sit $Y=0$ et $\alpha n=m$, ideoque

$$X=\frac{3L}{2pv}(m+n\cos 2\varphi), \quad \text{eritque} \quad \frac{v^3 dr}{r}(L+Y)=\frac{-3nLrdv\cos 2\varphi}{2v} \quad \text{seu} \quad \frac{dr}{rr}=\frac{-3ndv\cos 2\varphi}{2v^4}.$$

Deinde vero pro motu lineae absidum habemus in genere

$$d\varphi-ds=\frac{dt\sqrt{2gp}}{vv\sqrt{(L+Y)}}\left(\sqrt{(L+X)(L+Y)}-L-Y\sin^2 s-\frac{pX\cos s}{qv}\right)$$

$$+\frac{dp\sin s}{qv}-\frac{dp\sin s\cos s}{2p}+\frac{Ydt\cos s\sqrt{2gp}}{q\sqrt{(L+Y)}}+\frac{dY\sin s\cos s}{2(L+Y)}.$$

Cum nunc sit $Y=0$ et $\sqrt{(L+X)(L+Y)}=L+\frac{1}{2}X$, erit

$$d\varphi-ds=\frac{3dt(m+n\cos 2\varphi)\sqrt{2gLp}}{4pv^3}+\frac{3ndt\cos s\cos 2\varphi\sqrt{2gLp}}{4qv^4}+\frac{dp\sin s}{qv}-\frac{dp\sin s\cos s}{2p}.$$

existente $d\varphi=\frac{dt\sqrt{2gLp}}{vv}\left(1+\frac{3(m+n\cos 2\varphi)}{4pv}\right)$, unde fit

$$ds=\frac{dt\sqrt{2gLp}}{vv}-\frac{3ndt\cos s\cos 2\varphi\sqrt{2gLp}}{4qv^4}-\frac{dp\sin s}{qv}+\frac{dp\sin s\cos s}{2p}.$$

Est vero $dp=\frac{3L(m+n\cos 2\varphi)}{2(L+X)}\left(\frac{dp}{pv}-\frac{dv}{vv}\right)$ et $dv=\frac{qdt\sin s\sqrt{2gLp}}{p}$, ideoque

$$dp = \frac{3(m + n\cos 2\varphi)\, q\, dt \sin s \sqrt{2gLp}}{2p\upsilon\upsilon}.$$

Cum igitur sit proxime $\frac{dt\sqrt{2gLp}}{\upsilon\upsilon} = ds$, erit $dp = \frac{3(m + n\cos 2\varphi)\, q\, ds \sin s}{2p}$ atque

$$\frac{dr}{rr} = \frac{-3nq\, ds \sin s \cos 2\varphi}{2p\upsilon\upsilon} \quad \text{et}$$

$$d\varphi - ds = \frac{3mds}{4pp}\left(1 + 2\sin^2 s + q\cos s + q\sin^2 s\cos s\right) + \frac{3nds\cos 2\varphi}{4pp}\left(1 - 2\cos 2s - \frac{\cos s}{q} + 2q\sin^2 s\cos s\right),$$

in quibus formulis jam p et q ut constantes et $d\varphi = ds$ spectari possunt. Denique vero ope formulae $d\varphi = \ldots$ omnia ad tempus t revocari poterunt.

Alia solutio ejusdem problematis.

159. Cum ista solutio formulis differentialibus nimium sit implicata, quoniam eae ex differentio-differentialibus sunt immediate deductae, aliam viam tentemus ad hunc casum accommodatam. Cum enim aequationes principales sint

$$\text{I.} \quad \upsilon dd\varphi + 2d\upsilon d\varphi = \frac{-3ngLdt^2 \sin 2\varphi}{\upsilon^4},$$

$$\text{II.} \quad dd\upsilon - \upsilon d\varphi^2 = \frac{-2gLdt^2}{\upsilon\upsilon} - \frac{3gLmdt^2}{\upsilon^4} - \frac{9gLndt^2 \cos 2\varphi}{2\upsilon^4},$$

prima per υ multiplicata, prius membrum integrabile habebit, integrali existente $\upsilon\upsilon d\varphi$. Integrabilis ergo quoque fiet si multiplicetur per $2\upsilon^3 d\varphi$, quo pacto in altero membro elementum dt e signo integrali tolletur; prodibit enim

$$\upsilon^4 d\varphi^2 = 2gLdt^2\left(C - 3n\int \frac{d\varphi \sin 2\varphi}{\upsilon}\right).$$

Sit brevitatis gratia $\int \frac{d\varphi \sin 2\varphi}{\upsilon} = S$, ut habeamus $\upsilon^4 d\varphi^2 = 2gLdt^2(C - 3nS)$. Deinde prima per $2\upsilon d\varphi$ et altera per $2d\upsilon$ multiplicatae, in una summa efficiunt

$$2\upsilon\upsilon d\varphi dd\varphi + 2\upsilon d\upsilon d\varphi^2 + 2d\upsilon dd\upsilon = 2gLdt^2\left(-\frac{2d\upsilon}{\upsilon\upsilon} - \frac{3md\upsilon}{\upsilon^4} - \frac{3nd\varphi \sin 2\varphi}{\upsilon^3} - \frac{9nd\upsilon \cos 2\varphi}{2\upsilon^4}\right),$$

quae integrata dat

$$d\upsilon^2 + \upsilon\upsilon d\varphi^2 = 2gLdt^2\left(D + \frac{2}{\upsilon} + \frac{m}{\upsilon^3} + \frac{3n\cos 2\varphi}{2\upsilon^3}\right).$$

Cum ergo inde sit $2gLdt^2 = \frac{\upsilon^4 d\varphi^2}{C - 3nS}$, hinc commode tempus t eliminatur, obtineturque

$$(C - 3nS)(d\upsilon^2 + \upsilon\upsilon d\varphi^2) = \upsilon^4 d\varphi^2\left(D + \frac{2}{\upsilon} + \frac{2m + 3n\cos 2\varphi}{2\upsilon^3}\right)$$

$$\text{et} \quad d\varphi = \frac{d\upsilon\sqrt{(C - 3nS)}}{\upsilon\upsilon\sqrt{\left(D + \frac{2}{\upsilon} + \frac{(2m + 3n\cos 2\varphi)}{2\upsilon^3} - \frac{(C - 3nS)}{\upsilon\upsilon}\right)}}.$$

Jam quoties $d\upsilon$ evanescit, necesse est, ut formula irrationalis denominatoris evanescat, quod duobus casibus evenire debeat, quibus angulus quidem s fit vel 0 vel 180°, denominator factorem habebit $\sin s$. Statuamus ergo $v = \frac{p}{1 + q\cos s}$, et denominator erit

$$D + \frac{2}{p} + \frac{2m + 3n\cos 2\varphi}{2p^3} - \frac{(C - 3nS)}{pp}$$

$$+ \frac{2q\cos s}{p} + \frac{3q\cos s\,(2m + 3n\cos 2\varphi)}{2p^3} - \frac{2q\cos s\,(C - 3nS)}{pp}$$

$$+ \frac{3qq\cos^2 s\,(2m + 3n\cos 2\varphi)}{2p^3} - \frac{qq\cos^2 s\,(C - 3nS)}{pp}$$

$$+ \frac{q^3\cos^3 s\,(2m + 3n\cos 2\varphi)}{2p^3}.$$

Fiat nunc $D + \frac{2}{p} + \frac{2m + 3n\cos 2\varphi}{2p^3} - \frac{(C - 3nS)}{pp} + \frac{3qq(2m + 3n\cos 2\varphi)}{2p^3} - \frac{qq\,(C - 3nS)}{pp} = 0,$

$$1 + \frac{3\,(2m + 3n\cos 2\varphi)}{2pp} - \frac{2\,(C - 3nS)}{p} + \frac{qq\,(2m + 3n\cos 2\varphi)}{2pp} = 0,$$

eritque formula irrationalis in denominatore

$$\frac{q\sin s}{p}\sqrt{\left(C - 3nS - \frac{3\,(2m + 3n\cos 2\varphi)}{2p} - \frac{q\cos s\,(2m + 3n\cos 2\varphi)}{2p}\right)}$$

et $\frac{dv}{vv} = \frac{qd\varphi\sin s}{p}\sqrt{\left(1 - \frac{(3 + q\cos s)\,(2m + 3n\cos 2\varphi)}{2p\,(C - 3nS)}\right)}.$

Jam ex illis aequationibus quantitates p et q definiantur, quae si esset $m = 0$ et $n = 0$, prodirent: $p = C$ et $qq = 1 + CD$, atque hi erunt quasi valores medii ipsarum p et q, qui statuantur f et k, ut sit $C = f$ et $D = \frac{kk - 1}{f}$. Deinde cum m et n sint quantitates valde parvae, in terminis per m et n affectis scribere licebit $p = f$ et $q = k$, sicque habebimus

$$\frac{1}{p} = \frac{1}{f} + \frac{(3 + kk)\,(2m + 3n\cos 2\varphi)}{2f^3} + \frac{3nS}{ff} \quad \text{et} \quad \frac{qq}{pp} = \frac{kk}{ff} + \frac{(1 + 3kk)\,(2m + 3n\cos 2\varphi)}{2f^4} + \frac{3nS\,(1 + kk)}{f^3},$$

unde fit

$$p = f - \frac{(3 + kk)\,(2m + 3n\cos 2\varphi)}{4f} - 3nS \quad \text{et} \quad qq = kk + \frac{(1 - k^4)\,(2m + 3n\cos 2\varphi)}{2ff} + \frac{3n\,(1 - kk)\,S}{f}.$$

Quoniam nunc habemus valores litterarum p et q, ob $v = \frac{p}{1 + q\cos s}$ erit

$$S = \int\frac{d\varphi\,(1 + q\cos s)\sin 2\varphi}{p} \quad \text{et} \quad \frac{dv}{vv} = \frac{dp}{pp} - \frac{dq\cos s}{p} + \frac{qds\sin s}{p} + \frac{qdp\cos s}{pp}, \quad \text{seu}$$

$$\frac{dv}{vv} = \frac{dp}{pp} - \frac{(pdq - qdp)\cos s}{pp} + \frac{qds\sin s}{p}.$$

At est superioribus formulis differentiandis

$$\frac{dp}{pp} = \frac{3n\,(3 + kk)\,d\varphi\sin 2\varphi}{2f^3} - \frac{3n\,(1 + q\cos s)\,d\varphi\sin 2\varphi}{ffp}, \quad \text{seu} \quad = \frac{3n\,(1 - 2k\cos s + kk)\,d\varphi\sin\varphi}{2f^3} \quad \text{et}$$

$$\frac{2q\,(pdq - qdp)}{p^3} = \frac{-3n(1 + 3kk)\,d\varphi\sin 2\varphi}{f^4} + \frac{3n\,(1 + kk)\,d\varphi\,(1 + k\cos s)\sin 2\varphi}{f^4},$$

$$\frac{pdq - qdp}{pp} = \frac{3nd\varphi \sin 2\varphi\,(1 + kk\cos s - 2k)}{2f^3},$$

ex quibus colligitur

$$\frac{dv}{vv} = \frac{3n(1+kk)\,d\varphi \sin^2 s \sin 2\varphi}{2f^3} + \frac{qds \sin s}{p}.$$

At est

$$\frac{dv}{vv} = \frac{qd\varphi \sin s}{p}\sqrt{\left(1 - \frac{(3 + k\cos s)(2m + 3n\cos 2\varphi)}{2ff}\right)} = \frac{qd\varphi \sin s}{p} - \frac{k(3 + k\cos s)(2m + 3n\cos 2\varphi)\,d\varphi \sin s}{4f^3}$$

Ergo $\quad d\varphi - ds = \frac{pd\varphi}{4f^3 q}\left(6n(1+kk)\sin s \sin 2\varphi + k(3 + k\cos s)(2m + 3n\cos 2\varphi)\right).$

Hic jam spectatis p et q ut constantibus, nempe $p = f$ et $q = k$, erit

$$d\varphi - ds = \frac{d\varphi}{4ffk}\left(6mk + 2mkk\cos s + 9nk\cos 2\varphi + 3n(1 + \tfrac{3}{2}kk)\cos(2\varphi - s) - 3n(1 + \tfrac{1}{2}kk)\cos(2\varphi + s)\right)$$

Cum igitur proxime sit $d\varphi = ds$, erit integrando

$$\varphi - s = \text{Const.} + \frac{3m\varphi}{2ff} + \frac{mk\sin s}{2ff} + \frac{9n\sin 2\varphi}{8ff} + \frac{3n(2+3kk)\sin(2\varphi - s)}{8ffk} - \frac{n(2+kk)\sin(2\varphi + s)}{8ffk},$$

qua aequatione relatio inter longitudinem φ et anomaliam veram s exprimitur. Tum vero quia S per quantitatem minimam n multiplicatur, sufficiet posuisse

$$dS = \frac{d\varphi}{f}\left(\sin 2\varphi + \tfrac{1}{2}k\sin(2\varphi - s) + \tfrac{1}{2}k\sin(2\varphi + s)\right),$$

unde fit $\quad S = \frac{-\cos 2\varphi}{2f} - \frac{k\cos(2\varphi - s)}{2f} - \frac{k\cos(2\varphi + s)}{6f},$

hincque deducimus

$$p = f - \frac{m(3+kk)}{2f} - \frac{3n(1+kk)\cos 2\varphi}{4f} + \frac{3nk\cos(2\varphi - s)}{2f} + \frac{3nk\cos(2\varphi + s)}{6f},$$

$$qq = kk + \frac{m(1-k^4)}{ff} + \frac{3nkk(1-kk)\cos 2\varphi}{2ff} - \frac{3nk(1-kk)\cos(2\varphi - s)}{2ff} - \frac{3nk(1-kk)\cos(2\varphi + s)}{6ff}.$$

Denique ut omnia ad tempus t reducamus, habemus

$$dt\sqrt{2gL} = \frac{pp\,d\varphi}{(1 + q\cos s)^2\sqrt{(f - 3nS)}} = \frac{pp\,d\varphi}{(1 + q\cos s)^2}\left(\frac{1}{\sqrt{f}} + \frac{3nS}{2f\sqrt{f}}\right),$$

cujus integratio per praecedentes formulas in potestate est censenda. Cum enim sit

$$d\varphi = ds\left(1 + \frac{3m}{2ff} + \frac{mk\cos s}{2ff} + \frac{9n\cos 2\varphi}{4ff} + \frac{3n(2+3kk)\cos(2\varphi - s)}{8ffk} - \frac{3n(2+kk)\cos(2\varphi + s)}{8ffk}\right),$$

$$pp = ff - m(3+kk) - \tfrac{3}{2}n(1+kk)\cos 2\varphi + 3nk\cos(2\varphi - s) + nk\cos(2\varphi + s),$$

$$1 + \frac{3nS}{2f} = 1 - \frac{3n\cos 2\varphi}{4ff} - \frac{3nk\cos(2\varphi - s)}{4ff} - \frac{nk\cos(2\varphi + s)}{4ff}, \quad \text{erit}$$

$$pp\,d\varphi\left(1 + \frac{3nS}{2f}\right) = ffds\left(1 - \frac{m(3+2kk)}{2ff} + \frac{mk\cos s}{2ff} - \frac{3nkk\cos 2\varphi}{2ff} + \frac{3n(2+9kk)\cos(2\varphi - s)}{8ffk} - \frac{3n(2-kk)\cos(2\varphi + s)}{8ffk}\right)$$

Porro est

$$q = k + \frac{m(1-k^4)}{2ffk} + \frac{3nk(1-kk)\cos 2\varphi}{4ff} - \frac{3n(1-kk)\cos(2\varphi - s)}{4ff} - \frac{n(1-kk)\cos(2\varphi + s)}{4ff},$$

tandem concluditur $dt\sqrt{2fgL} = \frac{ff\,ds}{(1+k\cos s)^2} + \frac{ff\,W\,ds}{(1+k\cos s)^3}$ existente

$$W = \frac{-3m(2+kk)}{4ff} - \frac{m(1+kk)\cos s}{ffk} + \frac{mkk\cos 2s}{4ff} + \frac{n(8-5kk)\cos 2\varphi}{8ff}$$

$$+ \frac{3n(2+7kk)\cos(2\varphi - s)}{8ffk} + \frac{3n(6+5kk)\cos(2\varphi - 2s)}{16ff}$$

$$- \frac{3n(2+kk)\cos(2\varphi + s)}{8ffk} - \frac{n(2+kk)\cos(2\varphi + 2s)}{16ff}.$$

160. **Coroll. 1.** Formula $\varphi - s$ exprimit longitudinem absidis imae, unde si corpus N nunc in abside ima, ad absidem summam pertinget confecto angulo φ, ut ob $s = 180^0 = \pi$ sit

$$\varphi - \pi = \frac{3m\pi}{2ff} + \frac{9n\sin 2\varphi}{8ff} + \frac{3n(2+3kk)\sin(2\varphi - \pi)}{8ffk} - \frac{n(2+kk)\sin(2\varphi + \pi)}{8ffk},$$

$$\text{seu} \quad \varphi - \pi = \frac{3m\pi}{2ff} + \frac{9n\sin 2\varphi}{8ff} - \frac{n(1+2kk)\sin 2\varphi}{2ffk},$$

ubi ob $2\varphi = 2\pi$ proxime, et neglectis terminis binas dimensiones litterarum m et n involventibus, erit $\varphi = \pi + \frac{3m\pi}{2ff}$.

161. **Coroll. 2.** At dum absolvitur anomalia vera $s = 2\lambda\pi$, existente λ numero integro valde magno, ob $\varphi = 2\lambda\pi + \frac{3\lambda m\pi}{ff}$, proxime erit

$$\varphi = 2\lambda\pi + \frac{3\lambda m\pi}{ff} + \frac{9n}{8ff}\sin\frac{6\lambda m\pi}{ff} + \frac{n(1+2kk)}{2ffk}\sin\frac{6\lambda m\pi}{ff};$$

unde si sit $\frac{6\lambda m}{ff} = \frac{1}{2}$, seu $\lambda = \frac{ff}{12m}$, post $\frac{ff}{2m}$ revolutiones anomaliae verae, erit

$$\varphi = 2\lambda\pi + \frac{1}{4}\pi + \frac{9n}{4ff} + \frac{n(1+2kk)}{2ffk}.$$

162. **Coroll. 3.** Si esset $n = 0$, promotio lineae absidum in singulis revolutionibus anomaliae verae s foret eadem, scilicet $= \frac{3m\pi}{ff} = \frac{3(cc-aa)}{ff}\pi$, uti jam supra invenimus. Sed si n non est $= 0$, singulis revolutionibus anomaliae verae non amplius aequalis progressio lineae absidum respondet, quod tamen discrimen demum post plures revolutiones fit sensibile.

163. **Coroll. 4.** Relatio inter angulos φ et s ita definitur, ut sit

$$\varphi = \zeta + s + \frac{3ms}{2ff} + \frac{mk\sin s}{2ff} + \frac{9n\sin 2\varphi}{8ff} + \frac{3n(2+3kk)\sin(2\varphi - s)}{8ffk} - \frac{n(2+kk)\sin(2\varphi + s)}{8ffk},$$

ubi in posterioribus terminis pro φ scribi potest $\zeta + s + \frac{3ms}{2ff}$, neque vero hic terminum $\frac{3ms}{2ff}$ omittere licet, cum is crescente cum tempore angulo s ad valorem quantumvis magnum assurgere possit. Constans autem ζ non est arbitraria, sed denotat longitudinem absidis imae ab axe principali JA.

164. **Scholion 1.** Pro quavis ergo anomalia vera s et angulo constante ζ definiri po[...] longitudo corporis N seu angulus $AJN = \varphi$, qua cognita porro semiparameter p et excentricita[...] orbitae ellipticae variabilis innotescit, unde concluditur distantia $JN = v = \frac{p}{1 + q\cos s}$. Super[...] autem, ut relatio inter tempus t et angulum s assignetur, seu ut haec aequatio

$$dt\sqrt{2fgL} = \frac{ffds}{(1 + k\cos s)^2} + \frac{ffWds}{(1 + k\cos s)^3}$$

integretur, quod negotium, quia φ per s datur, concedendum est. Est enim

$$\int\frac{ds}{1 + k\cos s} = \frac{1}{\sqrt{(1 - kk)}}\,\text{Arc.}\cos\frac{k + \cos s}{1 + k\cos s},$$

$$\int\frac{ds}{(1 + k\cos s)^2} = \frac{1}{(1 - kk)^{\frac{3}{2}}}\,\text{Arc.}\cos\frac{k + \cos s}{1 + k\cos s} - \frac{k\sin s}{(1 - kk)(1 + k\cos s)},$$

$$\int\frac{ds}{(1 + k\cos s)^3} = \frac{2 + kk}{2(1 - kk)^{\frac{5}{2}}}\,\text{Arc.}\cos\frac{k + \cos s}{1 + k\cos s} - \frac{k\sin s}{2(1 - kk)(1 + k\cos s)^2} - \frac{3k\sin s}{2(1 - kk)^2(1 + k\cos s)}.$$

Reliquae partes exigunt integrationem hujusmodi formulae $\int\frac{ds\cos(\alpha s + \beta)}{(1 + k\cos s)^3}$, quae si k fuerit fractio valde parva, facile per seriem evolvitur; posito enim

$$\frac{1}{(1 + k\cos s)^3} = A + B\cos s + C\cos 2s + D\cos 3s + E\cos 4s + \text{etc.}$$

reperitur integrale $\int\frac{ds\cos(\alpha s + \beta)}{(1 + k\cos s)^3}$ ita expressum

$$\frac{A}{\alpha}\sin(\alpha s + \beta) + \frac{B\sin(\alpha s - s + \beta)}{2(\alpha - 1)} + \frac{C\sin(\alpha s - 2s + \beta)}{2(\alpha - 2)} + \text{etc.}$$
$$+ \frac{B\sin(\alpha s + s + \beta)}{2(\alpha + 1)} + \frac{C\sin(\alpha s + 2s + \beta)}{2(\alpha + 2)} + \text{etc.}$$

verum nisi k sit fractio valde parva, haec series parum juvat.

165. **Scholion 2.** At si k est quantitas valde exigua, aliud incommodum nascitur, quod in nostris formulis termini per k divisi nimis fiant magni, ideoque determinatio anguli $\varphi - s$ nimis incerta reddatur. Operae ergo pretium erit casum, quo $k = 0$, data opera evolvisse, quo fit

$$p = f - \frac{3(2m + 3n\cos 2\varphi)}{4f} - 3nS \quad \text{et} \quad qq = \frac{2m + 3n\cos 2\varphi}{2ff} + \frac{3nS}{f},$$

quae formula autem locum habere nequit, nisi sit positiva; si enim fieret negativa, hoc indicio esset quantitatem k evanescere non posse, vel tum etiam $\cos s$ imaginarium esse proditurum, ita ut hoc casu positio $v = \frac{p}{1 + k\cos s}$ contradictionem involveret. Casu autem, quo qq prodit positivum, reperitur

$$d\varphi = ds + \frac{3nd\varphi\sin s\sin 2\varphi}{f\sqrt{(4m + 6n\cos 2\varphi + 12fnS)}} \text{ existente } S = \int\frac{d\varphi\sin 2\varphi}{f} = \frac{-\cos 2\varphi}{2f},$$

ita ut sit

Erit $d\varphi = ds + \frac{3nd\varphi \sin s \sin 2\varphi}{2f\sqrt{m}}$ et $qq = \frac{m}{ff}$ et $p = f - \frac{3m}{2f} - \frac{3n\cos 2\varphi}{4f}$,

ideoque $q = \frac{\sqrt{m}}{f}$, sicque excentricitas q constans. Tum erit

$$dt\sqrt{2fgL} = \frac{d\varphi\,(ff - 3m - \frac{9}{4}n\cos 2\varphi)}{\left(1 + \frac{\sqrt{m}}{f}\cos s\right)^2}, \quad \text{seu} \quad = d\varphi\,(ff - 2f\cos s\sqrt{m} - 3m - \tfrac{9}{4}n\cos 2\varphi).$$

Solutio ergo hujus casus pendet a resolutione hujus aequationis $d\varphi = ds + \frac{3nd\varphi \sin s \sin 2\varphi}{2f\sqrt{m}}$, ex qua, si $\frac{n}{f\sqrt{m}}$ est quantitas valde parva, concluditur

$$\varphi = \zeta + s + \frac{3n\sin(2\varphi - s)}{4f\sqrt{m}} - \frac{n\sin(2\varphi + s)}{4f\sqrt{m}}.$$

Reliquis autem casibus, praecipue si m esset $= 0$, alia tractatio requireretur, in valorem scilicet ipsius S accuratius inquiri oporteret, quod difficultatibus haud esset cariturum.

166. **Scholion 3.** Solutio nostri problematis posterior ideo priori est anteferenda, quod binarum aequationum differentio-differentialium propositarum una integratio successerit. In genere igitur, si idem usu veniat, solutio facilior obtineri potest. Propositis enim his duabus aequationibus

$$v\,dd\varphi + 2\,dv\,d\varphi = -gLTdt^2 \quad \text{et} \quad ddv - v\,d\varphi^2 = -gLdt^2\left(\frac{2}{vv} + V\right),$$

multiplicetur prior per $2v^3 d\varphi$, ut prodeat

$$v^4 d\varphi^2 = 2gLdt^2\,(C - \int Tv^3 d\varphi) = 2gLdt^2\,(C - S),$$

posito $\int Tv^3 d\varphi = S$. Deinde priori per $2v\,d\varphi$, et posteriori per $2dv$ multiplicata, summa praebet

$$d\,.\,(vv\,d\varphi^2 + dv^2) = -2gLdt^2\left(Tv\,d\varphi + V\,dv + \frac{2dv}{vv}\right).$$

Quod si jam fuerit $Tv\,d\varphi + V\,dv$ integrabile, ponatur integrale $\int(Tv\,d\varphi + V\,dv) = \frac{R}{v^3}$, ut habeamus

$$dv^2 + vv\,d\varphi^2 = 2gLdt^2\left(D + \frac{2}{v} - \frac{R}{v^3}\right);$$

hinc eliminando dt^2 adipiscemur

$$(C - S)\,dv^2 = v^4 d\varphi^2\left(D + \frac{2}{v} - \frac{R}{v^3} - \frac{(C-S)}{vv}\right) \quad \text{et} \quad \frac{dv}{vv}\sqrt{(C - S)} = d\varphi\sqrt{\left(D + \frac{2}{v} - \frac{(C-S)}{vv} - \frac{R}{v^3}\right)}.$$

Statuamus $v = \frac{p}{1 + q\cos s}$, sitque

$$D + \frac{2}{p} - \frac{R(1 + 3qq)}{p^3} - \frac{(1+qq)(C-S)}{pp} = 0 \quad \text{et} \quad 2 - \frac{R(3 + qq)}{pp} - \frac{2(C-S)}{p} = 0,$$

ut fiat formula irrationalis

$$\sqrt{\left(D + \frac{2}{v} - \frac{(C-S)}{vv} - \frac{R}{v^3}\right)} = \frac{q\sin s}{p}\sqrt{\left(C - S + \frac{R(3 + q\cos s)}{p}\right)}, \quad \text{hincque}$$

$$\frac{dv}{vv} = \frac{q\,d\varphi\sin s}{p}\sqrt{\left(1 + \frac{R(3 + q\cos s)}{p\,(C - S)}\right)}.$$

Inde autem cum R et S sint quantitates valde parvae, posito $C=f$ et $D=\frac{kk-1}{f}$, ut fiat proxime $p=f$ et $q=k$, colligitur

$$\frac{1}{p}=\frac{1}{f}+\frac{S}{ff}-\frac{(3+kk)R}{2f^3} \quad \text{et} \quad p=f-S+\frac{(3+kk)R}{2f},$$

$$\frac{qq}{pp}=\frac{kk}{ff}+\frac{(1+kk)S}{f^3}-\frac{(1+3kk)R}{f^4},$$

unde fit

$$qq=kk+\frac{(1-kk)S}{f}-\frac{(1-k^4)R}{ff}.$$

Deinde ob $\frac{dp}{pp}=\frac{-dS}{ff}+\frac{(3+kk)dR}{2f^3}$ et $\frac{pdq-qdp}{pp}=\frac{(1+kk)dS}{2ffk}-\frac{(1+3kk)dR}{2f^3k}$, erit

$$\frac{dv}{vv}=\frac{qdS\sin s}{p}-\frac{dS}{ff}-\frac{(1+kk)dS\cos S}{2ffk}+\frac{(3+kk)dR}{2f^3}+\frac{(1+3kk)dR\cos s}{2f^3k}.$$

Est vero etiam $\frac{dv}{vv}=\frac{qd\varphi\sin s}{p}\left(1+\frac{(3+k\cos s)R}{2ff}\right)$, unde

$$\frac{q\sin s}{p}(d\varphi-ds)=\frac{-dS}{ff}-\frac{(1+kk)dS\cos s}{2ffk}+\frac{(3+kk)dR}{2f^3}+\frac{(1+3kk)dR\cos s}{2f^3k}-\frac{k(3+k\cos s)R\sin s}{2f^3}d\varphi.$$

Denique est

$$dt\sqrt{2fgL}=vvd\varphi\left(1+\frac{S}{f}\right)=\frac{ppd\varphi}{(1+q\cos s)^2}\left(1+\frac{S}{f}\right);$$

ubi notandum est esse ob $dv=\frac{kvvd\varphi\sin s}{f}$ in terminis minimis

$$dS=Tv^3d\varphi \quad \text{et} \quad dR=\frac{3kRvd\varphi\sin s}{f}+Tv^4d\varphi+\frac{kVv^5d\varphi\sin s}{f},$$

unde fit

$$\frac{q}{p}(d\varphi-ds)=\frac{(1+kk)Tv^4\sin s}{2f^3}d\varphi+\frac{Rvd\varphi}{2f^4}(6k+(3+5kk)\cos s+3k^3-k^3\cos^2 s)$$
$$+\frac{Vv^4d\varphi}{2f^4}(k(3+kk)+(1+3kk)\cos s).$$

167. **Scholion 4.** Aliam formam habitura esset solutio, si formula integralis

$$\int(Tvd\varphi+Vdv) \quad \text{non} \quad \frac{R}{v^3}, \quad \text{sed} \quad \frac{R}{v^\lambda},$$

vel aggregato ex pluribus hujusmodi formulis aequalis poneretur. Ponamus ergo

$$\int Tvd\varphi+Vdv)=-\mathfrak{A}-\frac{\mathfrak{B}}{v}-\frac{\mathfrak{C}}{vv}-\frac{D}{v^3}-\frac{E}{v^4}-\frac{F}{v^5}-\text{etc.}$$

existente

$$\int(Tv^3d\varphi=S \quad \text{et} \quad dt\sqrt{2fgL}=vvd\varphi\left(1+\frac{S}{2f}\right),$$

habebimus ergo

$$\frac{dv}{vv}\sqrt{(f-S)}=d\varphi\sqrt{\left(\frac{kk-1}{f}+\mathfrak{A}+\frac{2-\mathfrak{B}}{v}-\frac{(f-S-\mathfrak{C})}{vv}+\frac{D}{v^3}+\frac{E}{v^4}+\frac{F}{v^5}+\text{etc.}\right)},$$

quantitates $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, D, E, F et S ut valde parvae sunt spectandae. Ponamus brevitatis gratia

$$\frac{kk-1}{f}+\mathfrak{A}=A,\quad 2+\mathfrak{B}=B,\quad \text{et}\quad -f+S+\mathfrak{C}=C,$$

ut formula irrationalis sit

$$\sqrt{\left(A+\frac{B}{v}+\frac{C}{vv}+\frac{D}{v^3}+\frac{E}{v^4}+\frac{F}{v^5}\right)},$$

quae posito $v=\frac{p}{1+q\cos s}$ ita comparata esse debet, ut factorem obtineat $\sin s$, seu ut evanescat posito tam $s=0$ quam $s=180^0$. Quocirca efficiendum est, ut fiat

$$A+B\left(\frac{1\pm q}{p}\right)+C\left(\frac{1\pm q}{p}\right)^2+D\left(\frac{1\pm q}{p}\right)^3+\text{etc.}=0,$$

erunt ergo necesse est $\frac{1+q}{p}$ et $\frac{1-q}{p}$ binae radices hujus aequationis

$$A+Bz+Czz+Dz^3+Ez^4+Fz^5+\text{etc.}=0,$$

quae rejectis terminis minimis habebit hanc formam $\frac{kk-1}{f}+2z-fzz=0$, unde fit $z=\frac{1\pm k}{f}$, ita ut sit proxime $p=f$ et $q=k$. Ponatur jam in terminis minimis $p=f$ et $q=k$, et habebimus

$$\frac{kk-1}{f}+\mathfrak{A}+\frac{2(1\pm q)}{p}+\frac{\mathfrak{B}(1\pm k)}{f}-\frac{f(1\pm q)^2}{pp}+\frac{(S+\mathfrak{C})(1\pm k)^2}{ff}+\frac{D(1\pm k)^3}{f^3}+\frac{E(1\pm k)^4}{f^4}+\frac{F(1\pm k)^5}{f^5}=0,$$

quae ob signa ambigua resolvitur in has duas

$$\frac{kk-1}{f}+\mathfrak{A}+\frac{2}{p}+\frac{\mathfrak{B}}{f}-\frac{f(1+qq)}{pp}+\frac{(S+\mathfrak{C})(1+kk)}{ff}+\frac{D(1+3kk)}{f^3}+\frac{E(1+6kk+k^4)}{f^4}+\frac{F(1+10kk+5k^4)}{f^5}=0,$$

$$\frac{2q}{p}+\frac{\mathfrak{B}k}{f}-\frac{2fq}{pp}+\frac{2(S+\mathfrak{C})k}{ff}+\frac{D(3k+k^3)}{f^3}+\frac{E(4k+4k^3)}{f^4}+\frac{F(5k+10k^3+k^5)}{f^5}=0.$$

Ponamus jam $\frac{1}{p}=\frac{1+x}{f}$, et prior aequatio abit in hanc

$$\frac{kk}{f}+\mathfrak{A}+\frac{\mathfrak{B}}{f}-\frac{fqq}{pp}+\frac{(S+\mathfrak{C})(1+kk)}{ff}+\frac{D(1+3kk)}{f^3}+\text{etc.}=0,$$

unde deducimus

$$\frac{qq}{pp}=\frac{kk}{ff}+\frac{\mathfrak{A}}{f}+\frac{\mathfrak{B}}{ff}+\frac{(S+\mathfrak{C})(1+kk)}{f^3}+\frac{D(1+3kk)}{f^4}+\frac{E(1+6kk+k^4)}{f^5}+\frac{F(1+10kk+5k^4)}{f^6}+\text{etc.};$$

altera autem per q multiplicata, qui factor in terminis minimis abit in k, praebet

$$\frac{2x}{f}=\frac{\mathfrak{B}}{f}+\frac{2(\mathfrak{C}+S)}{ff}+\frac{D(3+kk)}{f^3}+\frac{E(4+4kk)}{f^4}+\frac{F(5+10kk+k^4)}{f^5},$$

unde deducimus

$$\frac{1}{p}=\frac{1}{f}+\frac{\mathfrak{B}}{2f}+\frac{\mathfrak{C}+S}{ff}+\frac{D(3+kk)}{2f^3}+\frac{E(4+4kk)}{2f^4}+\frac{F(5+10kk+k^4)}{2f^5}+\text{etc.}$$

$$p=f\left(1-\frac{\mathfrak{B}}{2}-\frac{\mathfrak{C}-S}{f}-\frac{D(3+kk)}{2ff}-\frac{E(4+4kk)}{2f^3}-\frac{F(5+10kk+k^4)}{2f^4}-\text{etc.}\right)$$

$$qq = kk + \mathfrak{A}f + \mathfrak{B}(1 - kk) + \frac{(\mathfrak{C}+S)(1-kk)}{f} + \frac{D(1-k^4)}{ff} + \frac{E(1+2kk-3k^4)}{f^3} + \frac{F(1+5kk\ldots}{f^4}$$

Tum autem formula irrationalis induit hanc formam

$$\frac{q\sin s}{p}\sqrt{\Big(f - S - \mathfrak{C} - \frac{D(3+k\cos s)}{f} - \frac{E(6+4k\cos s+kk(1+\cos^2 s))}{ff}}$$

$$-\frac{F(10+10k\cos s+5kk(1+\cos^2 s)+k^3\cos s(1+\cos^2 s))}{f^3}\Big),$$

unde concludimus

$$\frac{dv}{vv} = \frac{q\,d\varphi\sin s}{p} - \frac{k\,d\varphi\sin s}{2ff}\Big(\mathfrak{C} + \frac{D(3+k\cos s)}{f} + \frac{E(6+4k\cos s+kk(1+\cos^2 s))}{ff} + \text{etc.}\Big).$$

Est vero etiam $\frac{dv}{vv} = \frac{q\,ds\sin s}{p} + \frac{dp}{pp} - \frac{(pdq-qdp)}{pp}\cos s$, unde

$$\frac{q\sin s}{p}(d\varphi - ds) = \frac{k\,d\varphi\sin s}{2ff}\Big(\mathfrak{C} + \frac{D(3+k\cos s)}{f} + \frac{E(6+4k\cos s+kk(1+\cos^2 s))}{ff} + \frac{F(10+10k\cos s+kk(5+k\cos s)(1+\cos^2 s))}{f^3}\Big)$$

$$+\frac{dp}{pp} - \frac{(pdq-qdp)\cos s}{pp},$$

ubi quidem haec differentialia ipsarum dp et dq non tam commode exprimere licet, quam ... Quoties autem unico termino constat integrale $\int(Tv\,d\varphi + V\,dv)$, toties posterius membrum ... potest ad formam per $k\sin s$ multiplicatam. Est autem in genere

$$\frac{dp}{pp} - \frac{(pdq-qdp)\cos s}{pp} = \frac{-Tv^3\,d\varphi}{2ffk}(2k + (1+kk)\cos s) - \frac{d\mathfrak{A}\cos s}{2k} - \frac{d\mathfrak{B}(k+\cos s)}{2fk} - \frac{d\mathfrak{C}(2k+(1+kk)\cos s)}{2ffk}$$

$$-\frac{dD(3k+k^3+(1+3kk)\cos s)}{2f^3k} - \frac{dE(4k+4k^3+(1+6kk+k^4)\cos s)}{2f^4k} - \frac{dF(5k+10k^3+k^5+(1+10kk+5k^4)\cos s)}{2f^5k}\ \text{etc.}$$

quae expressio transmutatur in hanc formam

$$\frac{-(2k+(1+kk)\cos s)\,vv}{2ffk}\Big(Tv\,d\varphi + d\mathfrak{A} + \frac{d\mathfrak{B}}{v} + \frac{d\mathfrak{C}}{vv} + \frac{dD}{v^3} + \frac{dE}{v^4} + \frac{dF}{v^5}\ \text{etc.}\Big)$$

$$+\frac{vv\sin^2 s}{2ff}\left\{\begin{matrix} d\mathfrak{A}\left(1+\frac{f}{v}\right) + \frac{d\mathfrak{B}}{v} - \frac{dD}{fvv}(1+kk) - \frac{dE}{ffvv}\left(1+3kk+\frac{(1+kk)f}{v}\right) \\ -\frac{dF}{f^3vv}\left(1+6kk+k^4+\frac{(1+3kk)f}{v}+\frac{(1+kk)ff}{vv}\right) \end{matrix}\right\}$$

At ex aequatione assumta est

$$d\mathfrak{A} + \frac{d\mathfrak{B}}{v} + \frac{d\mathfrak{C}}{vv} + \frac{dD}{v^3} + \text{etc.} = -Tv\,d\varphi - V\,dv + \frac{\mathfrak{B}\,dv}{vv} + \frac{2\mathfrak{C}\,dv}{v^3}\ \text{etc.}$$

ita ut prius membrum superioris aequationis abeat in

$$+\frac{(2k+(1+kk)\cos s)}{2ffk}vv\,dv\Big(V - \frac{\mathfrak{B}}{vv} - \frac{2\mathfrak{C}}{v^3} - \frac{3D}{v^4} - \frac{4E}{v^5} - \frac{5F}{v^6} - \text{etc.}\Big).$$

Quare cum in terminis his minimis sit $dv = \frac{kd\varphi \sin s}{f} vv$, evidens est totam aequationem praecedentem dividi posse per sin s; reperitur enim

$$d\varphi - ds = \frac{d\varphi}{2f}\left(\mathfrak{C} + \frac{D(3 + k\cos s)}{f} + \frac{E(6 + kk + 4k\cos s + kk\cos^2 s)}{ff} + \frac{F(10 + 5kk + k(10 + kk)\cos s + 5kk\cos^2 s + k^3\cos^3 s)}{f^3}\right)$$

$$+ \frac{(2k + (1 + kk)\cos s)}{2ffk} v^4 d\varphi \left(V - \frac{\mathfrak{B}}{vv} - \frac{2\mathfrak{C}}{v^3} - \frac{3D}{v^4} - \frac{4E}{v^5} - \frac{5F}{v^6} - \text{etc.}\right) +$$

$$\left(d\mathfrak{A} + \frac{f}{v}\left(d\mathfrak{A} + \frac{d\mathfrak{B}}{f}\right) - \frac{(1+kk)}{fvv}\left(dD + \frac{dE}{v} + \frac{dF}{vv} + \text{etc.}\right) - \frac{(1+3kk)}{ffvv}\left(dE + \frac{dF}{v} + \text{etc.}\right) - \frac{(1+6kk+k^4)}{f^3 vv}(dF + \text{etc.})\right)$$

Atque haec est methodus generalis hujusmodi problemata tractandi, quoties formula $Tv\,d\varphi + V dv$ est integrabilis. Deinde etiam pro formula $\int Tv^3 d\varphi$, quam posuimus $= S$, sive sit integrabilis sive minus, poni poterit $\frac{S}{v^\lambda}$, pro numero dimensionum, quas v in ea obtinet, unde solutio saepe commodius reddi potest, ad quod haec solutio pariter extenditur.

Tertia solutio problematis propositi.

§ 168. Instituantur omnia ut in solutione secunda § 159, sed ponatur

$$\int \frac{d\varphi \sin 2\varphi}{v} = \frac{Q}{v} \quad \text{erit} \quad v^4 d\varphi^2 = 2gL dt^2 \left(f - \frac{3nQ}{v}\right),$$

ac porro per integrationem

$$\frac{dv}{vv}\sqrt{\left(f - \frac{3nQ}{v}\right)} = d\varphi\sqrt{\left(\frac{kk-1}{f} + \frac{2}{v} - \frac{f}{vv} + \frac{2m + 3n\cos 2\varphi + 6nQ}{2v^3}\right)}.$$

Hinc posito $v = \frac{p}{1 + q\cos s}$, ut fiat

$$\frac{dv}{vv}\sqrt{\left(f - \frac{3nQ}{v}\right)} = \frac{qd\varphi \sin s}{p}\sqrt{\left(f - \frac{(3 + q\cos s)(2m + 3n\cos 2\varphi + 6nQ)}{2p}\right)},$$

seu quia Q est valde parvum,

$$\frac{dv}{vv} = \frac{qd\varphi \sin s}{p}\sqrt{\left(1 - \frac{6nQ}{fp} - \frac{(2m + 3n\cos 2\varphi)(3 + q\cos s)}{2fp}\right)},$$

statui debet

$$\frac{1}{p} = \frac{1}{f} + \frac{(3 + kk)(2m + 3n\cos 2\varphi + 6nQ)}{2f^3} \quad \text{et} \quad \frac{qq}{pp} = \frac{kk}{ff} + \frac{(1 + 3kk)(2m + 3n\cos 2\varphi + 6nQ)}{2f^4},$$

unde fit

$$p = f - \frac{(3 + kk)(2m + 3n\cos 2\varphi + 6nQ)}{4f} \quad \text{et} \quad qq = kk + \frac{(1 - k^4)(2m + 3n\cos 2\varphi + 6nQ)}{2ff}.$$

Cum nunc sit $vdQ - Qdv = vd\varphi \sin 2\varphi$, ideoque

$$dQ = d\varphi \sin 2\varphi + \frac{Qdv}{v} = d\varphi \sin 2\varphi + \frac{kQvd\varphi \sin s}{f},$$

quia in terminis minimis est $dv = \frac{kvvd\varphi \sin s}{f}$, erit

$$\frac{dp}{pp} = \frac{-3nkQ\nu d\varphi \sin s\,(3+kk)}{f^4}, \quad \frac{2q\,(pdq-qdp)}{p^3} = \frac{-3nkQ\nu d\varphi \sin s\,(1+3kk)}{f^5}$$

ob $\frac{d\nu}{\nu\nu} = \frac{dp}{pp} - \frac{(pdq-qdp)\cos s}{pp} + \frac{qds \sin s}{p}$, habebimus

$$\frac{d\nu}{\nu\nu} = \frac{qds\sin s}{p} - \frac{3nQ\nu d\varphi \sin s}{2f^4}(2k(3+kk) - (1+3kk)\cos s).$$

Est vero ex superioribus

$$\frac{d\nu}{\nu\nu} = \frac{qd\varphi \sin s}{p} - \frac{3nkQd\varphi \sin s}{f^3} - \frac{k(2m+3n\cos 2\varphi)(3+k\cos s)\,d\varphi \sin s}{4f^3},$$

unde concludimus

$$d\varphi - ds = \frac{d\varphi\,(2m+3n\cos 2\varphi)(3+k\cos s)}{4ff} - \frac{3nQ\nu d\varphi}{2f^3 k}(2k(2+kk) - (1+5kk)\cos s).$$

Cum jam sit $\frac{Q}{\nu} = \frac{1}{f}\int d\varphi\,(\sin 2\varphi + \frac{1}{2}k \sin(2\varphi - s) + \frac{1}{2}k \sin(2\varphi + s))$, et proxime $d\varphi = ds$,

$$\frac{Q}{\nu} = \frac{-\cos 2\varphi}{2f} - \frac{k\cos(2\varphi - s)}{2f} - \frac{k\cos(2\varphi + s)}{6f} \quad \text{et}$$

$$\frac{\cos 2\varphi + 2Q}{\nu} = \frac{-k\cos(2\varphi - s)}{2f} + \frac{k\cos(2\varphi+s)}{6f} = \frac{-k}{3f}(\cos 2\varphi \cos s + 2\sin 2\varphi \sin s),$$

hincque

$$p = f - \frac{m(3+kk)}{2f} - \frac{nk(3+kk)(\cos(2\varphi+s) - 3\cos(2\varphi - s))}{8f(1+k\cos s)},$$

$$qq = kk + \frac{m(1-k^4)}{ff} + \frac{nk(1-k^4)(\cos(2\varphi+s) - 3\cos(2\varphi-s))}{4ff(1+k\cos s)}.$$

Invento valore ipsius Q, accuratius relatio inter $d\varphi$ et ds definitur, indeque vera relatio inter φ et s, qua cognita habebitur

$$dt\sqrt{2fgL} = \frac{\nu\nu d\varphi}{\sqrt{(1 + \frac{n}{2ff}(3\cos 2\varphi + 3k\cos(2\varphi - s) + k\cos(2\varphi+s))}}, \quad \text{seu}$$

$$dt\sqrt{2fgL} = \frac{pp\,d\varphi}{(1+q\cos s)^2} - \frac{nd\varphi\,(3\cos 2\varphi + 3k\cos(2\varphi - s) + k\cos(2\varphi+s))}{4ff(1+k\cos s)^2}.$$

Verum haec solutio minus idonea videtur quam secunda.

169. **Problema.** (Fig 183.) Si corpus N circa punctum quasi fixum J non in eodem plano moveatur, ad quod, praeter vim quadratis distantiarum reciproce proportionalem, sollicitetur viribus exiguis quibuscunque, ejus motum tam in longitudinem quam in latitudinem definire.

Solutio. Referatur motus ad planum fixum AJB, in quo sumta recta fixa JA, sint coordinatae orthogonales $JX = x$, $XY = y$ et $YZ = z$, ac ponatur distantia $JN = \sqrt{(xx+yy+zz)} =$ Quibus positis sumtoque elemento temporis dt constante, motus hujusmodi tribus aequationibus exprimetur:

$$ddx = -2gLdt^2\left(\frac{x}{\rho^3} + X\right)$$

$$ddy = -2gLdt^2\left(\frac{y}{\rho^3} + Y\right)$$

$$ddz = -2gLdt^2\left(\frac{z}{\rho^3} + Z\right),$$

ubi quantitates X, Y, Z ut valde parvae sunt spectandae. Consideretur elementum Nn seu directio motus, in qua nunc corpus movetur, quae cum puncto fixo J continet planum, cujus intersectio cum plano assumto AJB sit recta $J\Omega$, quae vocatur linea nodorum, ac terminus quidem Ω nodus ascendens, ubi corpus supra planum AJB ascendere incipit. Hic duae res notandae occurrunt, primo longitudo nodi ascendentis seu angulus $AJ\Omega = \psi$ et inclinatio plani ΩJN ad planum fixum AJB, quae sit $= \omega$. Ex Y ad $J\Omega$ ducatur normalis $Y\Omega$, junctaque $N\Omega$, quae etiam ad $J\Omega$ erit normalis, fiet angulus $Y\Omega N = \omega$. Statuatur nunc angulus $\Omega JN = \sigma$, erit $N\Omega = \rho\sin\sigma$ et $J\Omega = \rho\cos\sigma$, hincque $YN = \rho\sin\sigma\sin\omega = z$ et $\Omega Y = \rho\sin\sigma\cos\omega$, unde ob $XY\Omega = AJ\Omega = \psi$, concluditur $x = \rho\cos\sigma\cos\psi - \rho\sin\sigma\cos\omega\sin\psi$ et $y = \rho\cos\sigma\sin\psi + \rho\sin\sigma\cos\omega\cos\psi$. Quo autem facilius relationem inter hos angulos σ, ω, ψ eorumque differentialia investigemus, re ad trigonometriam sphaericam perducta, sit (fig. 183) arcus $A\Omega = \psi$, $\Omega\omega = d\psi$, $\Omega N = \sigma$, angulus $B\Omega N = \omega$, $B\omega n = \omega + d\omega$, et $\omega n = \sigma + d\sigma$. Ducto $\omega\pi$ perpendiculo in ΩN erit $\Omega\pi = d\psi\cos\omega$, et ob $\omega Y = \pi N$ habebimus $\sigma - d\psi\cos\omega = \sigma + d\sigma - Nn$, unde fit $d\sigma = Nn - d\psi\cos\omega$. Tum vero est

$$\sin\omega : \sin(\omega + d\omega) = \sin(\sigma - d\psi\cos\omega) : \sin\sigma, \text{ seu } \sin\omega : \sin\omega + d\omega\cos\omega = \sin\sigma - d\psi\cos\sigma\cos\omega : \sin\sigma,$$

hincque dividendo $\sin\omega : d\omega\cos\omega = \sin\sigma : d\psi\cos\sigma\cos\omega$, unde fit

$$d\omega\sin\sigma = d\psi\cos\sigma\sin\omega \quad \text{seu} \quad d\omega = \frac{d\psi\cos\sigma\sin\omega}{\sin\sigma}.$$

His notatis resumamus nostras aequationes differentio-differentiales ex quibus concludimus (fig. 183)

$$dx^2 + dy^2 + dz^2 = 2gLdt^2\left(D + \frac{2}{\rho} - 2\int(Xdx + Ydy + Zdz)\right),$$

ubi est $Nn = \sqrt{(dx^2 + dy^2 + dz^2)}$. At est angulus elementaris

$$NJn = \frac{\sqrt{(dx^2 + dy^2 + dz^2 - d\rho^2)}}{\rho} = d\sigma + d\psi\cos\omega,$$

unde concludimus

$$dx^2 + dy^2 + dz^2 = d\rho^2 + \rho\rho(d\sigma + d\psi\cos\omega)^2 = 2gLdt^2\left(2D + \frac{2}{\rho} - 2\int(Xdx + Ydy + Zdz)\right).$$

Statuamus brevitatis ergo $d\sigma + d\psi\cos\omega = d\varphi$, ut sit

$$d\rho^2 + \rho\rho d\varphi^2 = 2gLdt^2\left(2D + \frac{2}{\rho} - 2\int(Xdx + Ydy + Zdz)\right).$$

Tum vero ob $z = \rho\sin\sigma\sin\omega$ habebimus

$$\frac{x}{z} = \frac{\cos\sigma\cos\psi}{\sin\sigma\sin\omega} - \frac{\cos\omega\sin\psi}{\sin\omega} \quad \text{et} \quad \frac{y}{z} = \frac{\cos\sigma\sin\psi}{\sin\sigma\sin\omega} + \frac{\cos\omega\cos\psi}{\sin\omega},$$

unde per differentiationem ob $d\omega = \frac{d\psi\cos\sigma\sin\omega}{\sin\sigma}$, colligimus

$$\frac{zdx - xdz}{zz} = \frac{-d\varphi\cos\psi}{\sin^2\sigma\sin\omega} \quad \text{et} \quad \frac{zdy - ydz}{zz} = \frac{-d\varphi\sin\psi}{\sin^2\sigma\sin\omega},$$

hincque porro $zdx - xdz = -vvd\varphi\cos\psi\sin\omega$ et $zdy - ydz = -vvd\varphi\sin\psi\sin\omega$.

Est vero ex aequationibus principalibus:

$$zddx - xddz = 2gLdt^2(Zx - Xz) \quad \text{et} \quad zddy - yddz = 2gLdt^2(Zy - Yz),$$

quarum illa per $2(zdx - xdz)$, haec vero per $2(zdy - ydz)$ multiplicata et integrata dabit

$$(zdx - xdz)^2 = v^4 d\varphi^2\cos^2\psi\sin^2\omega = 4gLdt^2\int vvd\varphi\cos\psi\sin\omega(Xz - Zx),$$

$$(zdy - ydz)^2 = v^4 d\varphi^2\sin^2\psi\sin^2\omega = 4gLdt^2\int vvd\varphi\sin\psi\sin\omega(Yz - Zy),$$

quibus additis prodit

$$v^4 d\varphi^2\sin^2\omega = 4gLdt^2\int v^3 d\varphi\sin\omega(\sin\sigma\sin\omega(X\cos\psi + Y\sin\psi) - Z\cos\sigma).$$

At si illae aequationes differentientur, indeque differentiale ipsius $v^4 d\varphi^2\sin^2\omega$ eliminetur, obtinebimus

$$vd\varphi d\psi\sin\omega = 2gLdt^2\sin\sigma(\sin\omega(Y\cos\psi - X\sin\psi) - Z\cos\omega),$$

ita ut sit

$$d\psi = \frac{2gLdt^2\sin\sigma}{vd\varphi}(Y\cos\psi - X\sin\psi - Z\cot\omega).$$

Ponamus brevitatis gratia

$$\int v^3 d\varphi\sin\omega(\sin\sigma\sin\omega(X\cos\psi + Y\sin\psi) - Z\cos\sigma) = S,$$

ut sit $v^4 d\varphi^2\sin^2\omega = 4gLdt^2(C + S)$, fietque

$$dv^2 = 4gLdt^2\left(D + \frac{1}{v} - \int(Xdx + Ydy + Zdz) - \frac{C - S}{vv\sin^2\omega}\right), \quad \text{seu}$$

$$dv^2(C + S) = v^4 d\varphi^2\sin^2\omega\left(D + \frac{1}{v} - \int(Xdx + Ydy + Zdz) - \frac{C - S}{vv\sin^2\omega}\right),$$

ac praeterea $$d\psi = \frac{v^3 d\varphi\sin\omega\sin\sigma}{2(C + S)}(\sin\omega(Y\cos\psi - X\sin\psi) - Z\cos\omega).$$

Cum igitur X, Y, Z sint quantitates valde parvae, erit etiam S quantitas minima, et anguli ψ et ω fere constantes, ita ut sit proxime $d\varphi = d\sigma$, accuratius autem $d\sigma = d\varphi - d\psi\cos\omega$. Denique vero erit

$$\frac{d\omega}{\sin^2\omega} = \frac{v^3 d\varphi\cos\sigma}{2(C + S)}(\sin\omega(Y\cos\psi - X\sin\psi) - Z\cos\omega),$$

et aequationis hujus

$$\frac{dv}{vv}\sqrt{(C + S)} = d\varphi\sin\omega\sqrt{\left(D + \frac{1}{v} - \frac{C - S}{vv\sin^2\omega} - \int(Xdx + Ydy + Zdz)\right)}$$

resolutio est instituenda ut ante docuimus.

170. **Coroll. 1.** Uti ω inclinatio orbitae et ψ longitudo nodi ascendentis vocari solet, ita angulus ☊$JN = \sigma$ argumentum latitudinis et angulus φ longitudo in orbita appellatur, quae autem tantum ficta, cum tam linea nodorum quam inclinatio continuo mutetur.

171. **Coroll. 2.** Si vires exiguae ita fuerint comparatae, ut sit

$$\sin\omega\,(Y\cos\psi - X\sin\psi) - Z\cos\omega = 0,$$

tum ob $d\psi = 0$ et $d\omega = 0$, tam linea nodorum quam inclinatio nullam patitur mutationem, ideoque corpus N in eodem perpetuo plano feretur.

172. **Coroll. 3.** Cum autem angulus in plano AJB sumtus AJY vocetur corporis longitudo, erit $= \psi + \text{Ang.tang}\,(\text{tang}\,\sigma\cos\omega)$, tum vero latitudo corporis, quae est angulus YJN, est angulus cujus sinus est $\frac{z}{v} = \sin\sigma\sin\omega$.

173. **Scholion.** Haec methodus motum corporis ad planum fixum reducendi illi multum anteferenda videtur, qua ipsa corporis longitudo seu angulus AJY in calculum introducitur, quo pacto formulae satis intricatae redduntur. Hoc igitur incommodum hic maximam partem sustulimus, dum angulum σ, quo argumentum latitudinis denotatur, ac praeterea longitudinem in orbita seu angulum φ induximus, quoniam hoc modo formulae $zdx - xdz$ et $zdy - ydz$ tam commode exprimuntur, unde etiam fit

$$ydx - xdy = -vvd\varphi\cos\omega \quad \text{atque} \quad yddx - xddy = 2gLdt^2\,(Yx - Xy).$$

Haec ergo per $2\,(ydx - xdy)$ multiplicata et integrata dabit

$$(ydx - xdy)^2 = 4gLdt^2\textstyle\int vvd\varphi\cos\omega\,(Xy - Yx) = v^4 d\varphi^2\cos^2\omega,$$

quae etsi jam in praecedentibus contineatur, saepe ingentem usum praestat, uti in sequente problemate patebit. Hinc scilicet commode relatio inter dt et $d\varphi$ desumi poterit. Deinde etiam vis hujus methodi in hoc consistit, quod elementum temporis dt penitus e formulis integralibus exclusimus, quo deinceps commode ex calculo eliminari posset.

174. **Problema.** Si corpus M, cujus momenta inertiae respectu axium JA et JB sint aequalia, circa tertium axem JC utcunque gyretur, ac circa id corpus sphaericum N quomodocunque moveatur, hujus corporis N motum definire.

Solutio. (Fig. 183.) Plano axium JA et JB, quod quasi est corporis M planum aequatoris, pro plano fixo assumto, sit Maa momentum inertiae respectu axium JA et JB, at Mcc respectu axis JC. Pro motu ergo secundum problema praecedens definiendo habebimus ex § 128 has aequationes

$$ddx = \frac{-2g\,(M+N)\,xdt^2}{v^3}\left(1 + \frac{3\,(4aa+cc)}{2vv} - \frac{15\,(aaxx+aayy+cczz)}{2v^4}\right),$$

$$ddy = \frac{-2g\,(M+N)\,ydt^2}{v^3}\left(1 + \frac{3\,(4aa+cc)}{2vv} - \frac{15\,(aaxx+aayy+cczz)}{2v^4}\right),$$

$$ddz = \frac{-2g\,(M+N)\,zdt^2}{v^3}\left(1 + \frac{3\,(2aa+3cc)}{2vv} - \frac{15\,(aaxx+aayy+cczz)}{2v^4}\right),$$

quibus comparatis cum ante assumtis erit $L = M + N$ et

$$X = \frac{3x(4aa+cc)}{2v^5} - \frac{15x(aaxx+aayy+cczz)}{2v^7},$$

$$Y = \frac{3y(4aa+cc)}{2v^5} - \frac{15y(aaxx+aayy+cczz)}{2v^7},$$

$$Z = \frac{3z(2aa+3cc)}{2v^5} - \frac{15z(aaxx+aayy+cczz)}{2v^7},$$

hinc ob $xdx + ydy + zdz = vdv$, erit

$$Xdx + Ydy + Zdz = \frac{3(4aa+cc)(xdx+ydy)}{2v^5} + \frac{3(2aa+3cc)zdz}{2v^5} - \frac{15dv(aaxx+aayy+cczz)}{2v^6}$$

$$= \frac{3(4aa+cc)dv}{2v^4} - \frac{3(aa-cc)zdz}{v^5} - \frac{15aadv}{2v^4} + \frac{15(aa-cc)zzdv}{2v^6}.$$

Ergo $\int(Xdx + Ydy + Zdz) = \frac{(aa-cc)}{2v^3} - \frac{3(aa-cc)zz}{2v^5}$, hincque

$$dv^2 + vvd\varphi^2 = 4gLdt^2\left(D + \frac{1}{v} + \frac{(cc-aa)}{2v^3} - \frac{3(cc-aa)zz}{2v^5}\right).$$

Cum nunc ex § praecedente sit $yddx - xddy = 0$, erit

$$ydx - xdy = -vvd\varphi\cos\omega = -Edt\sqrt{4gL} \quad \text{et} \quad vvd\varphi^2 = \frac{4gLEEdt^2}{vv\cos^2\omega}, \quad \text{hincque}$$

$$dv^2 = 4gLdt^2\left(D + \frac{1}{v} + \frac{cc-aa}{2v^3} - \frac{3(cc-aa)zz}{2v^5} - \frac{EE}{vv\cos^2\omega}\right) \quad \text{et}$$

$$dv^2 = \frac{v^4d\varphi^2\cos^2\omega}{EE}\left(D + \frac{1}{v} + \frac{cc-aa}{2v^3} - \frac{3(cc-aa)zz}{2v^5} - \frac{EE}{vv\cos^2\omega}\right), \quad \text{seu}$$

$$\frac{Edv}{vv} = d\varphi\cos\omega\sqrt{\left(D + \frac{1}{v} + \frac{cc-aa}{2v^3} - \frac{3(cc-aa)\sin^2\sigma\sin^2\omega}{2v^3} - \frac{EE}{vv\cos^2\omega}\right)}$$

atque $2Edt\sqrt{gL} = vvd\varphi\cos\omega$. Deinde vero habemus

$$v^4d\varphi^2\cos^2\psi\sin^2\omega = 12gL(aa-cc)dt^2\int\frac{xzd\varphi\cos\psi\sin\omega}{v^3} \quad \text{et}$$

$$v^4d\varphi^2\sin^2\psi\sin^2\omega = 12gL(aa-cc)dt^2\int\frac{yzd\varphi\sin\psi\sin\omega}{v^3},$$

quibus additis fit

$$v^4d\varphi^2\sin^2\omega = 12gL(aa-cc)dt^2\int\frac{d\varphi\sin\sigma\cos\sigma\sin^2\omega}{v}.$$

Cum porro sit $v^4d\varphi^2\cos^2\omega = 4gLEEdt^2$, erit differentiando

$$2v^4d\varphi^2d\omega\sin\omega\cos\omega + \sin^2\omega d.(v^4d\varphi^2) = 12gL(aa-cc)dt^2\cdot\frac{d\varphi\sin\sigma\cos\sigma\sin^2\omega}{v}$$

$$\text{et} \quad -2v^4d\varphi^2d\omega\sin\omega\cos\omega + \cos^2\omega.d(v^4d\varphi^2) = 0,$$

unde concluditur

$$2v^4\,d\varphi^2 d\omega \sin\omega\cos\omega = 12gL\,(aa-cc)\,dt^2\cdot\frac{d\varphi\sin\sigma\cos\sigma\sin^2\omega\cos^2\omega}{v},\quad\text{seu}$$

$$d\varphi\,d\omega = \frac{6gL\,(aa-cc)\,dt^2\sin\sigma\cos\sigma\sin\omega\cos\omega}{v^5},\quad\text{ideoque}$$

$$d\omega = \frac{3\,(aa-cc)\,d\varphi\sin\sigma\cos\sigma\sin\omega\cos^3\omega}{2EEv}\quad\text{et}\quad d\psi = \frac{d\omega\sin\sigma}{\cos\sigma\sin\omega} = \frac{3\,(aa-cc)\,d\varphi\sin^2\sigma\cos^3\omega}{2EEv}.$$

Initio igitur ob $aa-cc$ minimum, elementa ψ et ω ut constantia spectantur, et cum sit $d\varphi = d\sigma - d\psi\cos\omega$, differentialia $d\varphi$ et $d\sigma$ pro aequalibus habentur. Ponatur jam $EE = F\cos^2\omega$ et $(cc-aa)(1-3\sin^2\sigma\sin^2\omega) = G$, ut habeamus

$$\frac{Edv}{vv} = d\varphi\cos\omega\,\sqrt{\left(D+\frac{1}{v}-\frac{F}{vv}+\frac{G}{v^3}\right)}.$$

Ponatur nunc $v = \frac{p}{1+q\cos s}$, fiatque $D+\frac{(1\pm q)}{p} - F\left(\frac{1\pm q}{p}\right)^2 + G\left(\frac{1\pm q}{p}\right)^3 = 0$, ut sit

$$D+\frac{1}{p}-\frac{F\,(1+qq)}{pp}+\frac{G\,(1+3qq)}{p^3} = 0\quad\text{et}\quad 1-\frac{2F}{p}+\frac{G\,(3+qq)}{pp} = 0,$$

ubi cum G sit valde parvum, sit $F = \frac{f}{2}+u$, ut prodeat valor prope verus $p = f$, eritque

$$EE = \frac{1}{2}f\cos^2\omega + u\cos^2\omega = \text{Constanti}.$$

Sit ε valor medius inclinationis et $EE = \frac{1}{2}f\cos^2\varepsilon$, erit

$$u = \frac{f(\cos^2\varepsilon-\cos^2\omega)}{2\cos^2\omega},\quad\text{atque}\quad 1-\frac{f}{p}-\frac{(\cos^2\varepsilon-\cos^2\omega)}{\cos^2\omega}+\frac{G\,(3+kk)}{ff} = 0,\quad\text{et hinc}$$

$$\frac{1}{p} = \frac{1}{f}-\frac{(\cos^2\varepsilon-\cos^2\omega)}{f\cos^2\omega}+\frac{G\,(3+kk)}{f^3}.$$

Tum vero prior aequatio erit

$$D+\frac{1}{p}-\frac{f(1+qq)}{2pp}-\frac{(1+kk)(\cos^2\varepsilon-\cos^2\omega)}{2f\cos^2\omega}+\frac{G\,(1+3kk)}{f^3} = 0.$$

Sit constans $D = \frac{kk-1}{2f}$, eritque

$$\frac{qq}{pp} = \frac{kk}{ff}-\frac{(1+kk)(\cos^2\varepsilon-\cos^2\omega)}{ff\cos^2\omega}+\frac{2G\,(1+3kk)}{f^4},\quad\text{ideoque}$$

$$p = f+\frac{f(\cos^2\varepsilon-\cos^2\omega)}{\cos^2\omega}-\frac{G\,(3+kk)}{f}\quad\text{et}\quad qq = kk-\frac{(1-kk)(\cos^2\varepsilon-\cos^2\omega)}{\cos^2\omega}+\frac{2G\,(1-k^4)}{ff},$$

unde formula irrationalis abit in

$$\frac{q\sin s}{p}\sqrt{\left(\frac{f}{2}+\frac{f(\cos^2\varepsilon-\cos^2\omega)}{2\cos^2\omega}-\frac{G\,(3+k\cos s)}{f}\right)},$$

ut ob $E = \frac{\cos\varepsilon\sqrt{f}}{\sqrt{2}}$ sit

$$\frac{d\nu}{\nu\nu} = \frac{qd\varphi \sin s \cos\omega}{p \cos\varepsilon} \sqrt{\left(1 + \frac{\cos^2\varepsilon - \cos^2\omega}{\cos^2\omega} - \frac{2G(3 + k\cos s)}{ff}\right)}, \quad \text{seu}$$

$$\frac{d\nu}{\nu\nu} = \frac{q \sin s}{p} d\varphi \sqrt{\left(1 - \frac{2G(3 + k\cos s)\cos^2\omega}{ff\cos^2\varepsilon}\right)} = \frac{q\sin s}{p}\left(d\varphi - \frac{Gd\varphi(3 + k\cos s)\cos^2\omega}{ff\cos^2\varepsilon}\right).$$

Per differentiationem autem obtinemus

$$\frac{dp}{pp} = \frac{3(cc - aa)\, d\varphi \sin\sigma\cos\sigma\sin^2\omega\,(1 - 2k\cos s + kk)}{f^3},$$

$$\frac{pdq - qdp}{pp} = \frac{3(cc - aa)\, d\varphi \sin\sigma\cos\sigma\sin^2\omega\,(\cos s - 2k + kk\cos s)}{f^3},$$

hincque concludimus

$$\frac{d\nu}{\nu\nu} = \frac{qds\sin s}{p} + \frac{3(cc - aa)\, d\varphi \sin\sigma\cos\sigma\sin^2\omega\,(1 - kk)\sin^2 s}{f^3}$$

$$= \frac{qd\varphi\sin s}{p} - \frac{k(cc - aa)(1 - 3\sin^2\sigma\sin^2\omega)\, d\varphi\,(3 + k\cos s)\cos^2\omega\sin s}{2f^3\cos^2\varepsilon},$$

ita ut sit

$$d\varphi - ds = \frac{3(cc - aa)\, d\varphi\sin\sigma\cos\sigma\sin^2\omega\,(1 - kk)\sin s}{ffk} + \frac{(cc - aa)(1 - 3\sin^2\sigma\sin^2\omega)(3 + k\cos s)\, d\varphi\cos^2\omega}{2ff\cos^2\varepsilon}.$$

Cum igitur in his terminis minimis liceat ponere $\omega = \varepsilon$, quae est inclinatio media, erit

$$d\varphi - ds = \frac{(cc - aa)(3 + k\cos s)d\varphi}{2ff} - \frac{3(cc - aa)(3 + k\cos s)\, d\varphi\sin^2\varepsilon\sin^2\sigma}{2ff} + \frac{3(cc - aa)(1 - kk)\, d\varphi\sin^2\varepsilon\sin s\sin\sigma\cos}{ffk}$$

ubi statuere licet $d\varphi = ds = d\sigma$. Tum vero habetur

$$p = \frac{f\cos^2\varepsilon}{\cos^2\omega} - \frac{(cc - aa)(1 - 3\sin^2\varepsilon\sin^2\sigma)(3 + kk)}{2f},$$

$$qq = \frac{kk\cos^2\varepsilon}{\cos^2\omega} + 1 - \frac{\cos^2\varepsilon}{\cos^2\omega} + \frac{(cc - aa)(1 - 3\sin^2\varepsilon\sin^2\sigma)(1 - k^4)}{ff}$$

ac praeterea

$$d\psi = \frac{-3(cc - aa)(1 + k\cos s)\, d\varphi\cos\varepsilon\sin^2\sigma}{ff}, \qquad d\omega = \frac{-3(cc - aa)(1 + k\cos s)\, d\varphi\sin\varepsilon\cos\varepsilon\sin\sigma\cos\sigma}{ff},$$

eritque $d\varphi = d\sigma + d\psi\cos\varepsilon$, ac tandem pro tempore

$$dt\sqrt{2fgL} = \frac{\nu\nu\, d\varphi\cos\omega}{\cos\varepsilon} = \frac{pp\, d\varphi\cos\omega}{\cos\varepsilon(1 + q\cos s)^2},$$

quae formulae omnes in terminis minimis sine difficultate integrari possunt; postrema tantum formul majorem solertiam postulat. Ponamus enim ad abbreviandum $\frac{cc - aa}{ff} = n$ et evolutis produc sinuum et cosinuum adipiscemur

$$d\psi = -\tfrac{3}{2} n d\varphi\cos\varepsilon\,(1 + k\cos s - \cos 2\sigma - \tfrac{1}{2}k\cos(2\sigma - s) - \tfrac{1}{2}k\cos(2\sigma + s)),$$

$$d\omega = -\tfrac{3}{2} n d\varphi\sin\varepsilon\cos\varepsilon\,(\sin 2\sigma + \tfrac{1}{2}k\sin(2\sigma - s) + \tfrac{1}{2}k\sin(2\sigma + s)),$$

$$ds =$$

$$n\,d\varphi\,(3+k\cos s) - \tfrac{3}{4}\,n\,d\varphi\,\sin^2\varepsilon\left(3 + k\cos s - 3\cos 2\sigma - \frac{(2-kk)}{2k}\cos(2\sigma - s) + \frac{2-3kk}{2k}\cos(2\sigma + s)\right),$$

$$d\varphi - d\sigma = -\tfrac{3}{2}\,n\,d\varphi\,\cos^2\varepsilon\left(1 + k\cos s - \cos 2\sigma - \tfrac{1}{2}k\cos(2\sigma - s) - \tfrac{1}{2}k\cos(2\sigma + s)\right).$$

Hic igitur totum negotium pendet ab integratione hujusmodi formulae $\int d\varphi\cos(\mu s + \nu\sigma)$, ad quam accurate evolvendam ponamus brevitatis gratia

$$d\varphi = ds + \alpha d\varphi + P d\varphi \quad \text{et} \quad d\varphi = d\sigma + \beta d\varphi + Q d\varphi, \quad \text{ut sit}$$

$$\alpha = \tfrac{3}{2}n - \tfrac{9}{4}n\sin^2\varepsilon, \qquad \beta = -\tfrac{3}{2}n\cos^2\varepsilon,$$

$$P = \tfrac{1}{2}nk\cos s - \tfrac{3}{4}\,n\sin^2\varepsilon\left(k\cos s - 3\cos 2\sigma - \frac{(2-kk)}{2k}\cos(2\sigma - s) + \frac{2-3kk}{2k}\cos(2\sigma + s)\right) \quad \text{et}$$

$$Q = -\tfrac{3}{2}n\cos^2\varepsilon\left(k\cos s - \cos 2\sigma - \tfrac{1}{2}k\cos(2\sigma - s) - \tfrac{1}{2}k\cos(2\sigma + s)\right).$$

Quare cum hinc conficiatur $d\varphi = \dfrac{\mu ds + \nu d\sigma + d\varphi(\mu P + \nu Q)}{\mu + \nu - \alpha\mu - \beta\nu}$, erit

$$\int d\varphi\cos(\mu s + \nu\sigma) = \frac{\sin(\mu s + \nu\sigma)}{\mu + \nu - \alpha\mu - \beta\nu} + \int\frac{d\varphi\,(\mu P + \nu Q)\cos(\mu s + \nu\sigma)}{\mu + \nu - \alpha\mu - \beta\nu}.$$

Tum vero cum $\mu P + \nu Q$ habeat hujusmodi formam

$$A\cos s + B\cos 2\sigma + C\cos(2\sigma - s) + D\cos(2\sigma + s),$$

haec per $\cos(\mu s + \nu\sigma)$ multiplicata denuo in simplices cosinus evolvitur, quorum singuli praebent formulas similes integrandas. Qui etsi videntur ob parvitatem rejiciendi, tamen si in iis fiat $\mu + \nu = 0$, ob denominatorem $-\alpha\mu - \beta\nu$ minimum ad notabilem valorem exsurgere possunt.

175. **Coroll. 1.** Cum sit $d\omega = -\tfrac{3}{2}\,n\,d\varphi\sin\varepsilon\cos\varepsilon\,(\ldots)$ (vide ult. lin. pag. praec.) patet duobus casibus inclinationem orbitae nullam pati mutationem, altero quo $\varepsilon = 0$, seu corpus N in ipso plano aequatoris AJB movetur, altero quo $\varepsilon = 90^0$, seu corpus N in plano ad aequatorem perpendiculari fertur; atque hoc casu etiam linea nodorum est fixa. Ceteris ergo paribus inclinatio obnoxia erit maximae variationi, quando inclinatio ε est 45^0.

176. **Coroll. 2.** Pro motu lineae nodorum invenimus longitudinem nodi ascendentis

$$\psi = \text{Const.} - \tfrac{3}{2}n\varphi\cos\varepsilon - \tfrac{3}{2}n\cos\varepsilon\left(k\int d\varphi\cos s - \int d\varphi\cos 2\sigma - \tfrac{1}{2}k\int d\varphi\cos(2\sigma - s) - \tfrac{1}{2}k\int d\varphi\cos(2\sigma + s)\right).$$

Pro motu autem lineae absidum erit longitudo absidis imae

$$\varphi - s = \text{Const.} + \tfrac{3}{2}n\left(1 - \tfrac{3}{2}\sin^2\varepsilon\right)\varphi + \tfrac{1}{2}nk\left(1 - \tfrac{3}{2}\sin^2\varepsilon\right)\int d\varphi\cos s$$

$$+ \tfrac{3}{4}n\sin^2\varepsilon\left(3\int d\varphi\cos 2\sigma + \frac{2-kk}{2k}\int d\varphi\cos(2\sigma - s) - \frac{(2-3kk)}{2k}\int d\varphi\cos(2\sigma + s)\right)$$

et pro argumento latitudinis σ habemus $\varphi = \sigma + \psi\cos\varepsilon$.

177. **Coroll. 3.** Si partes integrales rejiciamus, innotescet vero proxime motus medius lineae nodorum quam lineae absidum, ac si $n = \frac{cc - aa}{ff}$ sit numerus positivus, linea nodorum regreditur, idque eo minus, quo major fuerit inclinatio. Linea autem absidum progreditur, quamdiu $\sin^2 \varepsilon < \frac{2}{3}$, seu $\varepsilon < 54^0 45'$; sin autem fuerit $\varepsilon > 54^0 45'$, etiam linea absidum regreditur.

178. **Coroll. 4.** Cum sit proxime $d\varphi = ds = d\sigma$, erunt integralium valores proximi

$$\int d\varphi \cos s = \sin s, \quad \int d\varphi \cos 2\sigma = \tfrac{1}{2} \sin 2\sigma, \quad \int d\varphi \cos (2\sigma - s) = \sin (2\sigma - s) \quad \text{et}$$

$$\int d\varphi \cos (2\sigma + s) = \tfrac{1}{3} \sin (2\sigma + s),$$

unde praeter motum medium utriusque lineae nodorum et absidum, anomaliae periodicae definiri possunt.

179. **Scholion.** Hae determinationes recte se habere sunt censendae, dummodo fractio $n = \frac{cc - aa}{ff}$ satis fuerit parva, ut termini quadrato nn affecti pro nihilo haberi queant. Sin autem eveniat, ut haec fractio non sit adeo parva, tum jam superiores formulae accuratius evolvi deberent, ut termini per nn multiplicati simul comprehenderentur; hoc autem modo in formulas nimis prolixas incideremus. Verum hinc statim ii termini excludi poterunt, qui nullius plane momenti videbuntur, iis tantum retentis, qui per integrationem insignes coëfficientes adipiscuntur, cujusmodi est $\cos(2\sigma - 2s)$, unde per integrationem oritur

$$\int d\varphi \cos (2\sigma - 2s) = \frac{\sin (2\sigma - 2s)}{2\alpha - 2\beta} = \frac{2 \sin (2\sigma - 2s)}{3n (2 - 3\sin^2 \varepsilon + 2\cos^2 \varepsilon)},$$

qui terminus etsi ex ordine per nn multiplicato nascitur, tamen ob denominatorem exiguum ad ordinem per n multiplicatum elevatur. Deinde etiam si excentricitas k fuerit exigua, per integrationes ulterius productas anguli absoluti satis notabiles exsurgere possunt. Scilicet integratio $\int d\varphi \cos (2\sigma - s)$ ducit ad formam

$$\frac{\sin (2\sigma - s)}{1 + \alpha - 2\beta} + \frac{\int d\varphi (2Q - P) \cos (2\sigma - s)}{1 + \alpha - 2\beta},$$

at in $2Q - P$ continetur membrum

$$\tfrac{3}{2} nk \cos^2 \varepsilon \cos (2\sigma - s) - \frac{3n (2 - kk)}{8k} \sin^2 \varepsilon \cos (2\sigma - s),$$

quod per $\cos (2\sigma - s)$ multiplicatum praebet quantitatem constantem

$$\tfrac{3}{4} nk \cos^2 \varepsilon - \frac{3n (2 - kk)}{16k} \sin^2 \varepsilon,$$

ita ut inde oritur angulus absolutus

$$\left(\tfrac{3}{4} nk \cos^2 \varepsilon - \frac{3n (2 - kk)}{16k} \sin^2 \varepsilon\right) \varphi$$

ad motum medium adjiciendus. Simili modo ex formula

$$\int d\varphi \cos(2\sigma + s) = \frac{\sin(2\sigma + s)}{3-\alpha-2\beta} + \frac{\int d\varphi\,(2Q+P)\cos(2\sigma+s)}{3-\alpha-2\beta},$$

ob $2Q+P$ complectentem terminum $(\frac{3}{2}nk\cos^2\varepsilon - \frac{3n(2-3kk)}{8k}\sin^2\varepsilon)\cos(2\sigma+s)$, nascetur angulus absolutus $(\frac{1}{4}nk\cos^2\varepsilon - \frac{n(2-3kk)}{16k}\sin^2\varepsilon)\varphi$. Cum deinde in motu lineae absidum hi anguli denuo per $\frac{3n(2-kk)}{8k}\sin^2\varepsilon$ et $\frac{-3n(2-3kk)}{8k}\sin^2\varepsilon$ multiplicari debeant, fieri potest, ut inde motus medius non afficiatur. Verum si hi termini alicujus sint momenti, etiam ipsas formulas principales accuratius evolvi oporteret, quod autem negotium hic suscipi non convenit, cum nondum satis constet, quibusnam casibus id utilitatem esset habiturum. Quod denique ad integrationem formulae

$$\int \frac{pp\,d\varphi\cos\omega}{\cos\varepsilon\,(1+q\cos s)^2} = t\sqrt{2fgL}$$

attinet, in ea vires analyseos experiri oportet, ac tutissima quidem methodus videtur, postquam loco $d\varphi$ valor $ds + \alpha d\varphi + Pd\varphi$ est positus, formulam $\frac{pp\,ds\cos\omega}{\cos\varepsilon\,(1+q\cos s)^2}$ ita integrare, quasi p, q et ω essent constantes, tum vero invento integrali correctiones ex harum quantitatum variabilitate oriundas investigare. Atque haec de motu duorum corporum se mutuo attrahentium sufficere videntur, ex quo ad considerationem trium corporum progrediamur.

Caput VI.

De motu trium corporum sphaericorum, se mutuo attrahentium in genere.

180. **Problema.** (Fig. 185.) Si tria corpora sphaerica L, M, N, se mutuo attrahentia moveantur in eodem plano, eorum motum per calculum definire.

Solutio. Elapso tempore $= t$ versentur corpora in L, M, N in plano tabulae, in quo sumta recta fixa OV, ad quam eorum situs referatur, per puncta L, M, N agantur rectae $l\lambda$, $m\mu$, $n\nu$, ipsi OV parallelae, simulque ad eam perpendicula LP, MQ, NR. Quodsi jam longitudinem cujusque corporis ex altero spectati per angulum a recta OV in sensum Vv sumtum aestimemus, statuamus

longitudinem corporis M ex L spectati $lLM = \zeta$

longitudinem corporis N ex M spectati $mMN = \eta$

longitudinem corporis L ex N spectati $nNL = \vartheta$,

qui postremus angulus ϑ in figura duobus rectis major est intelligendus. Atque iidem anguli duobus rectis vel aucti vel minuti exhibebunt longitudinem corporum L, M, N ex M, N, L spectatorum. Ponamus nunc distantias $LM = x$, $MN = y$ et $NL = z$, erunt coordinatae