$$\int d\varphi \cos(2\sigma + s) = \frac{\sin(2\sigma + s)}{3 - \alpha - 2\beta} + \frac{\int d\varphi\,(2Q + P)\cos(2\sigma + s)}{3 - \alpha - 2\beta},$$

ob $2Q + P$ complectentem terminum $(\frac{3}{2} nk \cos^2\varepsilon - \frac{3n(2-3kk)}{8k}\sin^2\varepsilon)\cos(2\sigma + s)$, nascetur angulus absolutus $(\frac{1}{4} nk \cos^2\varepsilon - \frac{n(2-3kk)}{16k}\sin^2\varepsilon)\,\varphi$. Cum deinde in motu lineae absidum hi anguli denuo per $\frac{3n(2-kk)}{8k}\sin^2\varepsilon$ et $\frac{-3n(2-3kk)}{8k}\sin^2\varepsilon$ multiplicari debeant, fieri potest, ut inde motus medius non parum afficiatur. Verum si hi termini alicujus sint momenti, etiam ipsas formulas principales accuratius evolvi oporteret, quod autem negotium hic suscipi non convenit, cum nondum satis constet, quibusnam casibus id utilitatem esset habiturum. Quod denique ad integrationem formulae

$$\int \frac{pp\,d\varphi \cos\omega}{\cos\varepsilon\,(1 + q\cos s)^2} = t\sqrt{2fgL}$$

attinet, in ea vires analyseos experiri oportet, ac tutissima quidem methodus videtur, postquam loco $d\varphi$ valor $ds + \alpha d\varphi + P d\varphi$ est positus, formulam $\frac{pp\,ds\cos\omega}{\cos\varepsilon\,(1 + q\cos s)^2}$ ita integrare, quasi p, q et ω essent constantes, tum vero invento integrali correctiones ex harum quantitatum variabilitate oriundas investigare. Atque haec de motu duorum corporum se mutuo attrahentium sufficere videntur, ex quo ad considerationem trium corporum progrediamur.

Caput VI.

De motu trium corporum sphaericorum, se mutuo attrahentium in genere.

180. **Problema.** (Fig. 185.) Si tria corpora sphaerica L, M, N, se mutuo attrahentia moveantur in eodem plano, eorum motum per calculum definire.

Solutio. Elapso tempore $= t$ versentur corpora in L, M, N in plano tabulae, in quo sumta recta fixa OV, ad quam eorum situs referatur, per puncta L, M, N agantur rectae $l\lambda$, $m\mu$, $n\nu$, ipsi OV parallelae, simulque ad eam perpendicula LP, MQ, NR. Quodsi jam longitudinem cujusque corporis ex altero spectati per angulum a recta OV in sensum Vv sumtum aestimemus, statuamus

longitudinem corporis M ex L spectati $lLM = \zeta$
longitudinem corporis N ex M spectati $mMN = \eta$
longitudinem corporis L ex N spectati $nNL = \vartheta$,

qui postremus angulus ϑ in figura duobus rectis major est intelligendus. Atque iidem anguli duobus rectis vel aucti vel minuti exhibebunt longitudinem corporum L, M, N ex M, N, L spectatorum. Ponamus nunc distantias $LM = x$, $MN = y$ et $NL = z$, erunt coordinatae

$$OQ = OP + x\cos\zeta, \qquad QM = PL + x\sin\zeta$$
$$OR = OQ + y\cos\eta, \qquad RN = QM + y\sin\eta$$
$$OP = OR + z\cos\vartheta, \qquad PL = RN + z\sin\vartheta$$

hincque colligimus

$$x\cos\zeta + y\cos\eta + z\cos\vartheta = 0 \quad \text{et} \quad x\sin\zeta + y\sin\eta + z\sin\vartheta = 0$$

ac porro

$$x\sin(\zeta - \vartheta) + y\sin(\eta - \vartheta) = 0, \qquad x\sin(\zeta - \eta) + z\sin(\vartheta - \eta) = 0,$$
$$y\sin(\eta - \zeta) + z\sin(\vartheta - \zeta) = 0,$$

ideoque $\quad x : y : z = \sin(\eta - \vartheta) : \sin(\vartheta - \zeta) : \sin(\zeta - \eta),$

unde relatio inter distantias et angulos ita commodissime exhibetur, ut sit

$$x = v\sin(\eta - \vartheta), \quad y = v\sin(\vartheta - \zeta), \quad z = v\sin(\zeta - \eta),$$

ubi v denotat diametrum circuli triangulo LMN circumscripti. Si jam massae corporum litteris cognominibus L, M, N exprimantur, corpus L a reliquis sollicitatur

$$\text{sec. } OP \text{ vi} = \frac{LM\cos\zeta}{xx} - \frac{LN\cos\vartheta}{zz} \quad \text{et} \quad \text{sec. } PL \text{ vi} = \frac{LM\sin\zeta}{xx} - \frac{LN\sin\vartheta}{zz},$$

corpus vero M a reliquis sollicitatur

$$\text{sec. } OQ \text{ vi} = \frac{MN\cos\eta}{yy} - \frac{LM\cos\zeta}{xx} \quad \text{et} \quad \text{sec. } QM \text{ vi} = \frac{MN\sin\eta}{yy} - \frac{LM\sin\zeta}{xx}$$

et corpus N a reliquis sollicitatur

$$\text{sec. } OR \text{ vi} = \frac{LN\cos\vartheta}{zz} - \frac{MN\cos\eta}{yy} \quad \text{et} \quad \text{sec. } RN \text{ vi} = \frac{LN\sin\vartheta}{zz} - \frac{MN\sin\eta}{yy},$$

unde sequentes aequationes adipiscimur

$$dd\,.\,OP = 2gdt^2\left(\frac{M\cos\zeta}{xx} - \frac{N\cos\vartheta}{zz}\right), \qquad dd\,.\,PL = 2gdt^2\left(\frac{M\sin\zeta}{xx} - \frac{N\sin\vartheta}{zz}\right),$$
$$dd\,.\,OQ = 2gdt^2\left(\frac{N\cos\eta}{yy} - \frac{L\cos\zeta}{xx}\right), \qquad dd\,.\,QM = 2gdt^2\left(\frac{N\sin\eta}{yy} - \frac{L\sin\zeta}{xx}\right),$$
$$dd\,.\,OR = 2gdt^2\left(\frac{L\cos\vartheta}{zz} - \frac{M\cos\eta}{yy}\right), \qquad dd\,.\,RN = 2gdt^2\left(\frac{L\sin\vartheta}{zz} - \frac{M\sin\eta}{yy}\right),$$

ex quibus colligimus sequentes

$$dd.x\cos\zeta = 2gdt^2\left(-\frac{(L+M)\cos\zeta}{xx} + \frac{N\cos\eta}{yy} + \frac{N\cos\vartheta}{zz}\right), \quad dd.x\sin\zeta = 2gdt^2\left(-\frac{(L+M)\sin\zeta}{xx} + \frac{N\sin\eta}{yy} + \frac{N\sin\vartheta}{zz}\right)$$
$$dd.y\cos\eta = 2gdt^2\left(-\frac{(M+N)\cos\eta}{yy} + \frac{L\cos\vartheta}{zz} + \frac{L\cos\zeta}{xx}\right), \quad dd.y\sin\eta = 2gdt^2\left(-\frac{(M+N)\sin\eta}{yy} + \frac{L\sin\vartheta}{zz} + \frac{L\sin\zeta}{xx}\right)$$
$$dd.z\cos\vartheta = 2gdt^2\left(-\frac{(L+N)\cos\vartheta}{zz} + \frac{M\cos\zeta}{xx} + \frac{M\cos\eta}{yy}\right), \quad dd.z\sin\vartheta = 2gdt^2\left(-\frac{(L+N)\sin\vartheta}{zz} + \frac{M\sin\zeta}{xx} + \frac{M\sin\eta}{yy}\right)$$

quae porro transformantur in has

$$\text{I.}\quad 2dxd\zeta + xdd\zeta = 2gdt^2\left(\frac{N\sin(\eta-\zeta)}{yy} + \frac{N\sin(\vartheta-\zeta)}{zz}\right),$$

$$\text{II.}\quad ddx - xd\zeta^2 = 2gdt^2\left(-\frac{(L+M)}{xx} + \frac{N\cos(\eta-\zeta)}{yy} + \frac{N\cos(\vartheta-\zeta)}{zz}\right),$$

$$\text{III.}\quad 2dyd\eta + ydd\eta = 2gdt^2\left(\frac{L\sin(\vartheta-\eta)}{zz} + \frac{L\sin(\zeta-\eta)}{xx}\right),$$

$$\text{IV.}\quad ddy - yd\eta^2 = 2gdt^2\left(-\frac{(M+N)}{yy} + \frac{L\cos(\vartheta-\eta)}{zz} + \frac{L\cos(\zeta-\eta)}{xx}\right),$$

$$\text{V.}\quad 2dzd\vartheta + zdd\vartheta = 2gdt^2\left(\frac{M\sin(\zeta-\vartheta)}{xx} + \frac{M\sin(\eta-\vartheta)}{yy}\right),$$

$$\text{VI.}\quad ddz - zd\vartheta^2 = 2gdt^2\left(-\frac{(L+N)}{zz} + \frac{M\cos(\zeta-\vartheta)}{xx} + \frac{M\cos(\eta-\vartheta)}{yy}\right),$$

Ex harum aequationum I, III et V colligimus hanc integralem

$$LMxxd\zeta + MNyy\,d\eta + LNzz\,d\vartheta = Cdt,$$

deinde ex I et II deducimus

$$d.(dx^2 + xxd\zeta^2) = 4gdt^2\left(-\frac{(L+M)\,dx}{xx} + \frac{N(dx\cos(\eta-\zeta) + xd\zeta\sin(\eta-\zeta))}{yy} + \frac{N(dx\cos(\vartheta-\zeta) + xd\zeta\sin(\vartheta-\zeta))}{zz}\right),$$

quae ita repraesentetur

$$\frac{d.(dx^2 + xxd\zeta^2)}{4gN\,dt^2} = -\frac{(L+M)\,dx}{Nxx} + \frac{d.x\cos(\eta-\zeta) + xd\eta\sin(\eta-\zeta)}{yy} + \frac{d.x\cos(\vartheta-\zeta) + xd\vartheta\sin(\vartheta-\zeta)}{zz},$$

similesque ex reliquis ortae erunt

$$\frac{d.(dy^2 + yyd\eta^2)}{4gLdt^2} = \frac{-(M+N)\,dy}{Lyy} + \frac{d.y\cos(\vartheta-\eta) + yd\vartheta\sin(\vartheta-\eta)}{zz} + \frac{d.y\cos(\zeta-\eta) + yd\zeta\sin(\zeta-\eta)}{xx},$$

$$\frac{d.(dz^2 + zzd\vartheta^2)}{4gMdt^2} = \frac{-(L+N)\,dz}{Mzz} + \frac{d.z\cos(\zeta-\vartheta) + zd\zeta\sin(\zeta-\vartheta)}{xx} + \frac{d.z\cos(\eta-\vartheta) + zd\eta\sin(\eta-\vartheta)}{yy}.$$

Addantur hae tres aequationes, et cum sit

$$x\sin(\eta-\zeta) + z\sin(\eta-\vartheta) = 0, \qquad x\sin(\vartheta-\zeta) + y\sin(\vartheta-\eta) = 0,$$
$$y\sin(\zeta-\eta) + z\sin(\zeta-\vartheta) = 0,$$

summa erit

$$-\frac{(L+M)\,dx}{Nxx} - \frac{(M+N)\,dy}{Lyy} - \frac{(L+N)\,dz}{Mzz} + \frac{d(x\cos(\eta-\zeta) + z\cos(\eta-\vartheta))}{yy} + \frac{d(x\cos(\vartheta-\zeta) + y\cos(\vartheta-\eta))}{zz}$$
$$+ \frac{d(y\cos(\zeta-\eta) + z\cos(\zeta-\vartheta))}{xx}.$$

Ex aequationibus $x\cos\zeta + y\cos\eta + z\cos\vartheta = 0$ et $x\sin\zeta + y\sin\eta + z\sin\vartheta = 0$ colligimus

$$x\cos(\vartheta-\zeta) + y\cos(\vartheta-\eta) + z = 0, \qquad x\cos(\eta-\zeta) + z\cos(\eta-\vartheta) + y = 0,$$
$$y\cos(\zeta-\eta) + z\cos(\zeta-\vartheta) + x = 0,$$

quibus valoribus inductis consequimur

$$\frac{d.(dx^2 + xx\,d\zeta^2)}{4gN\,dt^2} + \frac{d.(dy^2 + yy\,d\eta^2)}{4gL\,dt^2} + \frac{d.(dz^2 + zz\,d\vartheta^2)}{4gM\,dt^2} = \frac{-(L+M+N)\,dx}{Nxx} - \frac{(L+M+N)\,dy}{Lyy} - \frac{(L+M \ldots}{Mzz}$$

hincque integrando

$$\frac{dx^2 + xx\,d\zeta^2}{N} + \frac{dy^2 + yy\,d\eta^2}{L} + \frac{dz^2 + zz\,d\vartheta^2}{M} = 4g\,(L+M+N)\,dt^2\left(D + \frac{1}{Nx} + \frac{1}{Ly} + \frac{1}{Mz}\right),$$

$$LM\,(dx^2 + xx\,d\zeta^2) + MN\,(dy^2 + yy\,d\eta^2) + LN\,(dz^2 + zz\,d\vartheta^2) =$$

$$4g\,(L+M+N)\,dt^2\left(E + \frac{LM}{x} + \frac{MN}{y} + \frac{LN}{z}\right)$$

ita ut jam habeamus duas aequationes integrales. Praeterea autem notasse convenit esse

$$LM\,(x\,ddx + dx^2) + MN\,(y\,ddy + dy^2) + LN\,(z\,ddz + dz^2) =$$

$$2g\,(L+M+N)\,dt^2\left(2E + \frac{LM}{x} + \frac{MN}{y} + \frac{LN}{z}\right),$$

etiamsi hinc nullā via ad novam integrationem aperiatur. Cum igitur septem habeamus quanti… scilicet tres distantias x, y, z, tres angulos ζ, η, ϑ et tempus t, quarum relationem mu… definiri oportet, ad hoc opus est sex aequationibus, ad quarum numerum complendum habem… primo has duas aequationes finitas

$$\text{I.}\quad x\cos\zeta + y\cos\eta + z\cos\vartheta = 0,\qquad \text{II.}\quad x\sin\zeta + y\sin\eta + z\sin\vartheta = 0,$$

deinde binas aequationes jam per integrationem erutas

$$\text{III.}\quad LMxx\,d\zeta + MNyy\,d\eta + LNzz\,d\vartheta = C\,dt \qquad \text{et}$$

$$\text{IV.}\quad LM\,(dx^2 + xx\,d\zeta^2) + MN\,(dy^2 + yy\,d\eta^2) + LN\,(dz^2 + zz\,d\vartheta^2) =$$

$$4g\,(L+M+N)\,dt^2\left(E + \frac{LM}{x} + \frac{MN}{y} + \frac{LN}{z}\right).$$

Loco duarum reliquarum binae trium sequentium commodissime accipientur

$$\text{V.}\quad 2\,dx\,d\zeta + x\,dd\zeta = 2gN\,dt^2\left(\frac{\sin(\eta-\zeta)}{yy} + \frac{\sin(\vartheta-\zeta)}{zz}\right),$$

$$\text{VI.}\quad 2\,dy\,d\eta + y\,dd\eta = 2gL\,dt^2\left(\frac{\sin(\vartheta-\eta)}{zz} + \frac{\sin(\zeta-\eta)}{xx}\right),$$

$$\text{VII.}\quad 2\,dz\,d\vartheta + z\,dd\vartheta = 2gM\,dt^2\left(\frac{\sin(\zeta-\vartheta)}{xx} + \frac{\sin(\eta-\vartheta)}{yy}\right),$$

quarum resolutio hoc modo tentanda videtur. Multiplicentur hae tres postremae aequationes seorsi… per certas formulas differentiales, ita ut membra posteriora fiant integrabilia seorsim, priorum aute… summa talis efficiatur. Ob

$$x : y : z = \sin(\eta - \vartheta) : \sin(\vartheta - \zeta) : \sin(\zeta - \eta),$$

prior conditio impletur si multiplicetur aequatio V. per $\frac{yz\sin(\vartheta-\zeta)\sin(\zeta-\eta)}{\sin^3(\vartheta-\zeta)-\sin^3(\zeta-\eta)}dP$,

aequatio VI. per $\frac{xz\sin(\zeta-\eta)\sin(\eta-\vartheta)}{\sin^3(\zeta-\eta)-\sin^3(\eta-\vartheta)}dQ$,

aequatio VII. per $\frac{xy\sin(\eta-\vartheta)\sin(\vartheta-\zeta)}{\sin^3(\eta-\vartheta)-\sin^3(\vartheta-\zeta)}dR$;

enim integralia posteriorum membrorum fiunt

$$2gNP\,dt^2,\quad 2gLQ\,dt^2,\quad 2gMR\,dt^2;$$

ergo, ut priorum membrorum aggregatum reddatur integrabile, quem in finem idoneos valores functionum P, Q, R investigari convenit. Verum hic calculi subsidiis destituti istud negotium deserere cogimur.

181. **Coroll. 1.** Posito $\sin(\eta-\vartheta)=\frac{x}{\varrho}$, $\sin(\vartheta-\zeta)=\frac{y}{\varrho}$, $\sin(\zeta-\eta)=\frac{z}{\varrho}$, aequationes V, VI, VII. in has abeunt formas

$$\text{V.}\quad d.(xx\,d\zeta)=\frac{2gNx\,dt^2}{\varrho}\left(\frac{y}{zz}-\frac{z}{yy}\right),$$

$$\text{VI.}\quad d.(yy\,d\eta)=\frac{2gLy\,dt^2}{\varrho}\left(\frac{z}{xx}-\frac{x}{zz}\right),$$

$$\text{VII.}\quad d.(zz\,d\vartheta)=\frac{2gMz\,dt^2}{\varrho}\left(\frac{x}{yy}-\frac{y}{xx}\right),$$

ita per x, y, z determinatur, ut sit

$$\varrho=\frac{2xyz}{\sqrt{(2xxyy+2xxzz+2yyzz-x^4-y^4-z^4)}}.$$

182. **Coroll. 2.** Si ex illis aequationibus eliminemus dt^2, obtinebimus

$$\frac{2g\,dt^2}{\varrho}=\frac{yyzz\,d.(xx\,d\zeta)}{Nx(y^3-z^3)}=\frac{xxzz\,d.(yy\,d\eta)}{Ly(z^3-x^3)}=\frac{xxyy\,d.(zz\,d\vartheta)}{Mz(x^3-y^3)}.$$

etiam in plures alias formas has aequationes transfundere licet, neque tamen methodus patet hinc novam aequationem integralem eliciendi.

183. **Scholion 1.** Hoc igitur problema, cui vera determinatio omnium motuum coelestium innititur, vires analyseos superat, etiamsi corpora se mutuo attrahentia sphaerica et in eodem plano moveri assumsimus; quae ergo conditiones, si secus se haberent, atque imprimis si numerus corporum ternarium excederet, multo minus de solutione cogitare liceret; ex quo intelligitur in subsidium Astronomiae ingentem analyseos promotionem desiderari. Neque etiam in genere ulla via ad approximationes patet, quibus uti non licet, nisi vel unum trium corporum sit valde parvum, vel vis ad motum reliquorum perturbandum nata fuerit vehementer exigua. Si enim corpus N evanescat, nostrae aequationes tantum ad has binas redeunt

$$LMxx\,d\zeta=C\,dt\quad\text{et}\quad LM(dx^2+xx\,d\zeta^2)=4g(L+M)\,dt^2\left(E+\frac{LM}{x}\right),$$

quibus motus duorum corporum continetur, unde si massa N sit valde parva, hinc idoneae approximationes peti poterunt. Deinde si corpus N sit infinite remotum, ut distantiae y et z fiant infinitae, aequationum differentio-differentialium primo expositarum binae priores jam totum negotium conficient, abeuntes in has formas:

$$2dxd\zeta + xdd\zeta = 0 \quad \text{et} \quad ddx - xd\zeta^2 = \frac{-2g(L+M)dt^2}{xx},$$

ita ut reliquas ne in computum quidem duci necesse sit, qui casus ex posterioribus aequationibus minus perspicitur, cum ibi reliquae quantitates praeter necessitatem calculo sint immixtae. Quodsi in mundo ejusmodi casus existeret, ut trium corporum se mutuo attrahentium neque unius massa sit prae reliquis valde parva, neque unius distantia a reliquis vehementer magna, fateri cogimur, talem motum nobis fore imperscrutabilem: verum commode in mundo usu venit, ut hujusmodi casus nobis nusquam deprehendatur, qua in re nostrae imbecillitati non parum consultum videtur. Quocirca obrem contenti simus in methodum inquisivisse, cujus beneficio proxime saltem motum trium corporum determinare valeamus, quando inter terna corpora se invicem attrahentia unum reperitur, cujus vis in reliqua sive ob massae parvitatem, sive ob ejus enormem distantiam, quasi evanescat, quippe qui solus casus relinquitur, in quo vires nostras experiri liceat.

184. **Scholion 2.** Cum igitur tam mundus alios motus non offerat, quam analysis ad illos investigandos non sit apta, nisi qui non multum a ratione motus in sectione conica recedant, omnem operam in inventione aberrationum ab hac motus lege collocari conveniet. Hanc ob rem motum regularem vocabimus, qui leges motus, quibus duo tantum corpora sphaerica se mutuo attrahentia sunt inventa, perfecte sequitur, cujusmodi motus, etiamsi forte nusquam in mundo locum habeat, tamen, quoniam discrimen nusquam est valde magnum, aberrationes seu perturbationes motus regularis per approximationes definire conabimur. In proposito igitur problemate motum trium corporum L, M, N ita comparatum assumamus, ut bina M et N respectu tertii L motu fere regulari revolvantur, unde hoc commodi consequimur, ut dum perturbationes alterius definire studemus, alterius motum tanquam regularem spectare queamus; cum enim perturbationes ab hoc in illo productae per se sint valde parvae, sive hoc posterius regulariter moveatur, sive parumper a regularitate recedat, nullum discrimen in perturbatione illius orietur. Ita quando in perturbationes motus lunae a sole oriundas inquirere volumus, motum solis respectu terrae tanquam regularem spectabimus, et vicissim, si errores in motu terrae ab actione lunae nati definiri debeant, qui terra ad quietem redacta in motum solis transferuntur, motum lunae tanquam regularem spectare licebit. Cum igitur propositis tribus corporibus unum semper in quiete considerari possit, problema ita tractabimus, ut binorum reliquorum unum motu regulari ferri censeatur, pro alteroque tantum perturbationes investigentur. Quod si praestiterimus, non amplius difficile erit, problemati pro corporibus quotcunque propositis satisfacere, quia enim perturbationes satis sunt exiguae, quantae a singulis seorsim producantur, assignavisse sufficiat, quae deinceps conjunctae omnes perturbationes ab omnibus simul ortas exhibebunt.

185. **Problema.** (Fig. 186.) Si corpus N circa corpus L, quod in quiete spectamus, motu regulari feratur, tum vero in eodem plano corpus M circa L ita moveatur, ut ejus motus ab actione corporis N perturbetur, hujus motus perturbationes assignare.

Solutio. Cum hic ad motum respectivum attendamus, corpore L in quiete spectato, ductis rectis LM, LN et MN, attractio mutua corporum L et M est $=\frac{L.M}{LM^2}$, corporum L et $N=\frac{L.N}{LN^2}$, atque corporum M et $N=\frac{M.N}{MN^2}$. Cum nunc corpus L sollicitetur secundum LM vi $=\frac{L.M}{LM^2}$, et secundum LN vi $=\frac{L.N}{LN^2}$, hae vires in sensum oppositum et in ratione massarum mutatae binis reliquis corporibus applicari debent. Ductis ergo rectis MT et NV ipsis NL et ML parallelis, corpus M praeter vires secundum $ML=\frac{L.M}{LM^2}$ et secundum $MN=\frac{M.N}{MN^2}$ sollicitari censendum est viribus secundum $ML=\frac{M.M}{LM^2}$ et secundum $MT=\frac{M.N}{LN^2}$; at corpus N praeter vires secundum $NL=\frac{L.N}{LN^2}$ et secundum $NM=\frac{M.N}{MN^2}$, a viribus secundum $NL=\frac{N.N}{LN^2}$ et secundum $NV=\frac{M.N}{LM^2}$. Sumtis nunc duabus directionibus fixis altera LA, altera ad hanc normali, corpus M sollicitabitur

$$\text{sec. } LQ \text{ vi } = \frac{-M(L+M)\,LQ}{LM^3} + \frac{M.N.MS}{MN^3} - \frac{M.N.LR}{LN^3},$$

$$\text{sec. } QM \text{ vi } = \frac{-M(L+M)\,QM}{LM^3} + \frac{M.N.SN}{MN^3} - \frac{M.N.RN}{LN^3}.$$

Corpus vero N sollicitabitur

$$\text{sec. } LR \text{ vi } = \frac{-N(L+N)\,LR}{LN^3} - \frac{M.N.MS}{MN^3} - \frac{M.N.LQ}{LM^3},$$

$$\text{sec. } RN \text{ vi } = \frac{-N(L+N)\,RN}{LN^3} - \frac{M.N.SN}{MN^3} - \frac{M.N.QM}{LM^3}.$$

Ponamus jam coordinatas pro corpore M

$$LQ=x,\quad QM=y,\quad LM=\sqrt{(xx+yy)}=v,$$

pro corpore N vero $\quad LR=\mathfrak{x},\quad RN=\mathfrak{y},\quad LN=\sqrt{(\mathfrak{x}\mathfrak{x}+\mathfrak{y}\mathfrak{y})}=\mathfrak{v},$

erit $MN=\sqrt{((\mathfrak{x}-x)^2+(\mathfrak{y}-y)^2)}=w$, et aequationes differentio-differentiales motum utriusque corporis exprimentes, posito elemento temporis dt constante, erunt

$$\text{I.}\quad ddx = 2gdt^2\left(\frac{-(L+M)\,x}{v^3}+\frac{N(\mathfrak{x}-x)}{w^3}-\frac{N\mathfrak{x}}{\mathfrak{v}^3}\right),$$

$$\text{II.}\quad ddy = 2gdt^2\left(\frac{-(L+M)\,y}{v^3}+\frac{N(\mathfrak{y}-y)}{w^3}-\frac{N\mathfrak{y}}{\mathfrak{v}^3}\right),$$

$$\text{III.}\quad dd\mathfrak{x} = 2gdt^2\left(\frac{-(L+N)\,\mathfrak{x}}{\mathfrak{v}^3}-\frac{M(\mathfrak{x}-x)}{w^3}-\frac{Mx}{v^3}\right),$$

$$\text{IV.}\quad dd\mathfrak{y} = 2gdt^2\left(\frac{-(L+N)\,\mathfrak{y}}{\mathfrak{v}^3}-\frac{M(\mathfrak{y}-y)}{w^3}-\frac{My}{v^3}\right).$$

Cum autem motus corporis M non adeo perturbari sumatur, hypothesis nostra exigit, ut termini $\frac{N}{ww}$ et $\frac{N}{\mathfrak{v}\mathfrak{v}}$ sint prae $\frac{L+M}{vv}$ valde parvi, atque eodem jure termini $\frac{M}{ww}$ et $\frac{M}{vv}$ prae $\frac{L+N}{\mathfrak{v}\mathfrak{v}}$ valde exigui esse debent; quia alioquin determinatio motus vires calculi superaret.

Cum igitur motus corporis N pro cognito habeatur, quantitates $\mathfrak{x}$, $\mathfrak{y}$ et $\mathfrak{v}$ tanquam functiones cognitae temporis t spectari possunt, sicque tantum duae aequationes priores relinquuntur, ex quibus colligimus

$$y\,ddx - x\,ddy = 2g\,dt^2\left(\frac{N(\mathfrak{x}y - x\mathfrak{y})}{w^3} - \frac{N(\mathfrak{x}y - x\mathfrak{y})}{\mathfrak{v}^3}\right), \quad \text{seu} \quad = 2gN(\mathfrak{x}y - x\mathfrak{y})\,dt^2\left(\frac{1}{w^3} - \frac{1}{\mathfrak{v}^3}\right)$$

$$\text{et} \quad 2\,dx\,ddx + 2\,dy\,ddy = 4g\,dt^2\left(\frac{-(L+M)\,dv}{vv} - \frac{Nv\,dv}{w^3} + N(\mathfrak{x}\,dx + \mathfrak{y}\,dy)\left(\frac{1}{w^3} - \frac{1}{\mathfrak{v}^3}\right)\right).$$

Ponamus nunc pro motu corporis M angulum $ALM = \varphi$, distantia existente $LM = v$, $x = v\cos\varphi$ et $y = v\sin\varphi$, hincque $y\,dx - x\,dy = -vv\,d\varphi$ et $dx^2 + dy^2 = dv^2 + vv\,d\varphi^2$; pro motu corporis N statuatur distantia $LN = u$, quae hactenus erat $= \mathfrak{v}$, et angulus $ALN = \vartheta$, ut sit $\mathfrak{x} = u\cos\vartheta$ et $\mathfrak{y} = u\sin\vartheta$, hincque

$$w = \sqrt{((u\cos\vartheta - v\cos\varphi)^2 + (u\sin\vartheta - v\sin\varphi)^2)} = \sqrt{(uu - 2uv\cos(\varphi - \vartheta) + vv)}$$

et $\quad \mathfrak{x}y - x\mathfrak{y} = uv\sin(\varphi - \vartheta)$, atque $\mathfrak{x}\,dx + \mathfrak{y}\,dy = u\,dv\cos(\varphi - \vartheta) - uv\,d\varphi\sin(\varphi - \vartheta)$.

Unde nostrae aequationes erunt

$$d.(vv\,d\varphi) = -2gNuv\,dt^2\sin(\varphi - \vartheta)\left(\frac{1}{w^3} - \frac{1}{u^3}\right),$$

$$d.(dv^2 + vv\,d\varphi^2) = 4g\,dt^2\left(\frac{-(L+M)\,dv}{vv} - \frac{Nv\,dv}{w^3} + N\big(u\,dv\cos(\varphi - \vartheta) - uv\,d\varphi\sin(\varphi - \vartheta)\big)\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\right)$$

seu si differentialia secundi gradus non reformidemus,

$$2\,dv\,d\varphi + v\,dd\varphi = -2gNu\,dt^2\sin(\varphi - \vartheta)\left(\frac{1}{w^3} - \frac{1}{u^3}\right) \quad \text{et}$$

$$ddv - v\,d\varphi^2 = -2g(L+M)\frac{dt^2}{vv} - 2gN\,dt^2\left(\frac{v}{w^3} - u\cos(\varphi - \vartheta)\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\right),$$

ubi u et ϑ tanquam quantitates per t datae sunt spectandae, terminique per N affecti tanquam valde parvi.

Verum illae aequationes ad integrationem magis sunt praeparatae, et posterior ob

$$w\,dw = v\,dv + u\,du - u\,dv\cos(\varphi - \vartheta) - v\,du\cos(\varphi - \vartheta) + uv(d\varphi - d\vartheta)\sin(\varphi - \vartheta)$$

transit in hanc formam

$$d.(dv^2 + vv\,d\varphi^2) = -4g(L+M)\,dt^2\frac{dv}{vv} - 4gN\,dt^2\left(\frac{dv\cos(\varphi - \vartheta) - v\,d\varphi\sin(\varphi - \vartheta)}{uu}\right)$$
$$+ 4gN\,dt^2\left(\frac{u\,du - v\,du\cos(\varphi - \vartheta) - uv\,d\vartheta\sin(\varphi - \vartheta) - w\,dw}{w^3}\right),$$

unde integrando quatenus licet obtinemus

$$dv^2 + vv\,d\varphi^2 = 4g(L+M)\,dt^2\left(D + \frac{1}{v}\right) - 4gN\,dt^2\left(\frac{v\cos(\varphi - \vartheta)}{uu} - \int\frac{v\,d\vartheta\sin(\varphi - \vartheta)}{uu} + 2\int\frac{v\,du\cos(\varphi - \vartheta)}{u^3}\right)$$
$$+ 4gN\,dt^2\left(\frac{1}{w} + \int\frac{du\,(u + v\cos(\varphi - \vartheta))}{w^3} - \int\frac{uv\,d\vartheta\sin(\varphi - \vartheta)}{w^3}\right),$$

sive

$$dv^2 + vv\,d\varphi^2 = 4g(L+M)\,dt^2\left(D + \frac{1}{v}\right) + 4gN\,dt^2\left(\frac{1}{w} - \frac{v\cos(\varphi - \vartheta)}{uu}\right)$$
$$+ 4gN\,dt^2\int du\left(\frac{u - v\cos(\varphi - \vartheta)}{w^3} - \frac{2v\cos(\varphi - \vartheta)}{u^3}\right)$$
$$- 4gN\,dt^2\int uv\,d\vartheta\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\sin(\varphi - \vartheta).$$

Prior vero aequatio per $2vv\,d\varphi$ multiplicata et integrata dat

$$v^4 d\varphi^2 = 4g\,(L+M)\,Cdt^2 - 4gNdt^2 \int uv^3 d\varphi \left(\frac{1}{w^3} - \frac{1}{u^3}\right) \sin(\varphi - \vartheta).$$

Ponamus brevitatis gratia

$$\int uv^3 d\varphi \left(\frac{1}{w^3} - \frac{1}{u^3}\right) \sin(\varphi - \vartheta) = P, \qquad \int uv\,d\vartheta \left(\frac{1}{w^3} - \frac{1}{u^3}\right) \sin(\varphi - \vartheta) = Q,$$

$$\int du \left(\frac{u - v\cos(\varphi - \vartheta)}{w^3} - \frac{2v\cos(\varphi - \vartheta)}{u^3}\right) = R,$$

ut habeamus has aequationes

$$v^4 d\varphi^2 = 4gdt^2\,(C\,(L+M) - NP)$$

$$dv^2 + vvd\varphi^2 = 4gdt^2 \left(D\,(L+M) + \frac{L+M}{v} + \frac{N}{w} - \frac{Nv\cos(\varphi - \vartheta)}{uu} + NR - NQ\right),$$

unde eliminando $4gdt^2$ nanciscimur

$$dv^2 \left(C(L+M) - NP\right) = v^4 d\varphi^2 \left(D(L+M) + \frac{L+M}{v} + \frac{N}{w} - \frac{Nv\cos(\varphi - \vartheta)}{uu} - NQ + NR - \frac{C(L+M)}{vv} + \frac{NP}{vv}\right).$$

Statuamus porro $\frac{N}{L+M} = n$, fietque

$$\frac{dv\sqrt{(C - nP)}}{vv} = d\varphi\sqrt{\left(D + \frac{1}{v} + \frac{n}{w} - \frac{nv\cos(\varphi - \vartheta)}{uu} - nQ + nR - \frac{C}{vv} + \frac{nP}{vv}\right)}$$

$$\text{et} \qquad vvd\varphi = 2\,dt\sqrt{g}\,(L+M)\,(C - nP),$$

ubi termini littera n affecti ut minimi spectantur. Illa autem aequatio etiam hoc modo exhiberi potest

$$\frac{dv\sqrt{(C - nP)}}{vv} = d\varphi\sqrt{\left(D + \frac{1}{v} - \frac{C}{vv} - 2n\int\frac{Pdv}{v^3} - n\int\frac{vdv}{w^3} + n\int udv\cos(\varphi - \vartheta)\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\right)}.$$

Ponatur $v = \frac{p}{1 + q\cos s}$ et $C = \frac{f}{2}$ atque $D = \frac{kk - 1}{2f}$, fietque $\frac{1}{p} = \frac{1}{f} + \frac{npp\cos(\varphi - \vartheta)}{f(1 - qq)\,uu} + \frac{2nP}{fp}$ et

$$\frac{qq}{pp} = \frac{kk}{ff} - \frac{2np\cos(\varphi - \vartheta)}{f(1 - qq)\,uu} + \frac{2n}{fw} - \frac{2nQ}{f} + \frac{2nR}{f} + \frac{2nP\,(1 + qq)}{fpp}$$

$$\text{et} \qquad \frac{dv\sqrt{(f - 2nP)}}{vv} = \frac{qd\varphi\sin s}{p}\sqrt{\left(f - 2nP + \frac{2np^3\cos(\varphi - \vartheta)}{(1 - qq)\,(1 + q\cos s)\,uu}\right)},$$

$$\text{seu} \qquad \frac{dv}{vv} = \frac{qd\varphi\sin s}{p}\sqrt{\left(1 + \frac{2npp\cos(\varphi - \vartheta)}{(1 - qq)\,(1 + q\cos s)\,uu}\right)}.$$

Vel si nullam approximationem admittamus, erit

$$\frac{1}{p} = \frac{1}{f} + \frac{npp\cos(\varphi - \vartheta)}{f(1 - qq)\,uu} + \frac{2nP}{fp},$$

$$\frac{qq}{pp} = \frac{kk}{ff} + \frac{2n}{fw} - \frac{2nQ}{f} + \frac{2nR}{f} - \frac{3np\cos(\varphi - \vartheta)}{f(1 - qq)\,uu} + \frac{npp\cos(\varphi - \vartheta)}{ff(1 - qq)\,uu} + \frac{2nP}{ffp} + \frac{2nPqq}{fpp},$$

hincque

$$\frac{dv\sqrt{(f - 2nP)}}{vv} = \frac{qd\varphi\sin s}{p}\sqrt{\left(f - 2nP + \frac{2np^3\cos(\varphi - \vartheta)}{(1 - qq)\,(1 + q\cos s)\,uu}\right)}, \quad \text{seu}$$

$$\frac{dv}{vv} = \frac{qd\varphi \sin s}{p} \sqrt{\left(1 + \frac{2np^3 \cos(\varphi - \vartheta)}{(f - 2nP)(1 - qq)(1 + q\cos s)uu}\right)}.$$

Est autem
$$\frac{dv}{vv} = \frac{dp}{pp} - \frac{(pdq - qdp)}{pp}\cos s + \frac{qds \sin s}{p}.$$

Cum nunc sint p et q proxime constantes, erit

$$\frac{dp}{pp} = \frac{nf(d\varphi - d\vartheta)\sin(\varphi - \vartheta)}{(1 - kk)uu} + \frac{2nfdu\cos(\varphi - \vartheta)}{(1 - kk)u^3} - \frac{2nuv^3 d\varphi}{ff}\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\sin(\varphi - \vartheta),$$

$$\frac{2q}{p}\cdot\frac{(pdq - qdp)}{pp} = -\frac{2n}{fu^3}\left(vdv - udv\cos(\varphi - \vartheta) + uvd\varphi\sin(\varphi - \vartheta) - \frac{(1 + kk)uvd\varphi}{(1 + k\cos s)^2}\sin(\varphi - \vartheta)\right)$$

$$- \frac{2n(1 + kk)vd\varphi\sin(\varphi - \vartheta)}{f(1 + k\cos s)^2 uu} + \frac{2nvd\vartheta\sin(\varphi - \vartheta)}{fuu} - \frac{4nvdu\cos(\varphi - \vartheta)}{fu^3}$$

$$+ \frac{2n(d\varphi - d\vartheta)\sin(\varphi - \vartheta)}{(1 - kk)uu} + \frac{4ndu\cos(\varphi - \vartheta)}{(1 - kk)u^3},$$

quibus valoribus substitutis, ob $dv = \frac{kvvd\varphi \sin s}{f}$ proxime, fit

$$\frac{dv}{vv} = \frac{qds \sin s}{p} + \frac{nv^3 d\varphi \sin s \cos s}{fw^3} - \frac{nuv^3 d\varphi \sin s}{ffw^3}(2\sin s\sin(\varphi - \vartheta) + \cos s\cos(\varphi - \vartheta) + k\cos s\cos(\varphi - \vartheta - s))$$

$$+ \frac{nv^3 d\varphi \sin^2 s\sin(\varphi - \vartheta)}{ff(1 - kk)uu}(3 + 3k\cos s - 2kk + kk\cos^2 s - k^3\cos s) - \frac{nvd\vartheta\sin^2 s\sin(\varphi - \vartheta)}{(1 - kk)uu} + \frac{2nvdu\sin^2 s\cos(\varphi - \vartheta)}{(1 - kk)u^3}.$$

Ex quibus colligimus

$$\frac{q(d\varphi - ds)}{p} = \frac{nv^3 d\varphi\cos s}{fw^3} - \frac{nuv^3 d\varphi}{ffw^3}\left(2\sin s\sin(\varphi - \vartheta) + \cos s\cos(\varphi - \vartheta) + k\cos s\cos(\varphi - \vartheta - s)\right)$$

$$- \frac{nkvd\varphi\cos(\varphi - \vartheta)}{(1 - kk)uu} + \frac{nv^3 d\varphi\sin s\sin(\varphi - \vartheta)}{ff(1 - kk)uu}(3 + 3k\cos s - 2kk + kk\cos^2 s - k^3\cos s)$$

$$- \frac{nvd\vartheta\sin s\sin(\varphi - \vartheta)}{(1 - kk)uu} + \frac{2nvdu\sin s\cos(\varphi - \vartheta)}{(1 - kk)u^3},$$

quae formula ita repraesentari potest:

$$\frac{q(d\varphi - ds)}{p} = \frac{nv^3 d\varphi\cos s}{fw^3} - nd.\left(\frac{v\sin s\cos(\varphi - \vartheta)}{(1 - kk)uu}\right)$$

$$- \frac{nuv^3 d\varphi}{ff}\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\left(2\sin s\sin(\varphi - \vartheta) + \cos s\cos(\varphi - \vartheta) + k\cos s\cos(\varphi - \vartheta - s)\right)$$

ita ut pro motu lineae absidum sit

$$\varphi - s = \text{Const.} - \frac{nfv\sin s\cos(\varphi - \vartheta)}{k(1 - kk)uu} + \frac{n}{k}\int\frac{v^3 d\varphi\cos s}{w^3}$$

$$- \frac{n}{fk}\int uv^3 d\varphi\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\left(2\sin s\sin(\varphi - \vartheta) + \cos s\cos(\varphi - \vartheta) + k\cos s\cos(\varphi - \vartheta - s)\right)$$

in quibus terminis minimis est $v = \frac{f}{1 + k\cos s}$.

Denique quo haec ad tempus revocari queant, erit $vv\,d\varphi = dt\sqrt{2g(L+M)(f-2nP)}$, ita ut proxime $vv\,d\varphi = dt\sqrt{2fg(L+M)}$ et $ds = d\varphi$.

186. **Coroll. 1.** Si corpus N motu regulari circa L circumferatur, in orbita, cujus semiparameter $= b$, excentricitas $= e$ et anomalia vera $= r$, ut sit $u = \frac{b}{1+e\cos r}$, erit

$$d\vartheta = dr, \qquad uu\,d\vartheta = dt\sqrt{2bg(L+N)} \qquad \text{et} \qquad du = \frac{euu\,d\vartheta \sin r}{b}.$$

Unde posito $\sqrt{\frac{b(L+N)}{f(L+M)}} = m$, erit proxime

$$uu\,d\vartheta = mvv\,d\varphi, \qquad \text{seu} \qquad d\vartheta = \frac{mvv\,d\varphi}{uu} = dr \qquad \text{et} \qquad du = \frac{mevv\,d\varphi \sin r}{b},$$

ita ut in superioribus formulis fractione n affectis omnia elementa ad $d\varphi$ reducantur.

187. **Coroll. 2.** His differentialibus introductis etiam differentiale $d\varpi$ ad $d\varphi$ perducemus, obtinebimus enim ob $dv = \frac{kvv\,d\varphi \sin s}{b}$ proxime

$$\varpi d\varpi = \frac{kvv\,d\varphi \sin s}{f}(v - u\cos(\varphi-\vartheta)) + \frac{mevv\,d\varphi \sin r}{b}(u - v\cos(\varphi-\vartheta)) + \frac{v\,d\varphi \sin(\varphi-\vartheta)}{u}(uu - mvv).$$

188. **Coroll. 3.** Ex relatione cognita, quae inter differentialia $d\varphi$, $d\vartheta$, ds, dr, dv et du locum habet, colligi poterunt valores formularum integralium P, Q et R, unde semiparameter variabilis p cum excentricitate q accuratius definientur.

189. **Scholion 1.** Haec solutio per approximationes instituenda isti innititur fundamento, quod termini littera n affecti sint valde parvi; quod duplici modo evenire potest, vel si ipse numerus n fuerit minimus, dum inter quantitates v, u, ϖ non enormis inaequalitas versatur, vel si saltem termini $\frac{n}{\varpi}$ et $\frac{n}{u}$ prae $\frac{1}{v}$ sint quam minimi, quod fieri potest, etiamsi n sit numerus valde magnus. Ita si L sit terra, M luna, et N sol, fractio $\frac{N}{L+M} = n$ quidem est maxima. Verum distantia terrae a sole u tantopere superat distantiam lunae a terra v, ut termini $\frac{n}{\varpi}$ et $\frac{n}{u}$ nihilominus sint perquam exigui prae $\frac{1}{v}$. At si L sit terra, M sol et N luna, ut perturbationes motus solis apparentis a luna ortae investigentur, erit n fractio minima, et distantiae v et ϖ praemagnae respectu distantiae u; interim tamen quantitas $\frac{n}{u}$ prae $\frac{1}{v}$ tanquam evanescens est spectanda, hocque casu terminus $\frac{1}{\varpi}$ prae $\frac{1}{u}$ rejici poterit. Quodsi porro L sit sol, M vero et N duo quicunque planetae primarii, erit n fractio minima, et quia distantiae u, v, ϖ non adeo sunt inaequales, ut una prae reliquis contemni queat, termini $\frac{n}{\varpi}$ et $\frac{n}{u}$ utique prae $\frac{1}{v}$ rejici poterunt.

190. **Scholion 2.** Terminos autem $\frac{n}{\varpi}$ et $\frac{n}{u}$ tam parvos prae $\frac{1}{v}$ esse oportet, ut termini inde nati per nn affecti nullius futuri essent momenti, quemadmodum etiam in solutione hic exposita omnes terminos, qui altiores ipsius n potestates essent complexuri, rejecimus. Sin autem etiam termini per nn affecti attentionem mereantur, in solutione quidem omnia manerent, donec ad differen-

*

tialia quantitatum p et q eruenda descendimus, quae accuratius usque ad terminos per... evolvi deberent, hoc autem modo in ambages inextricabiles incideremus. Verum hic labor etiam necessarius videtur, quando termini per nn affecti per se spectati sunt minimi, quoniam inde integrationem interdum termini multo majores nasci possunt; ita si in formula differentiali occ... hujusmodi terminus $nnd\varphi\cos(\alpha+n\varphi)$, is quidem ob factorem nn elidendus videri posset, sed integrationem inde emergit terminus $n\sin(\alpha+n\varphi)$ ad eum ordinem pertinens, quem minime neglig... volebamus. Ex quo perspicuum est hunc modum approximandi, quatenus hujusmodi termini ordinibus negligendis occurrunt, maxime esse lubricum, propterea quod termini haud levis mo... excludantur. Atque hoc potissimum in motus lunae investigatione observandum est, ubi ob... causam ejusmodi termini ingrediuntur, quorum valores a terminis quadrato nn affectis vel... altioribus potestatibus pendent, qui cum nonnisi difficillime per theoriam eruantur, expedit eorum valores ex observationibus definire.

191. Scholion 3. Formulae nostrae pro p et q inventae ideo non parum intricatae pro... quod in membro $\frac{nv\cos(\varphi-\vartheta)}{uu}$ naturam quantitatis v spectavimus, ejusque loco valorem $\frac{p}{1+q\cos...}$ ... stituimus, quod cum in formulis Q et R non fecerimus, etiamsi et hi ab v pendeant, etiam ... jure illi substitutioni supersedere poterimus. Statuamus ergo brevitatis gratia

$$\frac{1}{w}-\frac{v\cos(\varphi-\vartheta)}{uu}-Q+R=S \quad \text{et posito} \quad C=\frac{f}{2} \quad \text{et} \quad D=\frac{kk-1}{2f},$$

sequentes aequationes resolvendae proponuntur

$$vvd\varphi=dt\sqrt{2g(L+M)(f-2nP)} \quad \text{et}$$

$$\frac{dv\sqrt{(f-2nP)}}{vv}=d\varphi\sqrt{\left(\frac{kk-1}{f}+\frac{2}{v}+2nS-\frac{f}{vv}+\frac{2nP}{vv}\right)}\ (*).$$

Statuamus nunc $v=\frac{p}{1+q\cos s}$, et haec formula signo radicali implicata fit

$$\frac{kk-1}{f}+\frac{2}{p}+2nS-\frac{f}{pp}+\frac{2nP}{pp}+\frac{2q\cos s}{p}-\frac{2fq\cos s}{pp}+\frac{4nPq\cos s}{pp}-\frac{fqq\cos^2 s}{pp}+\frac{2nPqq\cos^2 s}{pp}.$$

Evanescant primo termini per $\cos s$ affecti, eritque

$$1-\frac{f}{p}+\frac{2nP}{p}=0, \quad \text{seu} \quad p=f-2nP;$$

hoc modo illa formula abit in

(*) Si excentricitas k evanescat, alio modo calculum tractari oportet; erit enim $\frac{1}{v}=\mathfrak{A}+\mathfrak{B}\cos\eta+\mathfrak{C}\cos^2\eta+\text{etc.}$ et $\frac{dv}{vv}=d\varphi\sqrt{(A+B\cos\eta+C\cos^2\eta+D\cos^3\eta+\text{etc.})}$. Cum nunc dv factorem obtineat $\sin\eta$, necesse est, ut sit $A\pm B+C\pm D+E$ etc. $=0$, ... $A+C+E+$ etc. $=0$ et $B+D+$ etc. $=0$. Simili modo poni debet $P=\ldots+\cos\eta+\cos^2\eta\ldots$ et $S=\ldots+\cos\eta+\cos^2\eta\ldots$ Haec methodus aptior videtur illa, qua omnes termini ad sinus ... cosinus angulorum multiplorum ipsius $\eta=\varphi-\vartheta$ reducuntur.

$$\frac{kk-1}{f} + \frac{1}{p} + 2nS - \frac{qq}{p}\cos^2 s.$$

Fiet ergo $\frac{qq}{p} = \frac{kk-1}{f} + \frac{1}{p} + 2nS$, eritque

$$\frac{dv\sqrt{p}}{vv} = \frac{qd\varphi\sin s}{\sqrt{p}}, \quad \text{seu} \quad \frac{dv}{vv} = \frac{qd\varphi\sin s}{p}.$$

Cum ergo sit $p = f - 2nP$ et $qq = 1 + \frac{(kk-1)p}{f} + 2nSp$,

erit differentiando $dp = -2ndP$ et $2qdq = \frac{(kk-1)\,dp}{f} + 2nSdp + 2npdS$, hincque

$$\frac{dv}{vv} = \frac{dp}{pp} + \frac{qdp\cos s}{pp} - \frac{(kk-1)\,dp\cos s}{2fpq} - \frac{nSdp\cos s}{pq} - \frac{ndS\cos s}{q} + \frac{qds\sin s}{p},$$

unde colligitur $\frac{q\,(d\varphi-ds)\sin s}{p} = \frac{dp}{p}\left(\frac{1}{v} - \frac{(kk-1)\cos s}{2fq} - \frac{nS\cos s}{q}\right) - \frac{ndS\cos s}{q}$,

seu $\frac{q\,(d\varphi-ds)\sin s}{p} = \frac{dp\,(\cos s + 2q + qq\cos s)}{2ppq} - \frac{ndS\cos s}{q}$.

Est vero $dS = \frac{-vdv + udv\cos(\varphi-\vartheta) - uvd\varphi\sin(\varphi-\vartheta)}{w^3} - \frac{dv\cos(\varphi-\vartheta)}{uu} + \frac{vd\varphi\sin(\varphi-\vartheta)}{uu}$,

seu $dS = \frac{-qvvd\varphi\sin s}{pw^3}(v - u\cos(\varphi-\vartheta)) - \frac{uvd\varphi\sin(\varphi-\vartheta)}{w^3} - \frac{qvvd\varphi\sin s\cos(\varphi-\vartheta)}{puu} + \frac{vd\varphi\sin(\varphi-\vartheta)}{uu}$,

quo valore substituto orietur

$$\frac{q\,(d\varphi-ds)\sin s}{p} = \frac{nv^3d\varphi\sin s\cos s}{pw^3} - \frac{nuvvd\varphi\sin s\cos s\cos(\varphi-\vartheta)}{p}\left(\frac{1}{w^3}-\frac{1}{u^3}\right) - \frac{nuvd\varphi\sin^2 s\sin(\varphi-\vartheta)}{(1+q\cos s)^2}\left(\frac{1}{w^3}-\frac{1}{u^3}\right)(2+q\cos s),$$

quae divisa per $\frac{q\sin s}{p}$ praebebit

$$d\varphi - ds = \frac{nvv\,d\varphi}{q}\left(\frac{v\cos s}{w^3} - u\cos s\cos(\varphi-\vartheta)\left(\frac{1}{w^3}-\frac{1}{u^3}\right) - \frac{(2+q\cos s)\,u\sin s\sin(\varphi-\vartheta)}{1+q\cos s}\left(\frac{1}{w^3}-\frac{1}{u^3}\right)\right),$$

sive hoc modo

$$d\varphi - ds = \frac{nvv\,d\varphi}{q}\left(\frac{v\cos s}{w^3} - u\left(\frac{1}{w^3}-\frac{1}{u^3}\right)\left(\cos s\cos(\varphi-\vartheta) - \frac{(2+q\cos s)\sin s\sin(\varphi-\vartheta)}{1+q\cos s}\right)\right),$$

ubi nullae plane approximationes sunt adhibitae. Tum vero erit

$$P = \int uv^3\,d\varphi\sin(\varphi-\vartheta)\left(\frac{1}{w^3}-\frac{1}{u^3}\right) \quad \text{et}$$

$$S = -\int\frac{qv^3\,d\varphi\sin s}{pw^3} + \int\frac{quvv\,d\varphi\sin s\cos(\varphi-\vartheta)}{p}\left(\frac{1}{w^3}-\frac{1}{u^3}\right) - \int uv\,d\varphi\sin(\varphi-\vartheta)\left(\frac{1}{w^3}-\frac{1}{u^3}\right),$$

sive

$$S = -\int\frac{qv^3\,d\varphi\sin s}{pw^3} + \int\frac{uvv\,d\varphi}{p}\left(\frac{1}{w^3}-\frac{1}{u^3}\right)\left(q\sin s\cos(\varphi-\vartheta) - (1+q\cos s)\sin(\varphi-\vartheta)\right).$$

vel etiam

$$S = -\int\frac{qv^3\,d\varphi\sin s}{pw^3} - \int\frac{uvv\,d\varphi}{p}\left(\frac{1}{w^3}-\frac{1}{u^3}\right)\left(\sin(\varphi-\vartheta) + q\sin(\varphi-\vartheta-s)\right).$$

Quibus valoribus integrabilibus definitis habebitur

$$p = f - 2nP, \quad q = \sqrt{\left(\frac{kkp}{f} + 1 - \frac{p}{f} + 2nSp\right)} \quad \text{et}$$

$$dt\sqrt{2g(L+M)(f-2nP)} = vvd\varphi = dt\sqrt{2gp(L+M)},$$

existente $v = \frac{p}{1+q\cos s}$, $dv = \frac{qvvd\varphi \sin s}{p}$ et $w = \sqrt{(vv - 2vu\cos(\varphi - \vartheta) + uu)}$.

Atque haec solutio praecedenti longe praeferenda videtur, cum quod nullis adhuc approximationibus sit restricta, tum vero quod ejus forma simplicior reperiatur.

192. **Problema.** (Fig. 187.) Si corpus N circa corpus L, quod in quiete spectamus, motu regulari feratur, tum vero corpus M non in eodem plano circa L ita moveatur, ut ejus motus ab actione corporis N perturbetur, definire has perturbationes.

Solutio. Ex corpore M, in planum orbitae a corpore N descriptae demittatur perpendiculum MP, et ex P ad rectam fixam LA agatur normalis PQ, vocenturque coordinatae pro corpore M $LQ = x$, $QP = y$ et $PM = z$, sitque distantia $LM = v = \sqrt{(xx + yy + zz)}$. Tum vero pro corpore N sint coordinatae $LR = \mathfrak{x}$, $RN = \mathfrak{y}$ et distantia $LN = u$. Posito ergo angulo $ALN = \vartheta$, erit $\mathfrak{x} = u\cos\vartheta$ et $\mathfrak{y} = u\sin\vartheta$. Deinde ponatur distantia $MN = w$, ut sit $w = \sqrt{((\mathfrak{x}-x)^2 + (\mathfrak{y}-y)^2 + zz)}$. Jam secundum directiones ternarum coordinatarum vires corpus M sollicitantes resolvantur, et cum primo M ad L trahatur vi $= \frac{M(L+M)}{vv}$, hinc nascitur vis

$$\text{sec. } LQ = \frac{-M(L+M)x}{v^3}, \quad \text{sec. } QP = \frac{-M(L+M)y}{v^3}, \quad \text{sec. } PM = \frac{-M(L+M)z}{v^3}.$$

Deinde ad corpus N urgetur vi $= \frac{MN}{ww}$, unde nascitur vis

$$\text{sec. } LQ = \frac{MN(\mathfrak{x}-x)}{w^3}, \quad \text{sec. } QP = \frac{MN(\mathfrak{y}-y)}{w^3}, \quad \text{sec. } PM = \frac{-MNz}{w^3}.$$

Denique cum corpus L ad N sollicitetur vi $= \frac{LN}{uu}$, hac rite in M translata prodit vis

$$\text{sec. } LQ = \frac{-MN\mathfrak{x}}{u^3} \quad \text{et} \quad \text{sec. } QP = \frac{-MN\mathfrak{y}}{u^3}.$$

Ex his viribus formulae motum continentes ita se habebunt

$$ddx = -2gdt^2\left(\frac{(L+M)x}{v^3} - \frac{N(\mathfrak{x}-x)}{w^3} + \frac{N\mathfrak{x}}{u^3}\right),$$

$$ddy = -2gdt^2\left(\frac{(L+M)y}{v^3} - \frac{N(\mathfrak{y}-y)}{w^3} + \frac{N\mathfrak{y}}{u^3}\right),$$

$$ddz = -2gdt^2\left(\frac{(L+M)z}{v^3} + \frac{Nz}{w^3}\right).$$

Ponamus brevitatis gratia $\frac{N}{L+M} = n$, ut habeamus

$$ddx = -2g(L+M)\,dt^2\left(\frac{x}{v^3}+\frac{nx}{w^3}-n\mathfrak{x}\left(\frac{1}{w^3}-\frac{1}{u^3}\right)\right),$$

$$ddy = -2g(L+M)\,dt^2\left(\frac{y}{v^3}+\frac{ny}{w^3}-n\mathfrak{y}\left(\frac{1}{w^3}-\frac{1}{u^3}\right)\right),$$

$$ddz = -2g(L+M)\,dt^2\left(\frac{z}{v^3}+\frac{nz}{w^3}\right).$$

His cum solutione problematis § 169 comparatis, quod ibi erat L hic nobis est $L+M$, ac praeterea

$$X=\frac{nx}{w^3}-nu\cos\vartheta\left(\frac{1}{w^3}-\frac{1}{u^3}\right),\quad Y=\frac{ny}{w^3}-nu\sin\vartheta\left(\frac{1}{w^3}-\frac{1}{u^3}\right),\quad Z=\frac{nz}{w^3}.$$

Sit nunc, solutionem secundum praecepta ibi data prosequendo, recta $L☊$ linea nodorum et ☊ nodus ascendens, ponaturque angulus $AL☊=\psi$ et inclinatio praesens orbitae a corpore M descriptae ad planum orbitae $N=\omega$; tum vocetur angulus $☊LM=\sigma$, eritque

$x=v(\cos\sigma\cos\psi-\sin\sigma\sin\psi\cos\omega)$, $y=v(\cos\sigma\sin\psi+\sin\sigma\cos\psi\cos\omega)$ et $z=v\sin\sigma\sin\omega$,

erit $d\omega=\frac{d\psi\cos\sigma\sin\omega}{\sin\sigma}$, atque fiat $d\sigma+d\psi\cos\omega=d\varphi$, ut sit φ longitudo corporis M in sua orbita. Quibus positis erit

$$dv^2+vv\,d\varphi^2=2g(L+M)\,dt^2\left(\frac{kk-1}{f}+\frac{2}{v}-2\int(Xdx+Ydy+Zdz)\right)$$

et $$v^4\,d\varphi^2\cos^2\omega=4g(L+M)\,dt^2\int vv\,d\varphi\cos\omega\,(Xy-Yx)$$

atque $$d\psi=\frac{2g(L+M)\,dt^2\sin\sigma}{v\,d\varphi}(Y\cos\psi+X\sin\psi-Z\cot\omega).$$

Cum autem sit

$$xdy-ydx=vv\,d\varphi\cos\omega,\quad xdz-zdx=vv\,d\varphi\cos\psi\sin\omega,\quad ydz-zdy=vv\,d\varphi\sin\psi\sin\omega,$$

erit $$dx=\frac{xdz}{z}-\frac{vv\,d\varphi\cos\psi\sin\omega}{z},\qquad dy=\frac{ydz}{z}-\frac{vv\,d\varphi\sin\psi\sin\omega}{z}$$

et $$\frac{dz}{z}=\frac{dv}{v}+\frac{d\sigma\cos\sigma}{\sin\sigma}+\frac{d\psi\cos\sigma\cos\omega}{\sin\sigma}=\frac{dv}{v}+\frac{d\varphi\cos\sigma}{\sin\sigma}.$$

Pro reductione formularum datarum habemus primo

$$Xy-Yx=nu(x\sin\vartheta-y\cos\vartheta)\left(\frac{1}{w^3}-\frac{1}{u^3}\right),\quad\text{seu}$$

$$Xy-Yx=nuv(\cos\sigma\sin(\vartheta-\psi)-\sin\sigma\cos\omega\cos(\vartheta-\psi))\left(\frac{1}{w^3}-\frac{1}{u^3}\right).$$

Deinde est

$$Xdx+Ydy+Zdz=\frac{nv\,dv}{w^3}-nu\,dv\left(\frac{1}{w^3}-\frac{1}{u^3}\right)(\cos\sigma\cos(\psi-\vartheta)-\sin\sigma\cos\omega\sin(\psi-\vartheta))$$

$$+nuv\,d\varphi\left(\frac{1}{w^3}-\frac{1}{u^3}\right)(\sin\sigma\cos(\psi-\vartheta)+\cos\sigma\cos\omega\sin(\psi-\vartheta)),$$

$$v^4\, d\varphi^2 \sin^2\omega = -4ng\,(L+M)\, dt^2 \int uv^3\, d\varphi \sin\sigma \sin^2\omega \cos(\psi-\vartheta)\left(\frac{1}{w^3}-\frac{1}{u^3}\right),$$

$$v^4\, d\varphi^2 \cos^2\omega = -4ng\,(L+M)\, dt^2 \int uv^3\, d\varphi \cos\omega\,(\cos\sigma \sin(\psi-\vartheta)+\sin\sigma\cos\omega\cos(\psi-\vartheta))\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$$

unde colligendo fit

$$v^4\, d\varphi^2 = -4ng\,(L+M)\, dt^2 \int uv^3\, d\varphi\,(\sin\sigma\cos(\psi-\vartheta)+\cos\sigma\cos\omega\sin(\psi-\vartheta))\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$$

Ponamus jam brevitatis gratia

$$\int uv^3\, d\varphi\,(\sin\sigma\cos(\psi-\vartheta)+\cos\sigma\cos\omega\sin(\psi-\vartheta))\left(\frac{1}{w^3}-\frac{1}{u^3}\right) = P,$$

$$\int\frac{udv}{w^3} - \int udv\,(\cos\sigma\cos(\psi-\vartheta)-\sin\sigma\cos\omega\sin(\psi-\vartheta))\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$$
$$+\int uv\,d\varphi\,(\sin\sigma\cos(\psi-\vartheta)+\cos\sigma\cos\omega\sin(\psi-\vartheta))\left(\frac{1}{w^3}-\frac{1}{u^3}\right) = Q,$$

eritque
$$v^4\, d\varphi^2 = 2g\,(L+M)\, dt^2\,(f-2nP)$$

et
$$dv^2 + v^2\, d\varphi^2 = 2g\,(L+M)\, dt^2\left(\frac{kk-1}{f}+\frac{2}{v}-2nQ\right),$$

unde fit
$$dv^2(f-2nP) = v^4\, d\varphi^2\left(\frac{kk-1}{f}+\frac{2}{v}-2nQ-\frac{f}{vv}+\frac{2nP}{vv}\right)$$

et
$$\frac{dv}{vv}\sqrt{(f-2nP)} = d\varphi\sqrt{\left(\frac{kk-1}{f}+\frac{2}{v}-2nQ-\frac{f}{vv}+\frac{2nP}{vv}\right)}.$$

Quare si ut supra ponamus $v = \frac{p}{1+q\cos s}$, obtinebimus

$$p = f-2nP,\qquad qq = 1+\frac{(kk-1)\,p}{f}-2nQp \quad\text{et}\quad \frac{dv}{vv} = \frac{qd\varphi\sin s}{p},$$

ac porro
$$\frac{q\,(d\varphi-ds)\sin s}{p} = \frac{dp\,(\cos s+2q+qq\cos s)}{2ppq}+\frac{ndQ\cos s}{q}.$$

Postea vero reperimus

$$d\psi = \frac{2ng\,(L+M)\,udt^2\sin\sigma\sin(\psi-\vartheta)}{vd\varphi}\left(\frac{1}{w^3}-\frac{1}{u^3}\right);$$

et ob $2g\,(L+M)\,dt^2 = \frac{v^4\, d\varphi^2}{p}$ erit

$$d\psi = \frac{nuv^3\, d\varphi\sin\sigma\sin(\psi-\vartheta)}{p}\left(\frac{1}{w^3}-\frac{1}{u^3}\right)\ \text{ et }\ \frac{d\omega}{\sin\omega} = \frac{nuv^3\, d\varphi\cos\sigma\sin(\psi-\vartheta)}{p}\left(\frac{1}{w^3}-\frac{1}{u^3}\right).$$

Praeterea ex his valoribus nanciscimur

$$w = \sqrt{\left(vv+uu-2uv\,(\cos\sigma\cos(\psi-\vartheta)-\sin\sigma\cos\omega\sin(\psi-\vartheta))\right)}.$$

Ponamus nunc brevitatis gratia

$$\cos\sigma\cos(\psi-\vartheta)-\sin\sigma\cos\omega\sin(\psi-\vartheta) = \cos\lambda,$$
$$\sin\sigma\cos(\psi-\vartheta)+\cos\sigma\cos\omega\sin(\psi-\vartheta) = \sin\mu,$$

$$P = \int uv^3\,d\varphi \sin\mu\left(\frac{1}{w^3}-\frac{1}{u^3}\right), \qquad Q = \int\frac{v\,dv}{w^3} + \int(uu\,d\varphi\sin\mu - u\,dv\cos\lambda)\left(\frac{1}{w^3}-\frac{1}{u^3}\right).$$

Cum jam sit $dp = -2nuv^3\,d\varphi\sin\mu\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$, erit

$$\frac{q(d\varphi-ds)\sin s}{p} = -\frac{nuv^3\,d\varphi\sin\mu(\cos s+2q+qq\cos s)}{ppq}\left(\frac{1}{w^3}-\frac{1}{u^3}\right)+\frac{nv\,dv\cos s}{qw^3}$$
$$+\frac{nuv\,d\varphi\sin\mu\cos s}{q}\left(\frac{1}{w^3}-\frac{1}{u^3}\right) \qquad -\frac{nu\,dv\cos\lambda\cos s}{q}\left(\frac{1}{w^3}-\frac{1}{u^3}\right),$$

seu

$$\frac{q(d\varphi-ds)}{p} = -\frac{nuv^3\,d\varphi\sin s\sin\mu}{pp}(2+q\cos s)\left(\frac{1}{w^3}-\frac{1}{u^3}\right)+\frac{nv^3\,d\varphi\cos s}{pw^3}-\frac{nuvv\,d\varphi\cos s\cos\lambda}{p}\left(\frac{1}{w^3}-\frac{1}{u^3}\right),$$

atque $w = \sqrt{(vv+uu-2uv\cos\lambda)}$, unde patet λ denotare angulum MLN. Cum ergo sit $d\sigma = d\varphi - d\psi\cos\omega$, erit

$$d\sigma = d\varphi - \frac{nuv^3\,d\varphi\sin\sigma\cos\omega\sin(\psi-\vartheta)}{p}\left(\frac{1}{w^3}-\frac{1}{u^3}\right) \qquad \text{et}$$

$$ds = d\varphi - \frac{nv^3\,d\varphi\cos s}{qw^3}+\frac{nuv^3\,d\varphi\sin s\sin\mu}{pq}(2+q\cos s)\left(\frac{1}{w^3}-\frac{1}{u^3}\right)+\frac{nuvv\,d\varphi\cos s\cos\lambda}{q}\left(\frac{1}{w^3}-\frac{1}{u^3}\right);$$

Cum vero ob $dv = \frac{qvv\,d\varphi\sin s}{p}$ fit

$$dQ = \frac{qv^3\,d\varphi\sin s}{pw^3}+uv\,d\varphi\left(\sin\mu-\frac{qv\cos\lambda\sin s}{p}\right)\left(\frac{1}{w^3}-\frac{1}{u^3}\right),$$

unde per integrationem valor ipsius Q colligi debet. Denique pro ratione temporis habemus

$$dt\sqrt{2g(L+M)} = \frac{vv\,d\varphi}{\sqrt{p}}.$$

Quodsi jam motus corporis N sit regularis ponaturque $u = \frac{b}{1+e\cos r}$, erit

$$dt\sqrt{2g(L+N)} = \frac{uu\,d\vartheta}{\sqrt{b}} \quad \text{et} \quad du = \frac{euu\,d\vartheta\sin r}{b};$$

quare posito $\sqrt{\frac{L+M}{L+N}} = \frac{1}{m}$, fit $\frac{1}{m} = \frac{vv\,d\varphi\sqrt{b}}{uu\,d\vartheta\sqrt{p}}$, hinc $d\vartheta = \frac{mvv\,d\varphi\sqrt{b}}{uu\sqrt{p}}$ et

$$du = \frac{mevv\,d\varphi\sin r\sqrt{b}}{b\sqrt{p}} = \frac{mevv\,d\varphi\sin r}{\sqrt{bp}} \quad \text{et} \quad dr = d\vartheta.$$

193. **Coroll. 1.** Cum termini littera n affecti sint minimi, primo his terminis penitus neglectis habebimus $p = f$, $q = k$, $ds = d\varphi$, $v = \frac{f}{1+k\cos s}$, $d\sigma = d\varphi$ et $d\psi = 0$, $d\omega = 0$, quibus valoribus corporis N motus regularis inducitur.

194. **Coroll. 2.** Deinde hi ipsi valores in terminis littera n affectis adhibeantur, ex quibus per integrationem primo quantitates P et Q, tum vero anguli s, σ, ψ et ω investigentur, quibus inventis erit accuratius $p = f - 2nP$ et $q = \sqrt{\left(\frac{kkp}{f}+\frac{2nP}{f}-2nQp\right)}$, hincque $v = \frac{p}{1+q\cos s}$.

195. **Coroll. 3.** Porro hi valores correcti in formulas integrales introducantur, ac denuo quantitates P et Q quam anguli s, σ, ψ et ω quaerantur, qui valores cum vero sint propiores etiam quantitates p, q et v indeque et w accuratius cognoscentur, unde similis operatio ad majorem consensum cum veritate obtinendum suscipi poterit.

196. **Scholion 1.** Hinc intelligitur istum calculum ob formularum complicationem non solum esse operosissimum, sed etiam alia via singulas harum formularum partes integrandi non patet, nisi ut eae in simplices sinus vel cosinus evolvantur, et integrationes omnes ad hujusmodi terminos $\int d\varphi \cos \xi$ perducantur, ubi relatio inter $d\varphi$ et $d\xi$ proxime saltem detur. Quodsi enim fuerit $d\xi = d\varphi(\alpha + \beta \cos x + \text{etc.})$, ubi terminus α sequentes plurimum superet, ob

$$d\varphi = \frac{d\xi}{\alpha} - \frac{\beta\, d\varphi \cos x}{\alpha} - \text{etc.}, \quad \text{fit} \quad \int d\varphi \cos \xi = \frac{1}{\alpha} \sin \xi - \frac{\beta}{\alpha} \int d\varphi \cos x \cos \xi \text{ etc.},$$

$$\text{at} \quad \int d\varphi \cos x \cos \xi = \tfrac{1}{2} \int d\varphi \cos(\xi - x) + \tfrac{1}{2} \int d\varphi \cos(\xi + x),$$

ita ut hic similis ratio integrationis sit adhibenda. Verum si eveniat, ut ipse numerus α sit perquam exiguus, hoc modo parum proficimus, hocque casu si fuerit $x = \mathfrak{b}\xi + \mathfrak{B}$, integrari oporteret hujusmodi formulam

$$\frac{d\xi \cos \xi}{\alpha + \beta \cos(\mathfrak{b}\xi + \mathfrak{B}) + \gamma \cos(\mathfrak{c}\xi + \mathfrak{C}) \text{ etc.}},$$

in qua coëfficientes β et γ prae α non sint exigui, sed potius valde magni. Quare si hujusmodi casus occurrant, ista consueta integrandi methodus minime ad scopum est accommodata. Praeterea quantitas irrationalis $w = \sqrt{(vv + uu - 2vu \cos \lambda)}$ maximum affert obstaculum, nisi insignis inaequalitas inter distantias v et u adsit, ita ut fractio $\frac{1}{w^3}$ facile in seriem valde convergentem transmutari possit. Ob has tantas difficultates optandum esset, ut geometrae potius in alias methodos integrandi quae non ad evolutionem in simplices sinus cosinusve adstringerentur, inquirerent, quod negotium si minus successerit, cognitio motuum coelestium non tam ob defectum Mechanicae, quam ob insufficientem Analyseos promotionem arceri est censenda.

197. **Scholion 2.** Quando autem resolutio formulae irrationalis w in seriem convergentem minus commode succedit, quemadmodum imprimis usu venit, quando perturbatio motus cujusdam planetae ab actione alius planetae vel etiam cometae oriunda definiri debet, ob calculi defectum vix alia via relinquitur, nisi ut pro singulis temporis momentis perturbationes ex formulis differentialibus definiantur, ac deinceps in unam summam colligantur. Planeta scilicet vel cometa assumitur, nisi alter planeta adesset, sectionem conicam circa solem secundum regulas Keplerianas esse descriptu[illegible] tum vero quasi singulis temporis momentis vis perturbans accedere concipitur, ubi quanta mutatio tam in ipsa orbita, quam in motu inde efficiatur, determinari oportet; id quod, quia tempus[illegible]mum accipiatur, ipsae formulae differentiales ostendent. Quodsi deinceps has perturbationes momentaneas in unam summam colligamus, evidens est conclusionem eo fore certiorem, quo minores fuerint temporis particulae, quamquam etiam hinc errores accumulari sunt censendi.

Caput VII.

De perturbatione motus momentanea a vi quacunque sollicitante oriunda.

98. **Problema.** (Fig. 188.) Si corpus, dum circa aliud corpus motu regulari sectionem conicam esset descripturum, per exiguum temporis intervallum a corpore quodam tertio in orbitae suae plano sito sollicitetur, determinare motus perturbationem momentaneam.

Solutio. Mente primum removeamus corpus perturbans et consideremus motum corporis M, qualis spectaretur ex corpore L, dum haec duo corpora L et M sola existerent ac se mutuo attrahent in ratione reciproca duplicata distantiarum. Describet ergo corpus M sectionem conicam BM, cujus alter focus erit in L, sitque B punctum orbitae ab L minime distans, seu absis ima, cujus longitudo a directione fixa LA computata, sit angulus $ALB = \alpha$. Orbitae vero vocetur semiparameter $= p$ et excentricitas $= q$, erit absidis imae distantia $LB = \frac{p}{1+q}$; absidis vero summae distantia ab $L = \frac{p}{1-q}$, unde fit axis transversus $= \frac{2p}{1-qq}$, cujus semissis $\frac{p}{1-qq}$ ponatur $= r$. Versetur jam corpus, cujus motum investigamus, in M, sitque angulus $BLM = s$, qui ejus anomalia vera appellatur, et distantia $LM = v$, erit $v = \frac{p}{1+q\cos s}$; ipsa vero longitudo a directione fixa LA computata sit angulus $ALM = \varphi$, erit utique $\varphi = \alpha + s$ et $\varphi - s = \alpha$. Quodsi jam tempusculo dt corpus ab M in m progredi sumamus, et litterae L et M massas corporum denotent, erit

$$vv\,ds = dt\sqrt{2gp\,(L+M)}, \qquad \text{ideoque} \qquad dt\sqrt{2gp\,(L+M)} = \frac{pp\,ds}{(1+q\cos s)^2},$$

ita ut sit angulus elementaris tempusculo dt confectus

$$MLm = d\varphi = ds = \frac{dt}{vv}\sqrt{2gp\,(L+M)},$$

ubi quidem litterae L et M massas ita denotare sunt intelligendae, ut $\frac{L}{vv}$ exprimat vim absolutam, qua corpora in distantia $= v$ ad L attrahuntur, posita gravitate absoluta $= 1$ in superficie terrae, ubi grave uno minuto secundo per altitudinem $= g$ delabi assumitur, ut tempus t in minutis secundis exprimatur. At quantitates L et M etiam ex tempore periodico colligere licet. Cum enim quantitates p et q sint constantes, erit

$$\int \frac{ds}{(1+q\cos s)^2} = \frac{1}{(1-qq)^{\frac{3}{2}}}\,\text{Arc.}\cos\frac{q+\cos s}{1+q\cos s} - \frac{q\sin s}{(1-qq)(1+q\cos s)},$$

erit integrando:

$$t\sqrt{2gp\,(L+M)} = \frac{pp}{(1-qq)^{\frac{3}{2}}}\,\text{Arc.}\cos\frac{q+\cos s}{1+q\cos s} - \frac{ppq\sin s}{(1-qq)(1+q\cos s)},$$

seu ob $\frac{p}{1-qq} = r$ habebitur

$$t\sqrt{2g\,(L+M)} = r\sqrt{r}\cdot\text{Arc.}\cos\frac{q+\cos s}{1+q\cos s} - qr\sqrt{p}\cdot\frac{\sin s}{1+q\cos s},$$

ubi t denotat tempus, quo corpus M ab abside ima B anomaliam veram $BLM = s$ absolvit. Quod si totum tempus periodicum vocetur $= \Theta$ min. sec. posito $s = 360^0 = 2\pi$, obtinebitur

$$\Theta \sqrt{2g(L+M)} = 2\pi r \sqrt{r}, \quad \text{ita ut sit} \quad \sqrt{2g(L+M)} = \frac{2\pi r\sqrt{r}}{\Theta}.$$

His definitis ponamus dum corpus in M versatur, unde motu assignato ulterius esset progressurum, quasi subito in N existere corpus in plano orbitae cujus massa $= N$, vocceturque distantia $LN = u$, angulus $ALN = \vartheta$, sitque distantia $MN = \sqrt{(uu - 2uv\cos(\varphi - \vartheta) + vv)} = w$ brevitatis gratia. Ob actionem hujus corporis N, cujus effectum tantum pro tempusculo dt hic definire instituimus, corpus M tempusculo dt non in m sed in μ perveniet, ejusque motus ita perturbabitur, ut, si corpus N, elapso tempusculo dt subito iterum tolleretur, aliam deinceps orbitam esset descripturum, a priori infinite parum recedentem, puta $\beta\mu$, pro qua statuamus longitudinem absidis imae $AL\beta = \alpha + d\alpha$, semiparametrum $= p + dp$, excentricitatem $= q + dq$, et semiaxem transversum $= r + dr$. Nunc autem elapso tempusculo dt erit anomalia vera $= \beta L\mu$, quas mutationes momentaneas ex problemate § 185 ac praecipue ejus scholio § 191 colligamus. Ponamus ergo, ut ibi, brevitatis gratia $\frac{N}{L+M} = n$, tum vero

$$dP = uv^3 d\varphi \sin(\varphi - \vartheta)\left(\frac{1}{w^3} - \frac{1}{u^3}\right) \quad \text{et}$$

$$dS = -\frac{qv^3 d\varphi \sin s}{pw^3} + \frac{q}{p} uvv\, d\varphi \sin s \cos(\varphi - \vartheta)\left(\frac{1}{w^3} - \frac{1}{u^3}\right) - uv\, d\varphi \sin(\varphi - \vartheta)\left(\frac{1}{w^3} - \frac{1}{u^3}\right)$$

atque § 191 invenimus fore, posito $v = \frac{p}{1 + q\cos s}$,

$$\text{I.} \quad vv\, d\varphi = dt\sqrt{2g(L+M)}\,(f - 2nP), \qquad \text{II.} \quad dv = \frac{qvv\, d\varphi \sin s}{p},$$

$$\text{III.} \quad p = f - 2nP, \qquad \text{IV.} \quad \frac{qq}{p} = \frac{kk - 1}{f} + \frac{1}{p} + 2nS,$$

$$\text{V.} \quad d\varphi - ds = \frac{nvv\, d\varphi}{q}\left(\frac{v\cos s}{w^3} - u\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\left(\cos s\cos(\varphi - \vartheta) + \frac{(2 + q\cos s)\sin s \sin(\varphi - \vartheta)}{1 + q\cos s}\right)\right)$$

ubi f denotat semiparametrum et k excentricitatem pro initio temporis t. Quoniam igitur hic istud initium in principio tempusculi dt constituimus, erit nobis $f = p$ et $k = q$, litterae autem p et q denotant earundem valores jam variatos $p + dp$ et $q + dq$, at $d\varphi$ angulum $ML\mu$. Ex quo colligitur

$$dp = -2n\,dP, \quad \text{et} \quad d\cdot\frac{1 - qq}{p} = -2n\,dS = d\cdot\frac{1}{r}, \quad \text{atque}$$

$$vv\, d\varphi = dt\sqrt{2g(L+M)}\,(p + dp), \quad \text{seu} \quad = dt\left(\sqrt{p} + \frac{dp}{2\sqrt{p}}\right)\sqrt{2g(L+M)}.$$

Variationes ergo tempusculo dt productae ita se habebunt:

1. semiparameter p augmentum capit dp, ut sit

$$dp = -2nuv^3 d\varphi \sin(\varphi - \vartheta)\left(\frac{1}{w^3} - \frac{1}{u^3}\right);$$

2. semiaxis transversus r, ob $\frac{dr}{rr} = 2ndS$, augmentum capit dr, ut sit

$$dr = \frac{-2nqrr\upsilon^3 d\varphi \sin s}{pw^3} + 2nrru\upsilon d\varphi \left(\frac{1}{w^3} - \frac{1}{u^3}\right)\left(\frac{q\upsilon}{p} \sin s \cos(\varphi - \vartheta) - \sin(\varphi - \vartheta)\right);$$

3. pro variatione excentricitatis q habemus

$$\frac{-2q\,dq}{p} - \frac{(1-qq)\,dp}{pp} = -2ndS, \quad \text{seu} \quad \frac{2q\,dq}{p} = 2ndS + \frac{2n(1-qq)\,dp}{pp},$$

unde fit

$$dq = \frac{-n\upsilon^3 d\varphi \sin s}{w^3} + \frac{nu\upsilon^3 d\varphi}{p}\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\Big((1+q\cos s)\sin s\cos(\varphi-\vartheta) - (2\cos s + q + q\cos^2 s)\sin(\varphi-\vartheta)\Big);$$

4. angulus autem elementaris $d\varphi$ tempusculo dt descriptus omissa particula infinite parva, ita definitur

$$d\varphi = \frac{dt}{\upsilon\upsilon}\sqrt{2gp(L+M)},$$

ubi si tempusculo dt valor notabilis tribuatur, quantitatibus p et υ valor medius inter eos, quos initio et fine obtinent, assignari poterit;

5. denique cum sit $\varphi - s = \alpha$, variatio momentanea ipsius α erit

$$d\alpha = \frac{n\upsilon\upsilon\, d\varphi}{q}\left(\frac{\upsilon\cos s}{w^3} - u\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\left(\cos s\cos(\varphi-\vartheta) + \frac{(2+q\cos s)\sin s\sin(\varphi-\vartheta)}{1+q\cos s}\right)\right),$$

vel etiam

$$d\alpha = \frac{n\upsilon^3 d\varphi}{q}\left(\frac{\cos s}{w^3} - \frac{u}{p}\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\Big((1+q\cos s)\cos s\cos(\varphi-\vartheta) + (2+q\cos s)\sin s\sin(\varphi-\vartheta)\Big)\right).$$

Posset hinc etiam variatio in distantia υ facta definiri, sed cum semper sit $\upsilon = \frac{p}{1+q\cos s}$, praestat quovis tempore ipsam distantiam υ definiri. Omnes ergo perturbationes momentaneae tempusculo dt productae ita determinabuntur:

1. Angulus elementaris interea confectus $d\varphi$ fit

$$d\varphi = \frac{dt}{\upsilon\upsilon}\sqrt{2gp(L+M)}.$$

2. Semiparameter orbitae p accipiet augmentum dp, ut sit

$$dp = -2nu\upsilon^3 d\varphi\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\sin(\varphi - \vartheta).$$

3. Semiaxis transversus orbitae $r = \frac{p}{1-qq}$ accipiet augmentum dr, ut sit

$$dr = \frac{2nrr\upsilon\upsilon\, d\varphi}{p}\left(\frac{-q\upsilon\sin s}{w^3} + u\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\Big(q\sin s\cos(\varphi-\vartheta) - (1+q\cos s)\sin(\varphi-\vartheta)\Big)\right),$$

sive $$dr = \frac{-2nrr\upsilon\upsilon\, d\varphi}{p}\left(\frac{q\upsilon\sin s}{w^3} + u\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\Big(\sin(\varphi-\vartheta) + q\sin(\varphi-\vartheta-s)\Big)\right).$$

4. Excentricitas q incrementum dq capiet, ut sit

$$dq = nv^3 d\varphi \left(\frac{-\sin s}{w^3} + \frac{u}{p}\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\left((1+q\cos s)\sin s\cos(\varphi-\vartheta) - (2\cos s + q + q\cos^2 s)\sin(\varphi-\vartheta)\right)\right)$$

5. Longitudo absidis α capiet augmentum $d\alpha$, ut sit

$$d\alpha = \frac{nv^3 d\varphi}{q}\left(\frac{\cos s}{w^3} - \frac{u}{p}\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\left((1+q\cos s)\cos s\cos(\varphi-\vartheta) + (2+q\cos s)\sin s\sin(\varphi-\vartheta)\right)\right)$$

Ex binis postremis formulis colligur fore

$$dq\cos s + q d\alpha\sin s = -2nuvv d\varphi\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\sin(\varphi-\vartheta) = \frac{dp}{v} \quad \text{et}$$

$$dq\sin s - q d\alpha\cos s = nv^3 d\varphi\left(-\frac{1}{w^3} + \frac{u}{p}\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\left((1+q\cos s)\cos(\varphi-\vartheta) - q\sin s\sin(\varphi-\vartheta)\right)\right)$$

quarum illa ex differentiatione aequalitatis $\frac{1}{v} = \frac{1+q\cos s}{p}$ sequitur, ob $\frac{dv}{vv} = \frac{q d\varphi\sin s}{p}$ et $d\varphi - ds = \ldots$ fit enim $dq\cos s + q d\alpha\sin s = \frac{dp}{v}$.

199. **Coroll. 1.** Ob actionem ergo corporis N singulis momentis elementa sectionis con... immutantur, ac si id subito annihilaretur, corpus M secundum ea elementa, quae ultimo momen... locum habuerint, moveri perget motu regulari.

200. **Coroll. 2.** Parameter nullam patitur mutationem, si fuerit vel $\sin(\varphi-\vartheta) = 0$, vel $w = u$. Illo casu corpus N cum corporibus L et M in directum est situm, ideoque ex L cum M vel in oppositione vel conjunctione conspicitur; hic vero casus locum habet, ubi fuerit $\cos(\varphi-\vartheta) = \ldots$

201. **Coroll. 3.** Si fuerit $\varphi - \vartheta = 0$ et $u > v$, erit $w = u - v$, et perturbationes momentaneae praeter $dp = 0$ inveniuntur:

$$dr = 2nrrvv d\varphi\cdot\frac{q\sin s}{p}\left(\frac{1}{ww} - \frac{1}{uu}\right), \quad dq = nvv d\varphi\sin s\left(\frac{1}{ww} - \frac{1}{uu}\right), \quad d\alpha = \frac{-nvv d\varphi\cos s}{q}\left(\frac{1}{ww} - \frac{1}{uu}\right)$$

202. **Coroll. 4.** Eodem porro casu, quo $\varphi - \vartheta = 0$, si sit $u < v$, ac propterea $w = v - u$, erunt perturbationes momentaneae:

$$dr = -2nrrvv d\varphi\cdot\frac{q\sin s}{p}\left(\frac{1}{ww} + \frac{1}{uu}\right), \quad dq = -nvv d\varphi\sin s\left(\frac{1}{ww} + \frac{1}{uu}\right), \quad d\alpha = \frac{nvv d\varphi\cos s}{q}\left(\frac{1}{ww} + \frac{1}{uu}\right)$$

203. **Coroll. 5.** Sin autem sit $\varphi - \vartheta = 180^0$, erit $\cos(\varphi-\vartheta) = -1$ et $w = v + u$, unde praeter $dp = 0$ reliquae perturbationes erunt

$$dr = 2nrrvv d\varphi\cdot\frac{q\sin s}{p}\left(-\frac{1}{ww} + \frac{1}{uu}\right), \quad dq = nvv d\varphi\sin s\left(-\frac{1}{ww} + \frac{1}{uu}\right), \quad d\alpha = \frac{-nvv d\varphi\cos s}{q}\left(-\frac{1}{ww} + \frac{1}{uu}\right)$$

204. **Coroll. 6.** Casu vero, quo fit $w = u$, ubi etiam $dp = 0$, reliquae perturbationes momentaneae sunt:

$$dr = \frac{-2nqrrv^3 d\varphi\sin s}{pu^3}, \quad dq = \frac{-nv^3 d\varphi\sin s}{u^3}, \quad d\alpha = \frac{nv^3 d\varphi\cos s}{qu^3}.$$

205. **Scholion 1.** Quando ergo motus corporis perturbantis N constat, ut ad singula temporis momenta ejus locus assignari possit, tum ope nostrarum formularum perturbationes singulis momentis productae assignari poterunt. Haec autem temporis momenta, etsi in calculo infinite parva assumta, tamen plerumque satis notabilia temporis intervalla, veluti horae, dies, quin etiam hebdomades eorum loco assumi licet, siquidem his intervallis exiguae mutationes oriuntur, vel potius quamdiu mutationes tempori fuerint proxime proportionales. Quatenus enim eae a ratione temporis recedunt, eatenus tempus in minores partes secari oportet. Ita hae formulae commode adhiberi poterunt, si quaestio fuerit, quantum motus cujuspiam planetae principalis ab actione alius planetae vel cometae perturbetur, siquidem utriusque motus in idem fere planum incidat. Ex eodem fonte Celeb. Clairaut perturbationem motus cometae jam apparituri, qui retro annis 1682 et 1607 fuerat observatus, feliciter determinavit, quod negotium etsi summopore laboriosum, eo felicius successit, quod perturbatio tantum, quoad in vicinia planetarum Jovis ac Saturni versabatur cometa, fuerat effecta.

206. **Scholion 2.** Expressiones inventae in alias formas transfundi possunt introducendo angulos trianguli LMN. Si enim ponamus hos angulos $MLN = \varphi - \vartheta = z$, $LMN = y$ et $LNM = x$, ut sit $x + y + z = 180^0$, erit $u = \frac{v \sin y}{\sin x}$ et $\varpi = \frac{v \sin z}{\sin x}$, quibus valoribus introductis ob

$$d\varphi = \frac{dt}{vv} \sqrt{2gp}\,(L + M) \quad \text{et} \quad v = \frac{p}{1 + q \cos s},$$

reperiuntur variationes tempusculo dt productae:

1. pro variatione semiparametri p,

$$dp = \frac{-2nv\,d\varphi \sin^2 x}{\sin^2 y \sin^2 z} (\sin^3 y - \sin^3 z);$$

2. pro variatione semiaxis transversi r,

$$dr = \frac{-2nrr\,d\varphi \sin^2 x}{p \sin^2 y \sin^2 z} \big((1 + q \cos s)(\sin^3 y - \sin^3 z) + q \sin s (\sin^2 y \cos y + \sin^2 z \cos z)\big);$$

vel etiam hoc modo

$$dr = \frac{-2nrr\,d\varphi \sin^2 x}{p \sin^2 y \sin^2 z} \big(\sin^3 y - \sin^3 z + q \sin^2 y \sin (y + s) - q \sin^2 z \sin (z - s)\big);$$

3. pro variatione excentricitatis q,

$$dq = \frac{-n\,d\varphi \sin^2 x}{\sin^2 y \sin^2 z} \Big(\sin s (\sin^2 y \cos y + \sin^2 z \cos z) + \frac{(2 \cos s + q + q \cos^2 s)}{1 + q \cos s} (\sin^3 y - \sin^3 z)\Big);$$

4. pro variatione longitudinis absidum α,

$$d\alpha = \frac{n\,d\varphi \sin^2 x}{q \sin^2 y \sin^2 z} \Big(\cos s (\sin^2 y \cos y + \sin^2 z \cos z) - \frac{\sin s (2 + q \cos s)(\sin^3 y - \sin^3 z)}{1 + q \cos s}\Big).$$

Cum hae formulae non parum sint complicatae, quovis casu oblato non tam facile dici potest, utrum hae variationes fuerint positivae, an negativae? antequam veros earum valores evolverimus. Interim quia ex istis formulis variationes casu $\varphi - \vartheta = z = 0$ colligere haud licet, priores formae in praxi anteferendae videntur.

207. **Scholion 3.** Effectus corporis N in motu corporis M perturbando est ceteris pa[ribus] maximus, si vel distantia $MN = \varpi$, vel $LN = u$ fuerit minima, hoc est si corpus N vel ad M, vel ad L proxime accedat; priori autem casu effectus major erit quam posteriori, quoniam ϖ tantum denominatore nostrarum formularum inest, u vero etiam numeratores afficit. Quodsi igitur L sol, M planeta quidam primarius et N cometa in plano orbitae planetae decurrens, motus planetae maxime turbabitur, quando cometa ad eum proxime accedit; verum etiam dum cometa prope solem praeterit, perturbatio erit eo major, quo vicinior fiat soli et quo major fuerit massa. Ita cometae non solum in perigaeo motum terrae perturbant, sed etiam in perihelio. Ceterum si fieri posset, ut alterutra distantiarum ϖ et u prorsus in nihilum abiret, formulae nostrae omni [usu] destituerentur, quandoquidem perturbationes fuerint infinitae. Casus hic locum esset habiturus si corpus N subito alteri corporum L vel M ita jungeretur, ut in unum coalesceret, qui etsi per formulas nostras inexplicabilis videtur, tamen in se est facillimus, propterea quod dum duo tantum aderunt corpora, motus erit regularis, in sectione conica procedens, quanquam haec sectio conica diversa erit ab illa, quae ante accessionem massae N fuerit descripta. Atque hic casus, etsi non nisi per miraculum locum habere potest, dum massa alterius corporum L vel M augeretur, expendi meretur.

208. **Problema.** Si dum corpora L et M se mutuo attrahentia motu regulari feruntur, alterius vel utriusque massa subito augeatur vel minuatur, definire motum subsecuturum.

Solutio. Hactenus ergo corpus M ex L visum descripserit sectionem conicam BM, cujus semiparameter sit $= p$, excentricitas $= q$ et longitudo absidis $ALB = \alpha$; nunc autem sit corporis M longitudo $ALM = \varphi$ et distantia $LM = v$, erit anomalia vera $BLM = \varphi - \alpha = s$ et $v = \frac{p}{1 + q \cos s}$, tum vero expositis horum corporum massis per litteras L et M, tempusculo dt describeretur angulus elementaris $MLm = d\varphi = ds = \frac{dt}{vv} \sqrt{2gp\,(L + M)}$. Jam hoc momento perpendatur corporis M situs ac motus; situs quidem cum distantia $LM = v$, tum angulo $ALM = \varphi$ definitur, ac motus primo directione seu angulo BML, tum vero celeritate ipsa per Mm determinatur. Sit ergo angulus $BML = \eta$ et celeritas in $M = \vartheta$, ita ut jam hae quatuor quantitates v, φ, η et ϑ tanquam datae sint spectandae, ex quibus praecedentia motus elementa definiri debent, ac primo quidem dum corporum massae sunt L et M, deinde vero dum massae sunt mutatae, puta L' et M'. Primo igitur habemus

$$\tang \eta = \frac{v\,d\varphi}{dv}, \quad \text{sed ob} \quad v = \frac{p}{1 + q \cos s} \quad \text{est} \quad dv = \frac{pq\,ds \sin s}{(1 + q \cos s)^2},$$

quia ergo est $ds = d\varphi$, erit

$$\tang \eta = \frac{v\,(1 + q \cos s)^2}{pq \sin s} = \frac{1 + q \cos s}{q \sin s}.$$

Deinde hinc est $Mm = \frac{v\,d\varphi}{\sin \eta} = \frac{v\,d\varphi}{1 + q \cos s} \sqrt{(1 + 2q \cos s + qq)}$, ideoque celeritas

$$\vartheta = \frac{Mm}{dt} = \frac{Mm}{vv\,d\varphi} \sqrt{2gp\,(L + M)} = \frac{\sqrt{2gp\,(L + M)\,(1 + 2q \cos s + qq)}}{v\,(1 + q \cos s)}, \quad \text{seu} \quad = \frac{\sqrt{2gp\,(L + M)}}{v \sin \eta}$$

unde colligimus $p = \frac{ssvv \sin^2 \eta}{2g(L+M)}$; hincque $1 + q\cos s = \frac{p}{v} = \frac{ssv \sin^2\eta}{2g(L+M)} = q \sin s \operatorname{tang} \eta$. Quocirca erit

$$q \cos s = \frac{ssv \sin^2 \eta}{2g(L+M)} - 1 \quad \text{et} \quad q \sin s = \frac{ssv \sin\eta \cos\eta}{2g(L+M)},$$

unde pro anomalia vera colligitur $\operatorname{tang} s = \frac{ssv \sin\eta\cos\eta}{ssv \sin^2\eta - 2g(L+M)}$, hincque ipsa excentricitas

$$q = \frac{\sqrt{(s^4 vv \sin^2\eta - 4g(L+M)\, ssv \sin^2\eta + 4gg(L+M)^2)}}{2g(L+M)}.$$

Quodsi nunc massae corporum L et M subito in L' et M' fuerint mutatae, his illarum loco positis hae formulae ostendent elementa orbitae deinceps descriptae, quae elementa sint: 1) semiparameter $= p'$, 2) excentricitas $= q'$ et 3) longitudo absidis imae $= \alpha'$, ita ut posita 4° anomalia vera $= s'$, sit $\alpha = \varphi - s'$. Nunc ergo iterum ex statu praecedente elidantur litterae s et η, scilicet

$$s = \frac{\sqrt{2gp(L+M)(1+2q\cos s+qq)}}{p}, \quad \sin\eta = \frac{1+q\cos s}{\sqrt{(1+2q\cos s+qq)}}, \quad \cos\eta = \frac{q \sin s}{\sqrt{(1+2q\cos s+qq)}},$$

eritque pro elementis variatis

$$p' = \frac{p(L+M)}{L'+M'}, \quad 1 + q'\cos s' = \frac{p(L+M)}{v(L'+M')}, \quad q' \sin s' = \frac{(L+M)\, q \sin s}{L'+M'}$$

et $d\varphi = ds' = \frac{dt}{vv}\sqrt{2gp'(L'+M')}$. Nova ergo elementa ita pendent a praecedentibus, ut sit

$$\frac{L+M}{L'+M'} = \frac{p'}{p} = \frac{1+q'\cos s'}{1+q\cos s} = \frac{q'\sin s'}{q \sin s},$$

ideoque quantitates p, $1 + q\cos s$ et $q \sin s$ in ratione reciproca massarum immutentur.

209. **Coroll. 1.** Si ergo massae L et M in L' et M' mutentur, dum corpus M in abside ima versatur, ob $s = 0$, erit etiam $s' = 0$, sicque linea absidum nullam patitur mutationem, tum vero erit

$$\frac{1+q'}{1+q} = \frac{L+M}{L'+M'}, \quad \text{ideoque} \quad q' = \frac{L+M}{L'+M'} q + \frac{L+M}{L'+M'} - 1, \quad \text{seu} \quad q' = \frac{p'}{p} q + \frac{p'-p}{p},$$

unde excentricitas vel crescit vel decrescit, semper autem parameter $2p$ in ratione reciproca massarum mutatur.

210. **Coroll. 2.** Si mutatio massarum eveniat, dum corpus M per absidem summam transit, quo p abeat in p', ob $s = 180°$ et $s' = 180°$, linea absidum non mutatur, sed excentricitas ita mutatur ut sit

$$\frac{p'}{p} = \frac{1-q'}{1-q}, \quad \text{ideoque} \quad q' = \frac{p'}{p} q + \frac{p-p'}{p}.$$

211. **Coroll. 3.** Si eadem mutatio oriatur dum $s = 90°$, erit

$$\frac{p'}{p} = 1 + q'\cos s' = \frac{q' \sin s'}{q},$$

unde si $\frac{p'}{p} = \lambda$, habebitur

$q'\sin s' = \lambda q$ et $q'\cos s' = \lambda - 1$, ideoque $q' = \sqrt{(\lambda\lambda qq + (\lambda - 1)^2)}$ et tang s' ...

Si mutatio eveniat dum $s = 270^0$, erit

$$q'\sin s' = -\lambda q \quad \text{et} \quad q'\cos s' = \lambda - 1, \quad \text{ideoque} \quad q' = \sqrt{(\lambda\lambda qq + (\lambda - 1)^2)} \quad \text{et} \quad \text{tang}\, s' = ...$$

212. **Coroll. 4.** Posito ergo $p' = \lambda p$, casu $s = 0$, erit

$$q' = \lambda q + \lambda - 1 \quad \text{et semiaxis transversus} \quad r' = \frac{p}{2(q+1) - \lambda(q+1)^2} = \frac{r(1-q)}{2 - \lambda(1+q)}.$$

Casu $s = 180^0$, ob $q' = \lambda q - \lambda + 1$ fit

$$r' = \frac{p}{2(1-q) - \lambda(1-q)^2} = \frac{r(1+q)}{2 - \lambda(1-q)}.$$

Casu $s = 90^0$, ob $q' = \sqrt{(\lambda\lambda qq + (\lambda - 1)^2)}$ fit

$$r' = \frac{p}{2 - \lambda(1+qq)} = \frac{r(1-qq)}{2 - \lambda(1+qq)}.$$

Casu $s = 270^0$ eadem mutatio in axe transverso oritur.

213. **Coroll. 5.** Si tempus periodicum prius ante mutationem sit Θ, et post mutationem $= \Theta'$, ob $\Theta = \frac{2\pi r\sqrt{r}}{\sqrt{2g(L+M)}}$ et $\Theta' = \frac{2\pi r'\sqrt{r'}}{\sqrt{2g(L'+M')}}$, erit

$$\frac{\Theta'}{\Theta} = \frac{r'\sqrt{r'}}{r\sqrt{r}}\sqrt{\frac{L+M}{L'+M'}} = \frac{r'\sqrt{\lambda r'}}{r\sqrt{r}},$$

unde ex variatione axis transversi variatio in tempore periodico orta definiri potest.

214. **Scholion 1.** Si secundum opinionem, quam Newtonus erat amplexus, massa solis ob lucis emissionem continuo imminueretur, hinc mutatio in motu planetarum facta definiri posset. Foret enim $L + M$ quantitas variabilis, qua posita $= S$, erit

$$d\varphi = \frac{dt}{vv}\sqrt{2gpS} \quad \text{et} \quad \frac{S}{S+dS} = \frac{p+dp}{p} = \frac{1 + q\cos s + d.q\cos s}{1 + q\cos s} = \frac{q\sin s + d.q\sin s}{q\sin s} = 1 - \frac{dS}{S}.$$

In hac autem variatione anomalia vera s eatenus tantum mutari est censenda, quatenus linea absidum mutatur; unde posita longitudine absidis imae $\varphi - s = \alpha$, erit $ds = -d\alpha$. Ne autem haec consideratio moram facessat, praestabit hunc casum ex primis principiis evolvisse. Habemus ergo

$$\text{I.} \quad 2dv\,d\varphi + v\,dd\varphi = 0 \quad \text{et} \quad \text{II.} \quad ddv - v\,d\varphi^2 = \frac{-2gS\,dt^2}{vv},$$

quarum illa dat $vv\,d\varphi = C\,dt$, seu $d\varphi = \frac{C\,dt}{vv}$, unde haec fiet

$$ddv = \frac{CC\,dt^2}{v^3} - \frac{2gS\,dt^2}{vv},$$

ubi S spectari debet tanquam functio temporis t. Quae aequatio quantumvis resolutu difficilis tamen solutio ex formulis superioribus petita ipsi satisfacere deprehenditur. Posito enim

$$v = \frac{p}{1+q\cos s}, \quad \text{fit primo} \quad p = \frac{bC}{S}, \quad \text{tum vero}$$

$$dq\cos s + qd\alpha\sin s = -\frac{dS}{S}(1+q\cos s) \quad \text{et} \quad dq\sin s - qd\alpha\cos s = -\frac{dS}{S}q\sin s,$$

unde colligitur

$$d\alpha = \frac{-dS\sin s}{Sq} \quad \text{et} \quad dq = \frac{-dS}{S}(\cos s + q), \quad \text{hincque porro}$$

$$d\varphi = \frac{qdt\sin s}{p}\sqrt{2gbC}.$$

Denique ob $d\varphi = \frac{dt}{vv}\sqrt{2gbC}$ fiet

$$ds = \frac{dt\sqrt{2gbC}}{vv} + \frac{dS\sin s}{Sq},$$

inde saltem variationes momentaneae innotescunt.

215. **Scholion 2.** Solutio hujus problematis suppeditat quoque enodationem quaestionis, qua motus planetae, si forte a quapiam causa ictum acceperit, quem deinceps erit prosecuturus, determinatur. Quemcunque enim motum ante ictum habuerit, si per ictum planetae M imprimatur celeritas $= s$ secundum directionem Mm, ut sit angulus $LMB = \eta$ et distantia $LM = v = \frac{p}{1+q\cos s}$, post ictum erit semiparameter $p = \frac{ssvv\sin^2\eta}{2g(L+M)}$, excentricitas vero q et anomalia vera s per has aequationes definientur

$$q\cos s = \frac{ssv\sin^2\eta}{2g(L+M)} - 1 \quad \text{et} \quad q\sin s = \frac{ssv\sin\eta\cos\eta}{2g(L+M)},$$

tum vero erit post ictum $d\varphi = ds = \frac{dt}{vv}\sqrt{2gp(L+M)}$, unde sectio conica cum ratione motus innotescit. Verum revertamur ad perturbationem motus planetarum investigandam, quae ab attractione tertii cujusdam corporis efficitur, quando hoc corpus extra planum orbitae est situm. Quanquam autem istud corpus quovis momento tanquam quiescens spectamus, ejus tamen loca successiva in plano quodam per L transeunte sita assumamus, quod planum tanquam fixum consideremus, cujus respectu planum orbitae planetae ob actionem continuo mutetur.

216. **Problema.** (Fig. 189.) Si corpus M, quod ad L attractum motu regulari esset progressurum, a tertio quodam corpore N extra planum motus sito attrahatur, determinare perturbationem motus momentaneam.

Solutio. Referat tabula planum, in quo corpus N perpetuo versetur, in eodem simul perpetuo existente corpore L, cujus respectu motum corporis M definiri oportet. Sit LA recta quaedam fixa, ac nunc quidem elapso tempore $= t$ versetur corpus perturbans in N, posito angulo $ALN = \vartheta$ et distantia $LN = u$; corpus vero, cujus motus quaeritur, sit extra planum ALN in M, unde si corpus N abesset, motu regulari in orbita quadam BM esset progressurum, cujus elementa sequenti modo denotentur. Primo sit $L☊$ intersectio ejus orbitae cum plano ALN, et longitudo nodi ascendentis

$AL☊ = \psi$, atque inclinatio orbitae ad planum $ALN = \omega$. Deinde ipsius orbitae BM sit semiparameter $= p$, excentricitas $= q$ et semiaxis transversus $r = \frac{p}{1-qq}$. Nunc autem sit anomalia vera BLM eritque distantia $LM = v = \frac{p}{1+q\cos s}$. Ponatur porro angulus $☊LM = \sigma$, qui vocatur argumentum latitudinis, erit pro abside ima B angulus $☊LB = \sigma - s$, ac posita longitudine corporis in orbita propria $= \varphi$, erit, uti supra § 192 vidimus, $d\varphi = d\sigma + d\psi\cos\omega$. Hinc denique quaerantur duo anguli λ et μ, ut sit

$\cos\sigma\cos(\vartheta-\psi) + \sin\sigma\cos\omega\sin(\vartheta-\psi) = \cos\lambda$ et $\sin\sigma\cos(\vartheta-\psi) - \cos\sigma\cos\omega\sin(\vartheta-\psi) = \sin\mu$,

erit $\lambda =$ angulo MLN, unde fiet distantia $MN = \sqrt{(vv + uu - 2uv\cos\lambda)}$, quae vocetur $= w$. Hinc in finem quaeratur angulus ν, ut sit $\operatorname{tang}\nu = \frac{v\sin\lambda}{u - v\cos\lambda}$, eritque $w = \frac{v\sin\lambda}{\sin\nu}$. Quodsi nunc ponamus brevitatis gratia

$$\frac{N}{L+M} = n \qquad \text{et} \qquad uv^3 d\varphi \sin\mu\left(\frac{1}{w^3} - \frac{1}{u^3}\right) = dP,$$

$$\frac{qv^3 d\varphi\sin s}{pw^3} + uv\,d\varphi\left(\sin\mu - \frac{qv\cos\lambda\sin s}{p}\right)\left(\frac{1}{w^3} - \frac{1}{u^3}\right) = dQ,$$

erit primo $d\varphi = \frac{dt}{vv}\sqrt{2gp(L+M)}$, ac perturbationes ab actione corporis N tempusculo dt productae ex § 192 sequenti modo se habere reperiuntur:

Primo pro variatione semiparametri p est $dp = -2nuv^3 d\varphi \sin\mu\left(\frac{1}{w^3} - \frac{1}{u^3}\right)$.

Deinde pro excentricitatis q variatione ob $\frac{qq-1}{p} = \frac{kk-1}{f} - 2nQ$, erit differentiando

$$\frac{2q\,dq}{p} + \frac{(1-qq)\,dp}{pp} = \frac{-2nqv^3 d\varphi\sin s}{pw^3} - 2nuv\,d\varphi\left(\sin\mu - \frac{qv\cos\lambda\sin s}{p}\right)\left(\frac{1}{w^3} - \frac{1}{u^3}\right),$$

unde fit $$dq = \frac{-nv^3 d\varphi\sin s}{w^3} + npuv\,d\varphi\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\left(\frac{\cos\lambda\sin s}{1+q\cos s} - \frac{(2\cos s + q + q\cos^2 s)\sin\mu}{(1+q\cos s)^2}\right),$$

quae reducitur ad hanc formam

$$dq = nv^3 d\varphi\left(\frac{-\sin s}{w^3} + \frac{u}{p}\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\left[(1+q\cos s)\cos\lambda\sin s - (2\cos s + q + q\cos^2 s)\sin\mu\right]\right).$$

Hinc cum sit $\frac{qq-1}{p} = -\frac{1}{r}$, erit $\frac{dr}{rr} = -2n\,dQ$; erit pro variatione semiaxis transversi r

$$dr = \frac{-2nqrrv^3 d\varphi\sin s}{pw^3} - 2nrruv\,d\varphi\left(\sin\mu - \frac{q\cos\lambda\sin s}{1+q\cos s}\right)\left(\frac{1}{w^3} - \frac{1}{u^3}\right),$$

seu $$dr = \frac{2nrrvv\,d\varphi}{p}\left(\frac{-qv\sin s}{w^3} + u\left(\frac{1}{w^3} - \frac{1}{u^3}\right)(q\cos\lambda\sin s - (1+q\cos s)\sin\mu)\right).$$

Praeterea consecuti sumus

$$ds = d\varphi - \frac{nv^3\,d\varphi\cos s}{qw^3} + \frac{nuvv\,d\varphi}{q}\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\left(\cos\lambda\cos s + \frac{(2+q\cos s)\sin\mu\sin s}{1+q\cos s}\right),$$

cum $\varphi - s$ denotet longitudinem absidis imae B in orbita, si ea dicatur $= \alpha$, erit $d\alpha = d\varphi - ds$,

$$d\alpha = \frac{nv^3\,d\varphi}{q}\left(\frac{\cos s}{w^3} - \frac{u}{p}\left(\frac{1}{w^3} - \frac{1}{u^3}\right)((1+q\cos s)\cos\lambda\cos s + (2+q\cos s)\sin\mu\sin s)\right).$$

Denique pro variatione orbitae respectu plani ALN invenimus primo pro longitudine nodi ☊

$$d\psi = \frac{-nuv^3\,d\varphi\sin\sigma\sin(\vartheta-\psi)}{p}\left(\frac{1}{w^3} - \frac{1}{u^3}\right),$$

Inde pro variatione inclinationis ω

$$d\omega = \frac{d\psi\sin\omega}{\operatorname{tang}\sigma} = \frac{-nuv^3\,d\varphi\cos\sigma\sin\omega\sin(\vartheta-\psi)}{p}\left(\frac{1}{w^3} - \frac{1}{u^3}\right),$$

Tum vero pro variatione anguli ☊$LM = \sigma$ habemus $d\sigma = d\varphi - d\psi\cos\omega$, ac proinde

$$d\sigma = d\varphi + \frac{nuv^3\,d\varphi\sin\sigma\cos\omega\sin(\vartheta-\psi)}{p}\left(\frac{1}{w^3} - \frac{1}{u^3}\right),$$

cum $\varphi - \sigma$ designet longitudinem nodi ☊ in orbita, si ea dicatur $= \beta$, erit

$$d\beta = \frac{-nuv^3\,d\varphi\sin\sigma\cos\omega\sin(\vartheta-\psi)}{p}\left(\frac{1}{w^3} - \frac{1}{u^3}\right).$$

Tandem vero ob $v = \frac{p}{1+q\cos s}$, erit $dv = \frac{qvv\,d\varphi\sin s}{p}$. Quare cum ex dato tempusculo dt habeatur

$$d\varphi = \frac{dt}{vv}\sqrt{2gp}\,(L+M),$$

Unde omnes perturbationes momentaneae pro tempusculo dt obtinentur. Quod quo facilius ad calculum revocetur, fingamus corpus M circa L in distantia $= c$ circulum describere, in eoque tempusculo dt angulum $d\zeta$ absolvere, eritque

$$d\zeta = \frac{dt}{c\sqrt{c}}\sqrt{2g}\,(L+M).$$

Unde cum detur angulus $d\zeta$ ex motu medio erit

$$dt\sqrt{2g}\,(L+M) = cd\zeta\sqrt{c}, \quad \text{ideoque} \quad d\varphi = \frac{cd\zeta\sqrt{cp}}{vv}.$$

217. **Coroll. 1.** Anguli λ et μ ita per trigonometriam sphaericam exhiberi possunt. In superficie sphaerica (Fig. 190) centro L descripta sint A, M, N, ☊ puncta, per quae rectae LA, LM, LN, L☊ transeant, erit $AN = \vartheta$, A☊ $= \psi$, ☊$N = \vartheta - \psi$, ☊$M = \sigma$ et M☊$N = \omega$, fietque $MN = z$, ac continuato arcu M☊ retro in O, ut OM sit quadrans, si ex O per N itidem ducatur quadrans ONR, erit $NR = \mu$.

218. **Coroll. 2.** Ducto arcu MR, quia ad utrumque quadrantem est normalis, resol… triangulum sphaericum ΩMN, in quo dantur latera $\Omega M = \sigma$, $\Omega N = \vartheta - \psi$ et angulus $M\Omega N$… inventoque latere MN cum angulo ΩMN, erit $\lambda = MN$ et $\sin\mu = \sin\lambda \cos \Omega MN$.

219. **Coroll. 3.** Loco tempusculi dt spatium non solum aliquot horarum sed etiam saep… dierum capi potest, nisi positio corporis N ratione ipsius M citissime varietur. Tum ex motu med… pro hoc temporis spatio colligatur angulus $d\zeta$, indeque erit $vv\,d\varphi = cd\zeta\sqrt{cp}$, quem valorem in singulis perturbationibus momentaneis substitui oportet.

220. **Scholion.** Ex his principiis perturbationes motus cujusque planetae principalis defini… poterunt, quatenus ab actione alius planetae vel etiam cometae oriuntur; ad planetas autem secun… darios, seu satellites, haec methodus minus commode accommodari potest, quandoquidem assumsimu… remoto corpore perturbante, motum futurum esse regularem; hinc itaque perturbationes in motu lunae, quae forte ab actione cujusdam planetae vel cometae proficiscuntur, determinare nequeunt. Sin autem ipse sol ut corpus perturbans consideretur, sine cujus actione luna motum regular… esset habitura, inaequalitates motus lunae hinc concludere licebit, sed quia actio solis est perenni… collectio perturbationum momentanearum conclusionem nimis lubricam reddit. Maximum autem usum haec methodus praestabit, si actio cujuspiam cometae in motum planetae principalis, per cujus viciniam cometa transit, investigari debeat: quoniam enim actio cometae non diutius manet sensibilis quam dum ejus distantia a planeta fuerit valde parva, omnino superfluum foret, totam actionem quam cometa per totum suum tempus periodicum exerit, exquirere velle, quem in finem integratio nostrarum formularum exhiberi opus esset. Sufficiet igitur per breve tempus effectum cometae in orbita cujuspiam planetae perturbanda cognovisse, id quod ope formularum differentialium haud difficulter praestabitur. Casus autem, quibus cometae ad planetas tam prope accedunt, ut pertur… bationem notabilem efficere queant, vehementer raro accidunt. Ac si cometa anni 1682 secundum praedictionem Cel. Clairaut hoc anno 1759 revertatur, phaenomena imprimis singularia in motu terrae ab ejus actione expectari possent, propterea quod in satis exigua a terra distantia praeter… batur. Operae ergo pretium erit, ope formularum traditarum in perturbationem motus terrae ejus… que orbitae, ab actione hujus cometae oriundam, inquirere, ut deinceps, quando elementa motu… istius cometae accuratius erunt definita, ad hoc exemplum plenior investigatio suscipi possit…

Digressio,

qua effectus Cometae A. 1759 expectati in motu terrae perturbando … investigatur.

I. Primo quidem assumo hunc cometam secundum eadem elementa latum iri, quae pro ej… apparitione A. 1682 sunt determinata. Etsi enim ob actionem Jovis et Saturni ejus tempus peri… dicum quasi biennio fuit retardatum, ob eandemque rationem ejus reliqua motus elementa haud lev… mutationes subiisse probabile, tamen quia de eorum valore praesente nihil certi constat, antequam