

## XII.

### Solutio duorum problematum, Astronomiam mechanicam spectantium.

**1. Problema.** (Fig. 192). Si corpus sphaeroidicum ex materia homogenea conflatum, attrahatur ad centrum virium  $O$ , cuius vis sit reciproce proportionalis quadratis distantiarum, invenire medium directionem, secundum quam hoc corpus urgebitur.

**Solutio.** Repraesentet circulus  $AGBH$  sectionem hujus corporis per ejus centrum  $C$  ad axem normaliter factam, seu sit iste circulus planum aequatoris hujus corporis sphaeroidici propositi, in plane tabulae exhibitum, et recta  $EF$ , quae huic plano normaliter insistere concipienda est, referat axem corporis, cuius idcirco poli sint in  $E$  et  $F$ . Ponatur radius aequatoris  $CA = CB = a$ , semissis axis  $CE = CF = b$ . Sit centrum virium ubicunque situm in  $O$ , unde ad planum aequatoris demittatur perpendicularum  $OD$ ; per  $D$  et centrum  $C$  agatur recta  $DACB$ , huicque diameter perpendicularis  $GH$ . Vocetur distantia  $CD = f$  et  $OD = g$ , ita ut sit  $\sqrt{(ff - gg)}$  distantia centri virium  $O$  a centro corporis  $C$ . Jam consideretur corporis quaecunque particula  $M$ , unde ad planum aequatoris demittatur perpendicularis  $MQ$ , et per  $Q$  diametro  $AB$  normalis ducatur corda  $NPN'$ . Vocentur nunc coordinatae  $CP = x$ ,  $PQ = y$  et  $QM = z$ ; per  $P$  quoque axi  $EF$  parallela agatur recta  $RPR'$ , et per  $M$  ipsi  $NN'$  parallela  $MRM'$ , atque per  $R$  trajiciatur  $TR$  ipsi  $DC$  parallela, erit

$$DT = PR = QM = z, \quad MR = PQ = y, \quad TR = DP = f - x \quad \text{et} \quad TO = g - z.$$

Hac fiet  $TM = \sqrt{(f - x)^2 + yy}$ , et distantia puncti  $M$  a centro virium  $O$ , nempe recta

$$MO = \sqrt{yy + (f - x)^2 + (g - z)^2},$$

propter brevitatis gratia ponatur  $= v$ . Urgebitur ergo punctum  $M$  in directione  $MO$  vi acceleratrice, quadrato  $v^2$  reciproce proportionali; sit ergo haec vis  $= \frac{kk}{vv}$ , qua punctum  $M$  in directione  $MO$  sollicitatur. Resolvatur haec vis secundum directiones  $Mn$  ipsi  $DO$  parallelam, et  $MT$ , eritque vis

in directione  $Mm = \frac{kk(g-z)}{\rho^3}$ , et vis in directione  $MT = \frac{kk\sqrt{(yy+(f-x)^2)}}{\rho^3}$ , quae ulterius resoluuntur secundum directiones  $M\mu$  ipsi  $RT$  vel  $CD$  parallelam, et  $MR$ , eritque vis in directione  $M\mu = \frac{kk(f-x)}{\rho^3}$ , et vis in directione  $MR = \frac{kky}{\rho^3}$ . Sicque quodlibet punctum  $M$  tribus urgetur viribus secundum directiones ternis coordinatis  $x, y, z$  parallelas, nimirum:

$$\text{secundum directionem } Mm, \text{ vi} = \frac{kk(g-z)}{\rho^3},$$

$$\text{secundum directionem } M\mu, \text{ vi} = \frac{kk(f-x)}{\rho^3},$$

$$\text{secundum directionem } MR, \text{ vi} = \frac{kky}{\rho^3}.$$

Sumta jam  $RM' = RM$  consideretur punctum  $M'$ , quod iisdem coordinatis definietur, quibus punctum  $M$  definitur nisi quod sit  $y$  negativa; erit enim demissio ex  $M'$  in planum aequatoris perpendiculo  $M'P = g$ ,  $PQ' = x$ ,  $PQ' = -y$  et  $Q'M' = z$ ; unde punctum  $M'$ , quia ejus distantia ab  $O$  quaque est  $\rho$ , urgebitur his viribus:

$$\text{secundum directionem } M'm', \text{ vi} = \frac{kk(g-z)}{\rho^3},$$

$$\text{secundum directionem } M'\mu', \text{ vi} = \frac{kk(f-x)}{\rho^3},$$

et secundum directionem  $M'R' = \frac{kky}{\rho^3}$ . Quodsi ergo haec duo puncta junctim considerentur, vires in directionibus  $MR$  et  $M'R'$  se mutuo destruent, et reliquae revocabuntur ad binas sequentes in punto  $R$  applicatas

$$\text{secundum directionem } Rr, \text{ vis} = \frac{2kk(g-z)}{\rho^3},$$

$$\text{secundum directionem } RT, \text{ vis} = \frac{2kk(f-x)}{\rho^3}.$$

Sumantur jam in inferiori hemisphaerio bina puncta  $M''$  et  $M'''$  his respondentia, ita ut  $QM'' = Q'M''' = QM$ , ideoque  $PR' = PR$ , eritque pro his punctis coordinata  $z$  negativa. Ponatur eorum distantia a centro virium  $O$

$$\sqrt{(yy+(f-x)^2+(g+z)^2)} = u,$$

atque ex istis binis punctis nascentur hae duae vires

$$\text{sec. directionem } R'r', \text{ vis} = \frac{2kk(g+z)}{u^3},$$

$$\text{sec. directionem } R'T', \text{ vis} = \frac{2kk(f-x)}{u^3}.$$

at munc abscissa  $x$  negativa, sen capiatur  $CP' = CP$ , atque ex reliquis coordinatis definiantur simil modo quaterna puncta  $M''V$ ,  $M'V$ ,  $M''T$  et  $M'T'$ , ponaturque

$\sqrt{yy + (f+x)^2 + (g+z)^2} = (v)$  et  $\sqrt{yy + (f-x)^2 + (g-z)^2} = (u)$ ,  
puncta haec quatuor praebent sequentes vires

$$\text{sec. directionem } R''r'', \text{ vis} = \frac{2kk(g-z)}{(v)^3},$$

$$R''T, \text{ vis} = \frac{2kk(f+x)}{(v)^3},$$

$$R'''r''', \text{ vis} = \frac{2kk(g+z)}{(u)^3},$$

$$R'''T', \text{ vis} = \frac{2kk(f-x)}{(u)^3},$$

Omnia ergo haec octo puncta, in singulis corporis octantibus similiter posita, conjunctim has praebent vires, quibus corpus sollicitabitur:

$$\text{sec. directionem } PR, \text{ vis} = 2kkg(v^{-3} + u^{-3}) - 2kkz(v^{-3} - u^{-3}),$$

$$P'R'', \text{ vis} = 2kkg((v)^{-3} + (u)^{-3}) - 2kkz((v)^{-3} - (u)^{-3}),$$

$$Ss, \text{ vis} = 2kkf(v^{-3} + (v)^{-3}) - 2kkx(v^{-3} - (v)^{-3}),$$

$$S's', \text{ vis} = 2kkf(u^{-3} + (u)^{-3}) - 2kkx(u^{-3} - (u)^{-3}).$$

Quemadmodum ergo hae vires sunt natae ex puncto  $M$  in primo sphaeroidis octante quadranti  $AG$  sursum imminentे assumto: si omnia istius octantis puncta hoc modo colligantur, prodibunt vires, quibus totum sphaeroides sollicitatur, eaeque jam habebuntur reductae ad binas directiones, unam alterae axi  $EF$ , alterae diametro aequatoris  $AB$  sint parallelae.

Quo autem hae vires facilius colligi queant, eae, quae directiones habent parallelas, primo unum vero earum momentum exprimi debet. Ita vires  $PR$  et  $P'R''$  dabunt vim  $Yy = 2kkg(v^{-3} + u^{-3} + (v)^{-3} + (u)^{-3}) - 2kkz(v^{-3} - u^{-3} - (v)^{-3} - (u)^{-3})$ ,

whose momentum respectu centri  $C$  seu axis  $GH$  sumptum erit

$$Yy \cdot CY = 2kkgx(v^{-3} + u^{-3} - (v)^{-3} - (u)^{-3}) - 2kkxz(v^{-3} - u^{-3} - (v)^{-3} + (u)^{-3}).$$

Hence vis  $Ss$  et  $S's'$  coalescent in unam vim

$$Xx = 2kkf(v^{-3} + u^{-3} + (v)^{-3} + (u)^{-3}) - 2kkx(v^{-3} + u^{-3} - (v)^{-3} - (u)^{-3}),$$

whose momentum respectu ejusdem axis  $GH$  erit

$$Xx \cdot CX = 2kkfx(v^{-3} + (v)^{-3} - u^{-3} - (u)^{-3}) - 2kkxz(v^{-3} - (v)^{-3} - u^{-3} + (u)^{-3}).$$

Inveniatur hunc puncto  $M$  massa elementaris  $dx dy dz$ , per eamque singulae istae expressiones multiplicandæ et integratione ter debito modo instituta prodibunt tam vires totales  $Yy$  et  $Xx$  ex attractione

totius sphaeroidis oriundae, quam earum momenta  $Yy \cdot CY$  et  $Xx \cdot CX$ ; quae deinceps in una toti attractioni aequivalentem conjungi poterunt. Quo autem hae integrationes commodius possint, transformemus formulas  $v^{-3}$ ,  $u^{-3}$ ,  $(v)^{-3}$  et  $(u)^{-3}$  in series, quae, si distingue virium  $V(fg + gg)$ , quam ponamus  $= h$ , a centro sphaeroidis  $C$  fuerit valde magna, convergent. Cum igitur sit  $v = \sqrt{hh - 2fx - 2gz + yy + xx + zz}$ , erit

$$v^{-3} = \frac{1}{h^3} - \frac{3fx + 3gz}{h^5} - \frac{3yy - 3xx - 3zz}{2h^5} + \frac{15ffxx + 30fgxz + 15ggzz}{2h^7},$$

$$u^{-3} = \frac{1}{h^3} - \frac{3fx - 3gz}{h^5} - \frac{3yy - 3xx - 3zz}{2h^5} + \frac{15ffxx - 30fgxz + 15ggzz}{2h^7},$$

$$(v)^{-3} = \frac{1}{h^3} - \frac{3fx + 3gz}{h^5} - \frac{3yy - 3xx - 3zz}{2h^5} + \frac{15ffxx - 30fgxz + 15ggzz}{2h^7},$$

$$(u)^{-3} = \frac{1}{h^3} - \frac{3fx - 3gz}{h^5} - \frac{3yy - 3xx - 3zz}{2h^5} + \frac{15ffxx + 30fgxz + 15ggzz}{2h^7}.$$

Hinc igitur erit vis tota  $Yy$  ex attractione totius sphaeroidis orta

$$Yy = \frac{8kkfg}{h^3} \int dx dy dz \left( 1 - \frac{3yy - 3xx - 9zz}{2hh} + \frac{15ffxx + 15ggzz}{2h^4} \right),$$

et vis tota  $Xx$  pro toto sphaeroide orta

$$Xx = \frac{8kkf}{h^3} \int dx dy dz \left( 1 - \frac{3yy - 9xx - 3zz}{2hh} + \frac{15ffxx + 15ggzz}{2h^4} \right).$$

Deinde vero erunt momenta totalia

$$Yy \cdot CY = \frac{24kkfg}{h^5} \int xx dx dy dz - \frac{120kkfg}{h^7} \int xxzz dx dy dz,$$

$$Xx \cdot CX = \frac{24kkfg}{h^5} \int zz dx dy dz - \frac{120kkfg}{h^7} \int xxzz dx dy dz.$$

Quoniam triplici integratione opus est, ponantur primo  $x$  et  $z$  constantes, ut obtineantur vires elementis secundum rectas  $RM$  sitis oriunda, eritque

$$Yy = \frac{8kkg}{h^3} \int y dx dz \left( 1 - \frac{yy - 3xx - 9zz}{2hh} + \frac{15ffxx + 15ggzz}{2h^4} \right),$$

$$Xx = \frac{8kkf}{h^3} \int y dx dz \left( 1 - \frac{yy - 9xx - 3zz}{2hh} + \frac{15ffxx + 15ggzz}{2h^4} \right),$$

$$Yy \cdot CY = \frac{24kkfg}{h^5} \int xxy dx dz - \frac{120kkfg}{h^7} \int xxzy dx dz,$$

$$Xx \cdot CX = \frac{24kkfg}{h^5} \int zzy dx dz - \frac{120kkfg}{h^7} \int xxzy dx dz.$$

Concipiatur jam recta  $RM$  usque ad superficiem sphaeroidis producta, atque  $y$  determinata debet ex aequatione locali pro hac superficie sphaeroidica, inter coordinatas  $x$ ,  $y$  et  $z$  expressa

$yy = aa - xx - \frac{aazz}{bb}$ . Ponatur nunc  $z$  constans, ut integrationes pateant ad sectiones hyperboloidis parallelas aequatori secundum  $MR$  factas: hunc in finem ponatur  $\sqrt{(aa - \frac{aazz}{bb})} = p$ , sit radius hujus sectionis, atque integrationem eousque extendi oportebit, donec fiat  $x = p$ .  
 $\frac{aa}{bb} = n$ , eritque pro hoc casu

$$\text{vis } Yy = \int \frac{8kkfdz}{h^3} \int dx \left( 1 - \frac{aa - 2xx + (n-9)zz}{2hh} + \frac{15ffxx + 15ggzz}{2h^4} \right) \sqrt{(pp - xx)},$$

$$\text{vis } Xx = \int \frac{8kkfdz}{h^3} \int dx \left( 1 - \frac{aa - 8xx + (n-3)zz}{2hh} + \frac{15ffxx + 15ggzz}{2h^4} \right) \sqrt{(pp - xx)},$$

$$\text{momentum } Yy \cdot CY = \int \frac{24kkfgdz}{h^5} \int xx dx \sqrt{(pp - xx)} - \int \frac{120kkfgdz}{h^7} \int xxzz dx \sqrt{(pp - xx)},$$

$$\text{momentum } Xx \cdot CX = \int \frac{24kkfgdz}{h^5} \int zz dx \sqrt{(pp - xx)} - \int \frac{120kkfgdz}{h^7} \int xxzz dx \sqrt{(pp - xx)}.$$

Posita autem ratione diametri ad peripheriam  $= 1 : \pi$ , si post integrationem fiat  $x = p$ , erit

$$\int dx \sqrt{(pp - xx)} = \frac{1}{4} \pi pp, \quad \int xx dx \sqrt{(pp - xx)} = \frac{1}{16} \pi p^4,$$

quibus valoribus substitutis erit

$$\text{vis } Yy = \int \frac{2\pi kkfpdz}{h^3} \left( 1 - \frac{aa - \frac{1}{2}pp + (n-9)zz}{2hh} + \frac{\frac{15}{4}ffpp + 15ggzz}{2h^4} \right),$$

$$\text{vis } Xx = \int \frac{2\pi kkfpdz}{h^3} \left( 1 - \frac{aa - 2pp + (n-3)zz}{2hh} + \frac{\frac{15}{4}ffpp + 15ggzz}{2h^4} \right),$$

$$\text{mom. } Yy \cdot CY = \int \frac{3\pi kkgp^4}{2h^5} dz - \int \frac{15\pi kkgp^4zz}{2h^7} dz,$$

$$\text{mom. } Xx \cdot CX = \int \frac{6\pi kkfpdz}{h^5} dz - \int \frac{15\pi kkfpdz}{2h^7} dz.$$

autem  $pp = aa - nzz = aa - \frac{aazz}{bb}$ , uti assumsumus, erit ergo  $aa = nbb$  et  $pp = n(bb - zz)$ .

Instituatur nunc ultima integratio, ac ponatur  $z = b$ , quoniam est

$$\int ppdz = n \int dz (bb - zz) = \frac{2}{3} nb^3, \quad \int p^4 dz = nn \int dz (bb - zz)^2 = \frac{8}{15} nn b^5,$$

$$\int ppzz dz = n \int zz dz (bb - zz) = \frac{2}{15} nb^5, \quad \int p^4 zz dz = nn \int zz dz (bb - zz)^2 = \frac{8}{105} nn b^7,$$

Ita quae sita ita se habebunt

$$\text{vis } Yy = \frac{2\pi kkg}{h^3} \left( \frac{2}{3}nb^3 - \frac{3n^5}{5hh} - \frac{2nnb^5}{5hh} + \frac{nnb^5ff + nb^5gg}{h^4} \right),$$

$$\text{vis } Xx = \frac{2\pi kkf}{h^3} \left( \frac{2}{3}nb^3 - \frac{nb^5}{5hh} - \frac{4nnb^5}{5hh} + \frac{nnb^5ff + nb^5gg}{h^4} \right),$$

$$\text{mom. } Yy \cdot CY = \frac{4\pi nnkkb^5fg}{5h^5} - \frac{4\pi nnkkb^7fg}{7h^7},$$

$$\text{mom. } Xx \cdot CX = \frac{4\pi nkkb^5fg}{5h^5} - \frac{4\pi nnkkb^7fg}{7h^7},$$

Massa autem totius sphaeroidis est  $= \frac{4}{3}\pi aa b = \frac{4}{3}\pi nb^3$ , quae si dicatur  $= M$ , eaque in formula inventas introducatur, reperietur

$$\text{vis } Yy = \frac{Mkkg}{h^3} \left( 1 - \frac{9bb}{10hh} - \frac{3aa}{5hh} + \frac{3aaff}{2h^4} + \frac{3bbgg}{2h^4} \right),$$

$$\text{vis } Xx = \frac{Mkkf}{h^3} \left( 1 - \frac{3bb}{10hh} - \frac{6aa}{5hh} + \frac{3aaff}{2h^4} + \frac{3bbgg}{2h^4} \right),$$

$$\text{mom. } Yy \cdot CY = \frac{3Mkkaafg}{5h^5} - \frac{3Mkkaabbfg}{7h^7},$$

$$\text{mom. } Xx \cdot CX = \frac{3Mkkbbfg}{5h^5} - \frac{3Mkkaaabbfg}{7h^7}.$$

Neglectis ergo in viribus  $Yy$  et  $Xx$  terminis praeter primum omnibus, erit

$$CY = \frac{3aaf}{5hh} \quad \text{et} \quad CX = \frac{3bbg}{5hh},$$

sicque cognitis punctis  $Y$  et  $X$ , in quibus applicatae sunt concipiendae vires  $Yy$  et  $Xx$ , quantum directiones sunt axi sphaeroidis  $CE$  et diametro aequatoris  $BCA$  respective parallelae, innotescunt media directio virium, quibus totum corpus ad centrum virium  $O$  sollicitatur. Ad hoc perficieatur concipiatur (fig. 194) sectio sphaeroidis per ejus axem  $ECF$  facta, in cuius plano sit centrum virium  $O$ , et  $AB$  sit diameter aequatoris in eodem plano ducta, erit  $CE = CF = b$ ,  $CA = CB = CD = f$ ,  $OD = g$  et  $CO = \sqrt{(ff+gg)} = h$ , atque tang  $DCO = \frac{g}{f}$ . Cum jam directiones binarum virium  $Xx$  et  $Yy$  se mutuo in  $z$  intersecent, media directio earum per punctum  $z$  transibit. Transibit vero etiam per centrum virium  $O$ , eritque ergo haec media directio  $zO$ . Quantum autem a centro  $C$  distet, fiat haec proportio

$$CD(f) : DO(g) = CY \left( \frac{3aaf}{5hh} \right) : Yt \left( \frac{3aag}{5hh} \right);$$

erit ergo

$$zt = Yt - CX = \frac{3(aa-bb)g}{5hh} = Cc \quad \text{proxime.}$$

Media ergo directio virium corpus sollicitantium transit non per centrum  $C$ , sed per

inferius quodpiam  $c$ , ut sit  $Cc = \frac{3(aa - bb)g}{5hh}$ , atque haec directio  $cO$  per centrum virium  $O$  transibit. Denique tota haec vis erit proxime  $cO = \frac{Mhk}{hh}$ , seu accuratius:  $cO = \frac{Mhk}{hh} \left(1 - \frac{3(aa - bb)(2gg - ff)}{10h^4}\right)$ , unde actio vis centripetae determinari poterit. Q. E. I.

**2. Coroll. 1.** Nisi ergo corpus sit sphaericum seu  $a = b$ , neque directio vis, qua id versus centrum virium  $O$  sollicitatur, per centrum corporis  $C$ , quod simul est ejus centrum gravitatis, transibit, neque vis ipsa  $cO$  quadrato distantiae  $CO$  amplius est reciproce proportionalis.

**3. Coroll. 2.** Cum igitur motus corporis progressivus perinde se habeat, ac si ipsi in centro gravitatis  $C$  applicata esset vis aequalis ipsi

$$cO = \frac{Mhk}{hh} \left(1 - \frac{3(aa - bb)(2gg - ff)}{10h^4}\right),$$

in directione ipsi  $cO$  parallela, haec vis neque per punctum  $O$  transibit, neque quadratis distantiae  $CO$  erit reciproce proportionalis. Quamobrem semita corporis non erit ellipsis, in cuius altero termino sit punctum  $O$ : haecque aberratio eo erit notabilior, quo magis figura sphaeroidica a sphaerica discrepet.

**4. Coroll. 3.** Hoc quoque casu axis  $EF$  non situm sibi parallelum tenebit, sed a momento sollicitantis continuo declinabitur. Quoniam vero momenta  $Yy \cdot CY$  et  $Xx \cdot CX$  sunt inter se contraria, illud praevalebit si  $a > b$ , ideoque vis  $cO$  momentum ad axem  $EF$  versus situm  $ef$  inclinandum erit  $= \frac{3Mhkfg(aa - bb)}{5h^5}$ . Interim tamen haec vis, quia per axem transit, motum vertiginis non afficiet.

**5. Coroll. 4.** Sit nunc angulus, quo axis sphaeroidis  $ECF$  ad rectam  $CO$  inclinatur,  $ECO = \varphi = COD$ , permanente distantia  $CO = h$ , erit  $CD = f = h \sin \varphi$  et  $OD = g = h \cos \varphi$ . Hinc itaque erit interdum  $Cc = \frac{3(aa - bb) \cos \varphi}{5h}$ , denotante  $a$  semidiametrum aequatoris  $AC$ , et  $b$  semiaxem sphaeroidis  $CE$ .

**6. Coroll. 5.** Angulo porro hoc  $ECO = \varphi$  loco rectangularium  $f$  et  $g$  introducto, erit vis, qua sphaeroides in puncto  $c$  ad centrum virium  $O$  sollicitatur,

$$= \frac{Mhk}{hh} \left(1 - \frac{3(aa - bb)(2 \cos^2 \varphi - \sin^2 \varphi)}{20hh}\right),$$

scilicet  $\cos^2 \varphi = \frac{1 + \cos 2\varphi}{2}$  et  $\sin^2 \varphi = \frac{1 - \cos 2\varphi}{2}$ , erit haec vis.

$$= \frac{Mhk}{hh} \left(1 - \frac{3(aa - bb)(1 + 3 \cos 2\varphi)}{20hh}\right).$$

**Coroll. 6.** Si haec vis in directione parallela centro gravitatis  $C$  concipiatur applicata, resolvatur secundum directiones  $CO$ , et  $Cy$ , ad  $CO$  in plano  $ECO$  normalem, reperietur

$$\text{vis } CO = \frac{Mhk}{hh} \left(1 - \frac{3(aa - bb)(1 + 3 \cos 2\varphi)}{20hh}\right) \text{ et vis } Cy = \frac{Mhk}{hh} \cdot \frac{3(aa - bb) \sin 2\varphi}{10hh}.$$

8. **Coroll. 7.** Ob illam igitur vim  $CO$ , quatenus quadratis distantiarum  $CO$  non exacte reciprocce proportionalis, orbita, quam centrum  $C$  describet, aliquantum ab elliptica discreta alterius autem vis  $C\gamma$  effectus in hoc consistet, ut punctum  $C$  non in eodem plane moveatur.

9. **Coroll. 8.** Momentum denique, quo haec vis pollet ad axem corporis  $EF$  inclinandum situm  $ef$  compellendum erit

$$= \frac{2Mkk(aa - bb) \sin \varphi \cos \varphi}{5h^3} = \frac{Mkk(aa - bb) \sin 2\varphi}{5h^3}.$$

Est itaque ceteris paribus reciproce ut cubus distantiae  $CO$ . Ratione anguli  $ECO = \varphi$  vero momentum erit maximum, si hic angulus  $ECO$  fiat semirectus.

10. **Scholion 1.** Cum igitur ex observationibus summa cura ab Illustris Academiae Regiae Parisinae Membris tam in Gallia quam in Lapponia et America institutis certissime evictum sit, terrae non esse sphaericam, sed sphaeroidicam compressam, cuius axis per polos ductus sit quam diameter aequatoris, hinc non levis mutatio tam in motu terrae quam in axis positione oriri debet. Quae ut definiri possit, non solum veram rationem inter axem terrae et diametrum aequatoris determinari oportet, sed etiam utriusque quantitatem absolutam, quod sequenti modo difficulter fieri poterit. Sit semidiameter aequatoris  $= a$ , et semiaxis per polos ductus  $= b$ , ponatur  $b : a = 1 : 1 + \omega$ , ut sit  $a = b + \omega b$ , erit  $\omega$  fractio valde parva. Sit in quaquam terrae regione elevatio poli  $= p$ , erit quantitas gradus meridiani in hac regione

$$= 0,017453292(b + \frac{1}{2}\omega b - \frac{3}{2}\omega b \cos 2p),$$

$$\text{seu } = \frac{b + \frac{1}{2}\omega b - \frac{3}{2}\omega b \cos 2p}{57,29577951}.$$

Gradus vero secundum longitudinem in circulo aequatori parallelo mensuratus erit

$$= 0,017453292(b + \frac{3}{2}\omega b - \frac{1}{2}\omega b \cos 2p) \cos p.$$

Cum jam in Gallia sub elevatione poli  $49^\circ 21' 24''$  mensura gradus in meridiano inventa sit 57438 hexapedarum parisinarum; hinc deducitur sequens aequatio

$$b + 0,7087569 \cdot \omega b = 3276344 \frac{1}{2}.$$

Sub circulo autem polari ab Illustri Praeside nostro de Maupertuis gradus meridiani definiti 57438 hexapedarum, pro elevatione poli  $66^\circ 30'$  ( $,19' 34''$ ), unde sequitur haec aequatio:

$$b + 1,5229976 \omega b = 3290955,$$

ex quibus duabus aequatoribus invenitur

$ab = 17943$  hexaped. paris.,  $b = 3263626$ , ac propterea  $a = 3281570^*$ ).

semiaxis terrae  $b = 3263626$  hexaped. paris. et semidiameter aequatoris  $a = 3281570$  hexaped. paris. illiusque numeri ad hunc ratio proxime erit ut 182 ad 183, ita ut sit  $\omega = \frac{1}{182}$  et  $a = \frac{183}{182} b$ .

**Scholion 2.** Definita ergo figura et quantitate terrae, si vim, qua ad solem urgetur, primum ejus orbita aliquantillum ab elliptica recedet, quia vis, qua centrum terrae ad centrum solis sollicitatur, non perfecte est quadratis distantiarum reciproce proportionalis. Erit  $a = \left(1 + \frac{1}{182}\right) b$ ,  $aa = \left(1 + \frac{1}{91}\right) bb$  proxime; ideoque vis, qua centrum terrae  $C$  ad solem per directum fit  $= \frac{Mkk}{hh} \left(1 - \frac{(1 + 3 \cos 2\varphi) bb}{607 hh}\right)$  proxime. Cum autem posita parallaxi solis horizontali sub horizonte  $= 10''$ , sit  $\frac{b}{h} = \sin 10''$ , ideoque  $\frac{bb}{hh} = 0,00000000235$ , erit haec vis  $= \frac{Mkk}{hh} \left(1 - \frac{(1 + 3 \cos 2\varphi)}{258249300000}\right)$ , quae differentia ab  $\frac{Mkk}{hh}$  tantilla est, ut ejus effectus omnino sentiri nequeat. Tum vero adest vis, qua terra de plano eclipticae detorquetur, cujus directio ad hoc planum normalis et sursum, seu versus terram sollicitans erit  $= \frac{Mkk}{hh} \cdot \frac{\sin 2\varphi}{129124650000}$ , quae maxima est quando sol proxime ad polum arcticum accedit, ubi fit  $\varphi = 66\frac{1}{2}^\circ$  ac

$$\sin 2\varphi = \sin 47^\circ = 0,7313537;$$

sicut autem in tropico capricorni versante, pari vi terra de ecliptica deorsum urgetur; cum autem haec vis sit minima, effectus erit imperceptibilis. Momentum autem, quo axis terrae inclinatur, postea soli propior ab eo detorquetur, erit  $= \frac{Mkk(aa - bb)\sin 2\varphi}{5h^3} = \frac{Mkkbb\sin 2\varphi}{455h^3}$ . Quia vero effectus minus ab hac vi oriundus in accuratissimis observationibus animadverti potest, eam negligere non licet.

**Scholion 3.** Terra deinde quoque ad lunam attrahitur, verum haec vis prae illa, qua ad solem urgetur, tam est exigua, ut in motu terrae vix perceptibilem alterationem efficiat. Quanquam ergo haec vis ad lunam tendens, ob figuram terrae sphaeroidicam, quadratis distantiarum non est reciproce proportionalis, sed ab hac proportione aliquantum recedit, tamen multo minus effectus in motu terrae oriundus ullo modo observabilis esse poterit. Aliter vero se res habet in illa vi, qua terra de plano eclipticae detruditur, quae ob lunae vicinitatem multo major est simili illa vi a sole. Sit enim distantia lunae a terra  $= H$ , et vis attractiva acceleratrix  $= \frac{KK}{HH}$ , erit vis lunae ad

motu de plano eclipticae depellendam tendens  $= \frac{3MKK(aa - bb)\sin 2\varphi}{10H^4}$ ; vis solis autem similem

*Script. autogr. ad marg.* Sub aequatore lat.  $1^\circ : 56725$  tois.  $b - wb = 3250103$ , et ex circ. polari  $wb = 16192$ ,  $b = 3266295$ ,  $a = 3282487$ , ergo

$$a:b = 203:202, \quad a:b = 201:200.$$

effectum edens  $= \frac{3Mkk(aa - bb) \sin 2\varphi}{10h^4}$ . Erit ergo vis lunae ad vim solis in similibus positionibus ut  $\frac{KK}{H^4}$  ad  $\frac{kk}{h^4}$ . Verum ex aetu maris Newtonus conclusit esse vim lunae ad mare movendum similem vim solis ut 4 ad 1, quam rationem quidem Cel. Dan. Bernoulli multo minorem statuit scilicet ut 5:2. Vires autem illae ad mare movendum sunt ut  $\frac{KK}{H^3}$  ad  $\frac{kk}{h^3}$ ; facto ergo  $\frac{KK}{H^3} =$  prodibit vis lunae ad terram de plano eclipticae deturbandam ad vim solis ut  $\frac{4}{H}$  ad  $\frac{1}{h}$ , hoc est  $4h$  ad  $H$ , quae ratio proxime erit ut 1333 ad 1, siquidem ponamus  $h = 20000$  semid. terrae,  $H = 60$ ; quare haec vis lunae plus quam millies excedit similem vim solis, ejusque effectus non erit negligendus. Tum vero vis lunae ad axem terrae inclinandum impensa erit  $= \frac{MKK(aa - bb) \sin 2\varphi}{5H^3}$ , quae propterea secundum Newtonum quadroplo major esse deberet quam vis solis; atque ex hoc fonte tam praecessio aequinoctiorum, quam nutatio quaepiam axis terrae sequi debet, quem utrumque effectum, quantum principia Mechanicae etiamnunc cognita id permittunt, determinare conabor.

**13. Problema III.** (Fig. 195). Determinare motum axis terrae, quatenus ista vi solis perturbatur, seu nutationem axis terrae a vi solis oriundam definire.

**Solutio.** Concipiamus centrum terrae in  $C$  quiescere, solemque in ellipsi circa id revolvitur, praesens enim propositum perinde est, sive motum annum soli tribuamus sive terrae. Repraesentemus ergo planum tabulae planum eclipticae, sitque  $AOB$  orbita, in qua sol moveri videtur; sit  $A$  eius apogaeum,  $B$  perigaeum, et post tempus quodpiam  $t$  sol ex apogaeo pervenerit in situm  $O$ ; vocetur semiaxis transversus orbitae solaris  $= c$ , excentricitas  $= n$ , erit  $CA = (1 + n)c$  et  $CB = (1 - n)c$ . Anomalia autem vera temporis  $t$  respondens, seu angulus  $ACO$  sit  $= \nu$ , et anomalia media  $= \vartheta$ ; distantia  $CO = z$ . Hoc autem tempore axis terrae teneat situm  $CE$ , ita ut sumto  $E$  pro polo boreali sit  $CE = b$ . Ex  $E$  in planum eclipticae demittatur perpendicularum  $EP$ , ductaque  $CP$  vocentur anguli  $ACP = \vartheta$  et  $ECP = \varphi$ , erit  $EP = b \sin \varphi$  et  $CP = b \cos \varphi$ . Jam axis  $EC$  cum directione  $ECO$  facit angulum  $ECO$ , ad quem inveniendum ex  $P$  in  $CO$  demittatur perpendicularum  $PQ$ , eritque ex  $EQ$  ad  $CO$  perpendicularis. Cum jam sit ang.  $OCP = \nu - \vartheta$ , erit  $PQ = b \cos \varphi \sin(\nu - \vartheta)$  et  $CQ = b \cos \varphi \cos(\nu - \vartheta)$ , unde fit  $\frac{CQ}{CE} = \cos OCE = \cos \varphi \cos(\nu - \vartheta)$ , qui est ille ipse angulus quem superius  $= \varphi$  vocavimus. Erit autem

$$\sin OCE = \sqrt{(1 - \cos^2 \varphi \cos^2(\nu - \vartheta))} \quad \text{et} \quad \sin 2OCE = 2 \cos \varphi \cos(\nu - \vartheta) \sqrt{(1 - \cos^2 \varphi \cos^2(\nu - \vartheta))}$$

Quoniam erit momentum vis solis ad hunc angulum  $OCE$  augendum

$$= \frac{2Mkk(aa - bb) \cos \varphi \cos(\nu - \vartheta) \sqrt{(1 - \cos^2 \varphi \cos^2(\nu - \vartheta))}}{5z^3},$$

pro quo brevitatis gratia scribatur  $Mp$ . Ducatur  $TEt$  normalis ad  $CE$ , eritque  $Et$  directio secundum quam punctum  $E$  ab ista vi detorquebitur. Quantum autem detorqueatur, cognoscetur ex momento inertiae totius terrae, respectu axis ad  $CE$  normalis, hoc est respectu diametri aequatoris. Si igitur terra ex materia homogena statuatur composita, respectu axis per aequatorem ducti repertur momentum inertiae  $= \frac{1}{5}M(aa + bb)$ . Quodsi jam angulus  $OCE$  brevitatis gratia ponatur  $=$

ratio ita erit comparata, ut tempusculo  $dt$  fiat

$$\frac{2ddz}{dt^2} = \frac{Mp}{\frac{1}{5}(aa + bb)M} = \frac{5p}{aa + bb}, \text{ ita ut sit } dds = \frac{5pdz^2}{2(aa + bb)}.$$

est  $p = \frac{2kk(aa - bb)\sin s \cos s}{5z^3}$ , ergo  $ddz = \frac{kkdt^2(aa - bb)\sin s \cos s}{(aa + bb)z^3}$ . Capiatur ergo  $Ee$  tantum, ut sit  $EEe = dds$ , erit  $e$  punctum, in quod polus  $E$  tempusculo  $dt$  detorqueretur, si ante quievisset. Cum autem polo motus jam impressus concipi debeat, is ita erit comparatus, ut, si a nullis viribus impinguatur, uniformiter secundum circulum maximum esset progressurus. Quantum ergo hic motus illa solis afficiatur, sequenti modo determinari poterit.

Concipiatur (fig. 196) in superficie spherae  $AO$  ecliptica, in eaque polus  $E$ , sumto  $A$  pro apogeo solis. Ducatur  $ER$  ad  $AO$  normalis, erit  $AR = \vartheta$  et  $ER = \varphi$ . Progrediatur motu jam concreto polus  $E$  tempusculo  $dt$  in  $e$ , erit  $Rr = d\vartheta$  et  $eG = d\varphi$ , atque si motu uniformi secundum circulum maximum progrederebetur, perveniret sequenti tempusculo in  $e'$ , ut esset  $rr' = d\vartheta + 2d\varphi d\vartheta \tan \varphi$  et  $eg = d\varphi - d\vartheta^2 \sin \varphi \cos \varphi$ , quarum formularum demonstrationem deinceps tradam. Jam capiatur  $\vartheta = \nu$ , junganturque circulo maximo puncta  $O$  et  $e'$ , erit arcus  $Oe' = s$ , sumto punto  $e'$  pro primo  $E$ , et  $r'O$  seu  $RO = \nu - \vartheta$ , atque  $r'e' = RE = \varphi$ , unde erit  $\cos s = \cos \varphi \cos(\nu - \vartheta)$ , atque  $\sin Oe'r' = \frac{\sin(\nu - \vartheta)}{\sin s}$ , seu  $\tan Oe'r' = \frac{\tan(\nu - \vartheta)}{\sin \varphi}$ , et  $\cos Oe'r' = \frac{\sin \varphi \cos(\nu - \vartheta)}{\sin s}$ . Nunc quia polus in hoc circulo  $Oe'$  pellitur, capiatur

$$e'e = dds = \frac{kkdt^2(aa - bb)\sin s \cos s}{(aa + bb)z^3},$$

unque  $e$  punctum, ad quod polus sine alterius tempusculi  $dt$  reperietur; ducatur perpendicularum  $e'g$  ad  $eq$  normalis, erit

$$ey = \frac{kkdt^2(aa - bb)\cos s \sin \varphi \cos(\nu - \vartheta)}{(aa + bb)z^3} \quad \text{et} \quad e'\gamma = \frac{kkdt^2(aa - bb)\cos s \sin(\nu - \vartheta)}{(aa + bb)z^3} = r'\rho \cos \varphi,$$

$$\text{sit } r'\rho = \frac{kkdt^2(aa - bb)\sin(\nu - \vartheta) \cos(\nu - \vartheta)}{(aa + bb)z^3}.$$

est  $r\rho = d\vartheta + dd\vartheta = rr' - r'\rho$  et  $e'g + ey = d\varphi + dd\varphi$ , unde fit

$$dd\vartheta = 2d\varphi d\vartheta \tan \varphi - \frac{kkdt^2(aa - bb)\sin(\nu - \vartheta) \cos(\nu - \vartheta)}{(aa + bb)z^3},$$

$$dd\varphi = -d\vartheta^2 \sin \varphi \cos \varphi + \frac{kkdt^2(aa - bb)\sin \varphi \cos \varphi \cos^2(\nu - \vartheta)}{(aa + bb)z^3},$$

us aequationibus motus poli  $E$  continetur, ita ut ex iis ad quodvis tempus positio axis  $CE$  queat.

Quodsi vero, loco temporis  $t$  anomaliam medium  $u$  in calculum introducamus, reperietur  $= 2c^3du^2$ , sicque simul quantitas  $kk$  ex calculo egreditur, eritque ergo

$$dd\vartheta = 2d\vartheta d\varphi \tang \varphi - \frac{2c^3 du^2 (aa - bb) \sin(v - \vartheta) \cos(v - \vartheta)}{(aa + bb) z^3},$$

$$dd\varphi = -d\vartheta^2 \sin \varphi \cos \varphi + \frac{2c^3 du^2 (aa - bb) \sin \varphi \cos \varphi \cos^2(v - \vartheta)}{(aa + bb) z^3}.$$

Posita autem anomalia media  $= u$ , quae anomaliae verae  $ACO = v$  respondeat, ponatur anomalia excentrica  $= r$ , erit

$$u = r + n \sin r,$$

$$\cos v = \frac{n + \cos r}{1 + n \cos r},$$

$$z = c(1 + n \cos r),$$

$$du = dr(1 + n \cos r) = \frac{z dr}{c}$$

$$\sin v = \frac{\sin r \sqrt{1 - nn}}{1 + n \cos r},$$

et

$$dv = \frac{dr \sqrt{1 - nn}}{1 + n \cos r} = \frac{du \sqrt{1 - nn}}{(1 + n \cos r)}$$

Cum jam  $du$  sit constans, erit introducendo  $r$

$$ddr(1 + n \cos r) - ndr^2 \sin r = 0 \quad \text{seu} \quad ddr = \frac{ndr^2 \sin r}{1 + n \cos r},$$

ideoque habebuntur hae duae aequationes

$$dd\vartheta = 2d\vartheta d\varphi \tang \varphi - \frac{2(aa - bb) dr^2 \sin(v - \vartheta) \cos(v - \vartheta)}{(aa + bb)(1 + n \cos r)},$$

$$dd\varphi = -d\vartheta^2 \sin \varphi \cos \varphi + \frac{2(aa - bb) dr^2 \sin \varphi \cos \varphi \cos^2(v - \vartheta)}{(aa + bb)(1 + n \cos r)},$$

multiplicetur prior per  $d\vartheta \cos^2 \varphi$  et posterior per  $d\varphi$ , ambaeque addantur, prodibit

$$d\vartheta dd\vartheta \cos^2 \varphi + d\varphi dd\varphi - d\varphi d\vartheta^2 \sin \varphi \cos \varphi = \frac{2(aa - bb) dr^2 \cos \varphi \cos(v - \vartheta) (d\varphi \sin \varphi \cos(v - \vartheta) - d\vartheta \cos \varphi \sin(v - \vartheta))}{(aa + bb)(1 + n \cos r)}$$

cujus pars prior est integrabilis; fiet enim

$$\frac{1}{2} d\varphi^2 + \frac{1}{2} d\vartheta^2 \cos^2 \varphi = \frac{2(aa - bb) du^2}{aa + bb} \int \frac{\cos \varphi \cos(v - \vartheta) (d\varphi \sin \varphi \cos(v - \vartheta) - d\vartheta \cos \varphi \sin(v - \vartheta))}{(1 + n \cos r)^3}$$

Ponatur  $\frac{aa - bb}{aa + bb} = m$ , eritque

$$dd\vartheta = 2d\vartheta d\varphi \tang \varphi - \frac{m du^2 \sin 2(v - \vartheta)}{(1 + n \cos r)^3}, \quad \frac{dd\varphi}{\sin \varphi \cos \varphi} + d\vartheta^2 = \frac{m du^2 (1 + \cos 2(v - \vartheta))}{(1 + n \cos r)^3}.$$

Quo clarius perspiciamus, quemadmodum has aequationes tractari conveniat, assumamus primo  $CE$  plano eclipticae normaliter insistere; et quia hoc casu angulus  $OCE$  est rectus, momentum inclinantis evanescit: quare si axis in hoc situ semel quieverit, in eodem perpetuo persistet quod etiam ex aequationibus inventis intelligitur; cum enim sit  $\cos \varphi = 0$  et  $\tang \varphi = \infty$ , aequatio dat  $d\vartheta d\varphi = 0$ , et altera  $dd\varphi = 0$ , quibus satisfit si  $d\varphi = 0$ , seu si axis  $CE$  perpendicular ad planum eclipticae maneat perpendicularis.

etiam nunc axem in ipsum planum eclipticae incidere; et quia is ab momento vis solis deplanato non depellitur, perpetuo erit  $\varphi = 0$ , atque motus axis ex priori aequatione sola determinatur, quae hoc casu abit in  $dd\vartheta = \frac{-mdu^2 \sin 2(u-\vartheta)}{(1+n \cos r)^3}$ .

In primo orbita circularis, seu  $n=0$  et  $r=u$ , erit  $dd\vartheta + mdu^2 \sin 2(u-\vartheta) = 0$ . Fingatur  $dP = mdu \cos 2(u-\vartheta) - Pdu$ , erit

$$dd\vartheta = -2\alpha du^2 \sin 2(u-\vartheta) + 2\alpha dud\vartheta \sin 2(u-\vartheta) + dPdu, \text{ seu}$$

$$-2\alpha dud\vartheta \sin 2(u-\vartheta) + \alpha du^2 \sin 4(u-\vartheta) + 2\alpha Pdu^2 \sin 2(u-\vartheta) + dPdu = -mdu^2 \sin 2(u-\vartheta).$$

ergo  $\alpha = \frac{1}{2}m$ , ut sit  $\frac{1}{4}mmdu \sin 4(u-\vartheta) + mPdu \sin 2(u-\vartheta) + dP = 0$ . Ponatur

$$mm \cos 4(u-\vartheta) + Q, \text{ ob } du - d\vartheta = du - \frac{1}{2}mdu \cos 2(u-\vartheta) - \frac{1}{16}mmdu \cos 4(u-\vartheta) - Qdu,$$

$$\begin{aligned} R &= -\frac{1}{4}mmdu \sin 4(u-\vartheta) - \frac{1}{8}m^3du \sin 4(u-\vartheta) \cos 2(u-\vartheta) - \frac{1}{64}m^4du \sin 4(u-\vartheta) \cos 4(u-\vartheta) \\ &\quad + \frac{1}{4}mmQdu \sin 4(u-\vartheta) + dQ \\ &= -\frac{1}{4}mmdu \sin 4(u-\vartheta) - \frac{1}{16}m^3du \sin 2(u-\vartheta) \cos 4(u-\vartheta) - mQdu \sin 2(u-\vartheta), \end{aligned}$$

nde apparet  $Q$  habiturum esse coefficientem  $m^3$ , ideoque ejus valorem tam fore exiguum, ut rejiciatur. Erit ergo vero proxime

$$d\vartheta = \frac{1}{2}mdu \cos 2(u-\vartheta) - \frac{1}{16}mmdu \cos 4(u-\vartheta),$$

inque integrando ponatur

$$\vartheta = C - \frac{1}{4}m \sin 2(u-\vartheta) + \frac{1}{64}mm \sin 4(u-\vartheta) + R,$$

$$\begin{aligned} d\vartheta &= \frac{1}{2}mdu \cos 2(u-\vartheta) + \frac{1}{16}mmdu \cos 4(u-\vartheta) + dR, \\ &\quad - \frac{1}{2}md\vartheta \cos 2(u-\vartheta) - \frac{1}{16}mmd\vartheta \cos 4(u-\vartheta), \end{aligned}$$

valore substituto habebitur

$$dR = \frac{1}{4}mmdu \cos^2 2(u-\vartheta) + \frac{1}{16}m^3du \cos 2(u-\vartheta) \cos 4(u-\vartheta) + \frac{1}{256}m^4du \cos^2 4(u-\vartheta),$$

$$\begin{aligned} dR &= \frac{1}{8}mmdu + \frac{1}{8}mmdu \cos 4(u-\vartheta) - \frac{1}{32}m^3du \cos 2(u-\vartheta) \\ &\quad - \frac{1}{32}m^3du \cos 6(u-\vartheta) + \frac{1}{512}m^4du + \frac{1}{512}m^4du \cos 8(u-\vartheta), \end{aligned}$$

$$R = \frac{1}{8}mmu + \frac{1}{512}m^4u - \frac{1}{32}mm \sin 4(u-\vartheta). \text{ Consequenter habebitur}$$

$$\vartheta = C - \frac{1}{4}m \sin 2(u-\vartheta) - \frac{3}{64}m^2 \sin 4(u-\vartheta) - \frac{1}{8}m^2u.$$

Potest autem hoc casu aequatio proposita  $dd\vartheta + mdu^2 \sin 2(u - \vartheta) = 0$ , absolute integrari, sive  
plicetur per 2 ( $du - d\vartheta$ ), ut sit

$$2du dd\vartheta - 2d\vartheta dd\vartheta + 2mdu^2 (du - d\vartheta) \sin 2(u - \vartheta) = 0,$$

erit enim  $2dud\vartheta - d\vartheta^2 = Cdu^2 + mdu^2 \cos 2(u - \vartheta)$ , vel posito  $u - \vartheta = s$ , seu  $\vartheta = u - s$   
habebitur  $du^2 - ds^2 = Cdu^2 + mdu^2 \cos 2s$ , seu  $ds^2 = du^2 (\alpha - m \cos 2s)$ , hincque  $du = \sqrt{\alpha - m \cos 2s} ds$ ,  
ubi  $\alpha$  est constans a motu axis ipsi primum impresso pendens. Quoniam igitur assumimus  
momentum vis solis, seu littera  $m$  evanescat, axem esse quietum, posito  $m = 0$ , erit  $ds = du$ ,  
ideoque  $\alpha = 1$ , ita ut sit  $du = \frac{ds}{\sqrt{1 - m \cos 2s}}$ , ex qua aequatione promotionem axis a vi solis  
dam definiri oportet. Cum jam sit  $m$  fractio valde parva, erit

$$\frac{1}{\sqrt{1 - m \cos 2s}} = 1 + \frac{1}{2}m \cos 2s + \frac{1 \cdot 3}{2 \cdot 4}m^2 \cos^2 2s + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}m^3 \cos^3 2s + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}m^4 \cos^4 2s + \dots$$

Potestatis autem  $\cos 2s$  ad cosinus angulorum multiplorum reductis, fiet

$$\begin{aligned} \frac{1}{\sqrt{1 - m \cos 2s}} &= +1 + \frac{1}{2}m \cos 2s + \frac{1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4}m^2 \cos 4s + \frac{1}{4} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}m^3 \cos 6s + \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot 7}{8 \cdot 2 \cdot 4 \cdot 6 \cdot 8}m^4 \cos 8s \\ &+ \frac{1}{2} \cdot \frac{1 \cdot 3}{2 \cdot 4}m^2 + \frac{3}{4} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}m^3 + \frac{4}{8} \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}m^4 \\ &+ \frac{3}{8} \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}m^4 \end{aligned}$$

etc.

Integrando ergo habebitur

$$\begin{aligned} u &= g + (1 + \frac{3}{16}m^2 + \frac{105}{1024}m^4)s + \frac{1}{4}m(1 + \frac{15}{32}m^2)\sin 2s + \frac{3}{64}m^2(1 + \frac{35}{48}m^2)\sin 4s \\ &\quad + \frac{15}{384}m^3 \sin 6s + \frac{35}{8192}m^4 \sin 8s, \end{aligned}$$

rejiciantur termini, in quibus  $m$  ultra duas obtinet dimensiones, eritque

$$u = g + u - \vartheta + \frac{3}{16}m^2u - \frac{3}{16}m^2\vartheta + \frac{1}{4}m \sin 2(u - \vartheta) + \frac{3}{64}m^2 \sin 4(u - \vartheta),$$

$$\text{seu } \vartheta = g + \frac{3}{16}m^2u + \frac{1}{4}m \sin 2(u - \vartheta) + \frac{3}{64}m^2 \sin 4(u - \vartheta),$$

axis ergo durante quavis solis revolutione modo progredietur, modo regredietur per arcum

ita ut si  $m = \frac{1}{200}$ , hoc spatium futurum sit  $= \frac{1}{400} = 0^\circ, 14 = 8' 24''$ . Tum vero qualibet revolutione  
solis, seu singulis annis, axis in ecliptica progredietur per spatium  $= \frac{3}{16}m^2 \cdot 360^\circ$ , quod ergo  
 $m = \frac{1}{200}$ , erit  $= \frac{3.360^\circ}{16.40000} = 6''$ .

Aliter autem res se habebit, si axis terrae ad eclipticam fuerit inclinatus; tum enim  
orbita solis circulari, ut sit  $n = 0$  et  $\varphi = u$ , manente  $\vartheta = u - s$ , haec duo habebuntur aequationes  
resolvendae

$$dds = 2(ds - du)d\varphi \tan\varphi + mdu^2 \sin 2s,$$

$$\frac{dd\varphi}{\sin\varphi \cos\varphi} + (du - ds)^2 = mdu^2 (1 + \cos 2s).$$

Multiplicetur aequatio prior per  $2qds$  et posterior per  $-dq$ , ambaeque invicem addantur, erit

$$\left. \begin{aligned} 2qdsddds - 4qds(ds - du)d\varphi \tang \varphi \\ - \frac{dqdd\varphi}{\sin \varphi \cos \varphi} - dq(du - ds)^2 \end{aligned} \right\} = 2mqdu^2ds \sin 2s - mdu^2dq - mdu^2dq \cos 2s$$

partem posteriorēm integrando fiet

$$C - mqdu^2 - mqdu^2 \cos 2s = \int \left( \frac{2qdsddds - 4qds(ds - du)d\varphi \tang \varphi}{\sin \varphi \cos \varphi} - dq(du - ds)^2 \right).$$

Si minime  $q = \cos^2 \varphi$ , erit  $dq = -2d\varphi \sin \varphi \cos \varphi$ , ideoque

$$\begin{aligned} Cdu^2 - mdu^2 \cos^2 \varphi (1 + \cos 2s) &= \int (2dsddds \cos^2 \varphi - 4ds(ds - du)d\varphi \sin \varphi \cos \varphi) \\ &= \int (2d\varphi dd\varphi + 2dsddds \cos^2 \varphi - 2ds^2 d\varphi \sin \varphi \cos \varphi + 2du^2 d\varphi \sin \varphi \cos \varphi) \\ &= d\varphi^2 + ds^2 \cos^2 \varphi - du^2 \cos^2 \varphi. \end{aligned}$$

Quocirca erit  $Cdu^2 = d\varphi^2 + (ds^2 - du^2) \cos^2 \varphi + mdu^2 \cos^2 \varphi (1 + \cos 2s)$ .

Si jam sumamus casu, quo  $m = 0$ , axem quiescere, ut sit  $ds = du$  et  $d\varphi = 0$ , fiet  $C = 0$  et  $ds^2 = du^2 = mdu^2 (1 + \cos 2s)$ , hincque

$$du = \sqrt{\frac{ds^2 + \frac{d\varphi^2}{\cos^2 \varphi}}{1 - m(1 + \cos 2s)}}.$$

Verum constantem  $C$  potius convenit definiri ex statu quopiam axis initiali. Si igitur assumamus principia, ubi axis primum a vi solis comitari coepit, fuisse angulum  $s = u - \vartheta = \varepsilon$ , et inclinationem  $\varphi = \gamma$ ; in hoc statu motum axis nullum statui oportet, seu erit  $d\vartheta = 0$  et  $d\varphi = 0$ , ideoque  $ds = du$ , quibus substitutis fiet  $Cdu^2 = mdu^2 \cos^2 \gamma (1 + \cos 2\varepsilon)$ , unde hanc obtinemus aequationem

$$mdu^2 \cos^2 \gamma (1 + \cos 2\varepsilon) = d\varphi^2 + ds^2 \cos^2 \varphi - du^2 \cos^2 \varphi + mdu^2 \cos^2 \varphi (1 + \cos 2s),$$

ex qua oritur

$$du^2 = \frac{d\varphi^2 + ds^2 \cos^2 \varphi}{\cos^2 \varphi + m \cos^2 \gamma (1 + \cos 2\varepsilon) - m \cos^2 \varphi (1 + \cos 2s)}.$$

Quoniam inclinatio  $\varphi$  minime a primitiva  $\gamma$  discrepat, ponatur  $\varphi = \gamma + \omega$ , erit  $\omega$  quantitas minima, et  $d\omega$  prae  $ds$  pro evanescente haberi potest. Fiet ergo  $d\varphi = d\omega$  et  $\cos \varphi = \cos \gamma - \omega \sin \gamma$ , que  $\cos^2 \varphi = \cos^2 \gamma - \omega \sin 2\gamma$ , quo valore substituto erit

$$du^2 = \frac{d\omega^2 + ds^2 \cos^2 \gamma - \omega ds^2 \sin 2\gamma}{\cos^2 \gamma - \omega \sin 2\gamma + m \cos 2\varepsilon + m \omega \sin 2\gamma - m \cos^2 \gamma \cos 2s + m \omega \sin 2\gamma \cos 2s},$$

$$du^2 = \frac{ds^2}{1 - m \cos 2s + \frac{m \cos 2\varepsilon + m \omega \sin 2\gamma}{\cos^2 \gamma - \omega \sin 2\gamma}} + \frac{d\omega^2}{\cos^2 \gamma + m \cos 2\varepsilon - \omega \sin 2\gamma - m \cos^2 \gamma \cos 2s},$$

vel approximando sit  $\frac{\cos 2\varepsilon}{\cos^2 \gamma} = \alpha$ , erit

$$du^2 = \frac{ds^2}{1 + m\alpha - m \cos 2s + 2m(1+\alpha)\omega \tan \gamma} + \frac{d\omega^2}{\cos^2 \gamma + m \cos 2\varepsilon - \omega \sin 2\gamma - m \cos^2 \gamma \cos 2s}$$

seu  $du^2 = \frac{ds^2}{1 + m\alpha} + \frac{m ds^2 \cos 2s}{(1+m\alpha)^2} + \frac{mm ds^2 \cos^2 2s}{(1+m\alpha)^3} - \frac{2m(1+\alpha)\omega ds^2 \tan \gamma}{(1+m\alpha)^2} + \frac{d\omega^2}{\cos^2 \gamma + m \cos 2\varepsilon}$

Ponatur  $\omega = A \cos 2\varepsilon - A \cos 2s$ , quo posito  $s = \varepsilon$  fiat  $\varphi = \gamma$ , erit  $d\omega = 2A ds \sin 2s$   
 $dd\omega = 2Add s \sin 2s + 4Ad s^2 \cos 2s$ . At ob  $dd\varphi = dd\omega$  et  $\sin \varphi = \sin \gamma + \omega \cos \gamma$ ,

$$\sin \varphi \cos \varphi = \sin \gamma \cos \gamma + \omega \cos 2\gamma,$$

et  $\frac{dd\varphi}{\sin \varphi \cos \varphi} = \frac{dd\omega}{\sin \gamma \cos \gamma} - \frac{\omega dd\omega \cos 2\gamma}{\sin^2 \gamma \cos^2 \gamma} = -(du - ds)^2 + mdu^2(1 + \cos 2s)$ .

Ergo habebitur

$$-(du - ds)^2 \sin \gamma \cos \gamma + mdu^2 \sin \gamma \cos \gamma (1 + \cos 2s) = 2Add s \sin 2s + 4Ad s^2 \cos 2s.$$

At prior aequatio dat

$$dd s = 4A(ds - du)ds \sin 2s (\tan \gamma + \frac{\omega}{\cos^2 \gamma}) + mdu^2 \sin 2s,$$

quo valore ibi substituto fiet

$$-(du - ds)^2 \sin \gamma \cos \gamma + mdu^2 \sin \gamma \cos \gamma (1 + \cos 2s) = 8AA(ds - du)ds \sin^2 2s (\tan \gamma + \frac{\omega}{\cos^2 \gamma}) + 2Amdu^2 \sin^2 2s + 4Ad s^2 \cos 2s.$$

At ex superiori aequatione est

$$du^2 = \frac{ds^2}{1 + m\alpha} + \frac{m^2 ds^2}{2(1+m\alpha)^3} + \frac{m ds^2 \cos 2s}{(1+m\alpha)^2} - \frac{2am(1+\alpha)Ad s^2 \sin \gamma \cos \gamma}{(1+m\alpha)^2} + \frac{2m(1+\alpha)Ad s^2 \tan \gamma \cos 2s}{(1+m\alpha)^2} + \frac{2AA ds^2}{(1+m\alpha) \cos^2 \gamma} - \frac{2AA ds^2 \cos 4s}{(1+m\alpha) \cos^2 \gamma} + \frac{mm ds^2 \cos 4s}{2(1+m\alpha)^3}$$