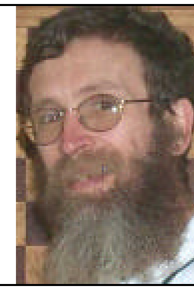
How Euler Did It

by Ed Sandifer



The Euler-Pythagoras theorem

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Euler didn't do a lot of geometry. Most of what he did falls into one of two categories. One category includes papers that were part of now-forgotten research agendas of the 1700's. Euler would usually do several papers on such a topic. His work on reciprocal trajectories and on the quadrature of Lunes both fall into this category. The other category includes papers that are solitary gems in the field, topics Euler visited once, created a masterpiece, and then moved on. His work on the Euler Line and on the so-called Euler Formula ($V - E + F = 2$, actually two papers) are examples here.

This month we look at a rare example from a third category of Euler's geometry, a topic he visited once, but the single paper that resulted from that visit is mostly forgotten. It is a beautiful result that I learned from Bill Dunham at a meeting of the Ohio Section of the MAA. Though Euler did not bother to mention it, the Pythagorean theorem is an easy corollary of the main result, hence the title of this column.

In 1748 Euler had been in the employ of Frederick II in Berlin for seven years. He published one of his most influential works, the *Introductio in analysin infinitorum*, often heralded as the world's first precalculus book, though that description is a gross oversimplification. He finished writing his first calculus book, the *Institutiones calculi differentialis*, though it would not be published until 1755. He "only" published nine books and papers in 1748, but he wrote more than 25. Most of his papers were about astronomy, especially the motions of the moon and planets, and about optics and the tools necessary to observe those motions. Euler's paper, *Variae demonstrationes geometricae*, or "Several proofs in geometry," is a bit of an oddball in the midst of all this astronomy. It is number 135 on Eneström's list of Euler's works.

As its title suggests, *Variae demonstrationes geometricae* presents several results, not all that closely related, and not all that new. First, he solves a fairly simple and not very interesting problem of Fermat. Then he does two theorems in triangle geometry. The first of this pair is the theorem relating the area of a triangle to its perimeter and the radius of its inscribed circle. The second is the theorem often known as Heron's theorem, giving the area of a triangle in terms of the lengths of its three sides. He also proves what we sometimes call Brahmagupta's theorem, giving the area of a quadrilateral inscribed in a circle in terms of the four sides of the quadrilateral. Euler attributes this theorem to one of his contemporaries, Philipp Naudé.

Finally, just three pages from the end of this 18-page article, he gets to the theorems that interest us today. Euler refers us to Figure 7 when he states the first of these theorems:

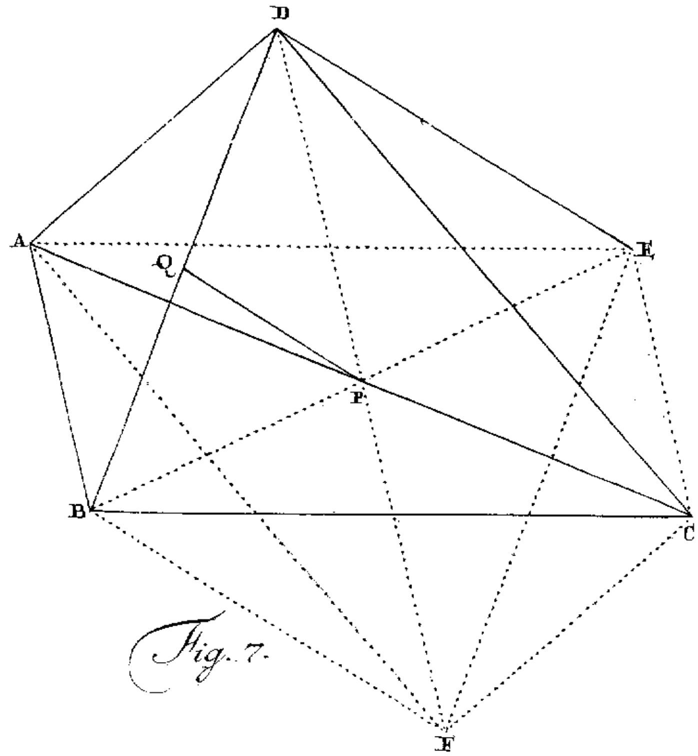


Fig. 7.

Theorem: Given any convex quadrilateral (*trapezio*) *ABCD* with diagonals *AC*, *BD*, if a parallelogram is completed about the two sides *AB*, *BC* to give the parallelogram *ABCE*, and if the two points *D* and *E* are joined to form the segment *DE*, then the sum of the squares of the four sides of the quadrilateral $AB^2 + BC^2 + CD^2 + DA^2$ will be greater than the sum of the squares of the diagonals $AC^2 + BD^2$ by the square of the segment *DE*, that is

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + DE^2$$

Proof: First, complete the three points *A*, *B*, *C* to form the parallelogram *ABCE*, as suggested in the wording of the theorem, and draw the diagonal *BE*. Further, construct *F* so that *CF* is parallel to *AD* and *BF* is parallel to *ED*. Since *BC=AE*, we get that the triangles *CBF* and *AED* are congruent.

Now, draw the lines *AF*, *DF* and *EF* and look at the two parallelograms *ADCF* and *BDEF* with diagonals *AC*, *DF* and *BE*, *DF* respectively. Euler cites a "property of parallelograms" to tell us that, in *ADCF* we have

$$2AD^2 + 2CD^2 = AC^2 + DF^2$$

and in *BDEF* we have

$$2BD + 2DE^2 = BE^2 + DF^2.$$

This is just an application of the Law of Cosines. We know that angles *ADC* and *DCF* are supplementary, so their cosines are negatives of one another. Then the Law of Cosines tells us that $AC^2 = AD^2 + CD^2 - 2\cdot AD\cdot CD\cdot\cos(ADC)$. Meanwhile, $DF^2 = AD^2 + CD^2 + 2\cdot AD\cdot DC\cdot\cos(ADC)$. Add these together to get Euler's property of parallelograms.

Solve each of these equations for DF^2, set the two parts equal to each other and add AC^2 to get

$$2AD^2 + 2CD^2 = 2BD^2 + 2DE^2 + AC^2 - BE^2$$

We have one parallelogram left, *ABCE*, where we know that

$$2AB^2 + 2BC^2 = AC^2 + BE^2.$$

Add this to the last equation to get

$$2AD^2 + 2CD^2 + 2AB^2 + 2BC^2 = 2BD^2 + 2DE^2 + 2AC^2.$$

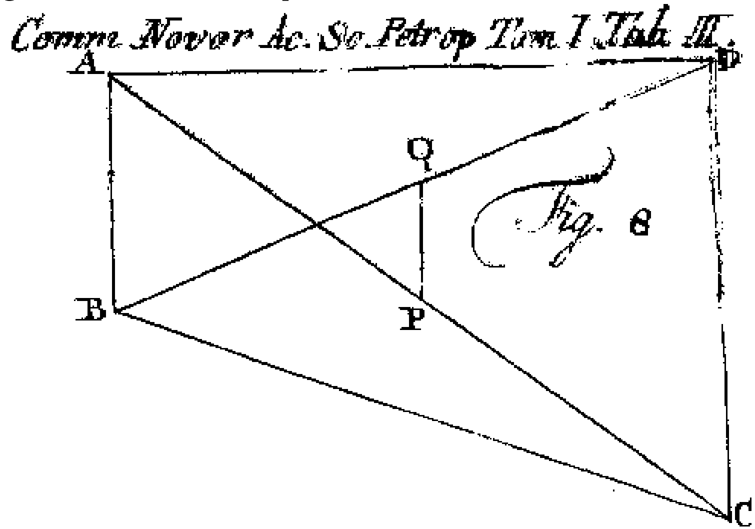
Divide by two and rearrange a little bit to get

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + DE^2,$$

as promised. Q. E. D.

Euler gives us four corollaries. The first three are fairly routine. First, if the quadrilateral is a parallelogram, then the interval *DE* vanishes so the sum of the squares of the sides exactly equals the sum of the squares of the diagonals. This isn't really fair, since this is exactly the "property of parallelograms" that Euler used to prove the theorem itself. Euler's second corollary is that the sum of the squares of the sides of a quadrilateral is always greater than or equal to the sum of the squares of the diagonals, with equality exactly when the quadrilateral is a parallelogram.

In Euler's third corollary he bisects diagonal AC at P and BD at Q, and draws segment QP. Then he shows that this last segment is half the length of DE, and so its square is one fourth of DE^2. This is satisfying because it gives us a way to avoid using that awkward auxiliary point E and to replace it with two points P and Q that are more naturally associated with the original quadrilateral.



Substituting $4PQ^2$ for DE^2 in the original theorem leads to Euler's fourth, and most interesting corollary, illustrated in Figure 8:

Corollary 4: In any quadrilateral *ABCD*, if its diagonals AC and BD are bisected by points P and Q, which are joined by segment PQ, then the sum of the squares of the four sides, $AB^2 + BC^2 + CD^2 + DA^2$ is equal to the sum of the squares of the two diagonals, $AC^2 + BD^2$ plus four times the square of the line PQ. That is to say,

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4PQ^2 .$$

Euler stopped here, but we don't have to. In the special case that *ABCD* is a rectangle, besides knowing that all four angles are right angles and that opposite sides are equal, we also know that $P=Q$ and $AC=BD$. Making appropriate substitutions, this tells us that in the right triangle *ABC*, we have

$$AB^2 + BC^2 + AB^2 + BC^2 = AC^2 + AC^2 + 4 \cdot 0 ,$$

which gives immediately the well known:

Theorem (Euler-Pythagoras): If *ABC* is a triangle with right angle at *B*, then

$$AB^2 + BC^2 = AC^2 .$$

In the same volume of the *Novi Commentarii* as Euler published E-135, his enthusiastic but less talented friend George Wolfgang Krafft (1701-1754) an article, *Demonstrationes duorum theorematum geometricorum*, "Proofs of two geometric theorems." One of Krafft's theorems is Euler's Corollary 4, proved in an almost entirely algebraic manner, based on the Law of Cosines. Today, most of us probably find Euler's geometric method more appealing. But in the 1740's the fashion in mathematics was becoming more and more algebraic, at the expense of geometry. There is every chance that people in Euler's and Krafft's time found Krafft's the more attractive proof.

We leave the reader with two thoughts. First, how much of Euler's work is still true if the quadrilateral *ABCD* is not convex, or is even self-intersecting?

The second thought is based on some remarks by Eisso Atzema. What if the four points *A, B, C* and *D* are not in a plane, so that they form the vertices of a tetrahedron. Then *ABCD* describes a circuit on that tetrahedron. The two edges that the circuit does not use, *AC* and *BD* are the segments that are the diagonals in the planar case. How much of Euler's work survives this excursion into three dimensions?

References:

[E35] Euler, Leonhard, Variae demonstrations geometricae, *Novi commentarii academiae scientiarum Petropolitanae*, **1** (1747/48) 1750, pp. 49-66, reprinted in *Opera Omnia* Series I vol 26 pp. 15-32. Available through The Euler Archive at , www.EulerArchive.org

[K] Krafft, George Wolfgang, Demonstrationes duorum Theorematum Geometricorum, , *Novi commentarii academiae scientiarum Petropolitanae*, 1 (1747/48) 1750, pp. 131-136.

Illustrations from The Euler Archive at www.EulerArchive.org.

Ed Sandifer (SandiferE@wcsu.edu) is Professor of Mathematics at Western Connecticut State University in Danbury, CT. He is an avid marathon runner, with 32 Boston Marathons on his shoes, and he is Secretary of The Euler Society (www.EulerSociety.org)

How Euler Did It is updated each month.