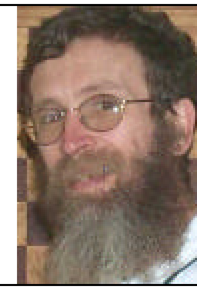
How Euler Did It

by Ed Sandifer



Knight's Tour

April 2006

It is sometimes difficult to imagine that Euler had a social life, but it is not surprising that he could find mathematics in what other people did for fun. He begins the article we are considering this month by writing:

> "I found myself one day in a company where, on the occasion of a game of chess, someone proposed this question:
>
> *To move with a knight through all the squares of a chess board, without ever moving two times to the same square, and beginning with a given square*."

This is the problem now known as the Knight's Tour, and is an early special case of a Hamiltonian path on a graph, a problem that still occupies graph theorists.

Euler wrote this in the early 1750's, a time when a chess fad swept the courts of Europe. In 1751 the great chess master and good composer François-André Danican Philidor (1726-1795), whose games are still studied today, played before Frederick the Great at Potsdam and went on to visit Berlin. Euler might have met Philidor, or maybe not. Either way, it seems that Euler caught the Chess Bug, too. There are stories that he took up the game but was disappointed with how well he played. So he took some lessons and became "very good." I have been unable to verify these stories, and none of his chess games seem to have survived, so it is hard to know how good he might have been.[1]

Euler apparently wrote this article in 1758, though he had mentioned the Knight's tour in a letter to Goldbach in 1757. [J+W] The article was published in the 1759 volume of the Berlin *Mémoires*, which, because of the Seven Years War (1756-1763), was not actually published until 1766. It was published again in 1849 in a posthumous collection of Euler's works. He mentions that this paper is based on "a particular idea that Mr. Bertrand[2] of Geneva gave me." After this paper, Euler did not return to mathematical problems in chess. He came very close, though. Knight's tours are closely related to a kind of magic square called "pandiagonal," and Euler wrote about pandiagonal magic squares in 1779, when he wrote *Recherches sur un nouvelle espèce de quarrés magiques* (Researches on

[1] There are also stories that Euler composed some music, or maybe designed an algorithm for composing music, and that the results were awful. I have been unable to find any substance to these stories, either.

[2] Louis Bertrand (1731-1812)

a new kind of magic squares)[E530]. This is a very long paper, but I cannot find that Euler mentions its connections to the Knight's tour.

A bit later in E 309, Euler writes:

> "3. To make this question a bit clearer, I show here a route where, in beginning in one corner of the chess board, one moves through all the squares:"

42	59	44	9	40	21	46	7
61	10	41	58	45	8	39	20
12	43	60	55	22	57	6	47
53	62	11	30	25	28	19	38
32	13	54	27	56	23	48	5
63	52	31	24	29	26	37	18
14	33	2	51	16	35	4	49
1	64	15	34	3	50	17	36

Fig. 1. Euler's first example of an open Knight's tour

The numbers here mark the order of the squares that the Knight visits. In this example, the Knight starts in the lower left hand corner, and finishes in the square just to the right of the starting point. This method of describing a tour is a bit hard to follow, so we will substitute a more modern and more graphical notation for the same tour, though Euler did not use this kind of notation:

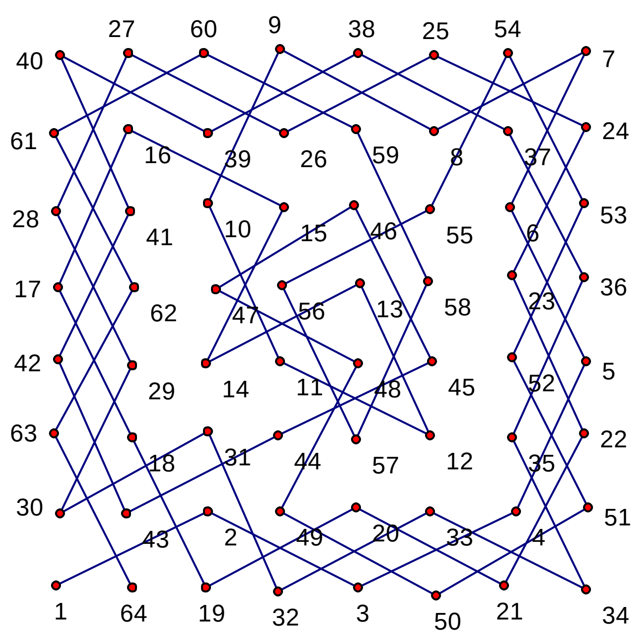


Fig 2. A graphical description of the same tour.

Since the Knight cannot move directly from its ending position back to its starting position, Euler says that this tour is “not re-entrant upon itself.” We would call it an “open” tour or a Hamiltonian path, as opposed to a “closed” tour or a Hamiltonian circuit.

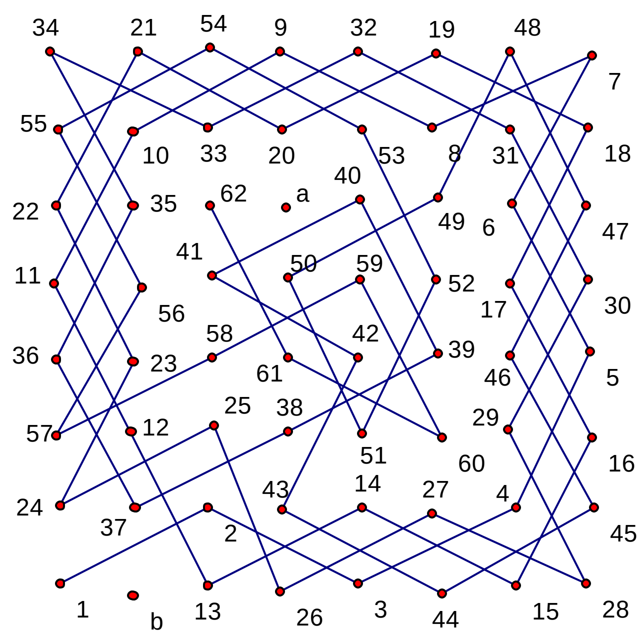
He gives us an example of a closed tour, “a route that is re-entrant upon itself,” and notes that this gives a great many equivalent tours, starting this tour in any square, and traversing the numbers either forward or backward.

After these introductory comments and examples, the paper can be divided into several parts.

In paragraphs 9 to 14, he shows how new tours can be made from old ones by a technique that reconnects some of the steps in the path. This is probably the technique that he had learned from Bertrand.

Paragraphs 15 to 17 are devoted to using the reconnecting technique to extend a path that does not visit all the squares into a tour. Since the resulting tour is likely to be open, he spends paragraphs 18 to 24 showing how to reconnect an open tour to make it into a closed tour. This technique involves a lot of branching, and probably isn’t very computationally efficient.

Starting with paragraph 25, Euler looks for tours with certain kinds of symmetries, like visiting first one half of the board, then the other half. Paragraphs 35 to 41 consider tours on boards that are not standard 8x8 chess boards. He looks at rectangular boards in paragraphs 42 and 43, and his last paragraph gives examples of four tours on boards that are shaped like crosses.



To get a sense of the main technique of the paper, rather than look at all the details, let’s look at how Euler completes an incomplete tour.

Euler chooses the example at the right, starting in the lower left corner and ending at step 62, missing the squares labeled *a* and *b*.

First, make a list of the squares that can be reached from the last square, number 62:

9, 53, 59, 61, 23, 11, 55 and 21.

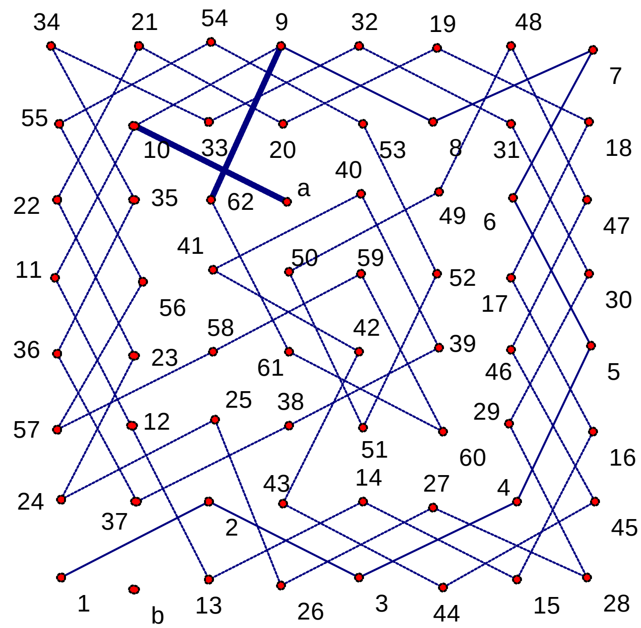
Now, also make a list of the squares that can be reached from the missing square *a*:

32, 8, 52, 42, 58, 56, 10 and 54.

We notice that there are some squares in the first list for which the next square also appears on the second list, 9 and 10, 53 and 54, and 55 and 56. Euler could have selected any of these pairs, but he picks the pair 9 and 10 to reconnect the path.

Euler revises the path as follows. Instead of going from 9 to 10, he goes from 9 to 62. Then he traverses the path backwards until he gets to square 10, and from there he can reach the missing square *a*. He writes the new path as:

1 … 9 – 62 … 10 – *a*.



The resulting path is shown at the right, with the connection from 9 to 10 that Euler removes shown with a dashed line, and the two new connections, 9 to 62 and 10 to *a*, shown with thick lines.

This leaves *b* still to be added to the tour. Euler doesn't tell us why, but he doesn't try to connect *b* at the beginning of the tour. That would not be immediate. From *b* we can reach squares 57, 25 and 43, and 1 can reach only 2 and 12, and none of the resulting pairs are consecutive.

So, Euler lists again all the squares that can be reached from the new last square *a*:

32, 8, 52, 42, 58, 56, 10 and 54.

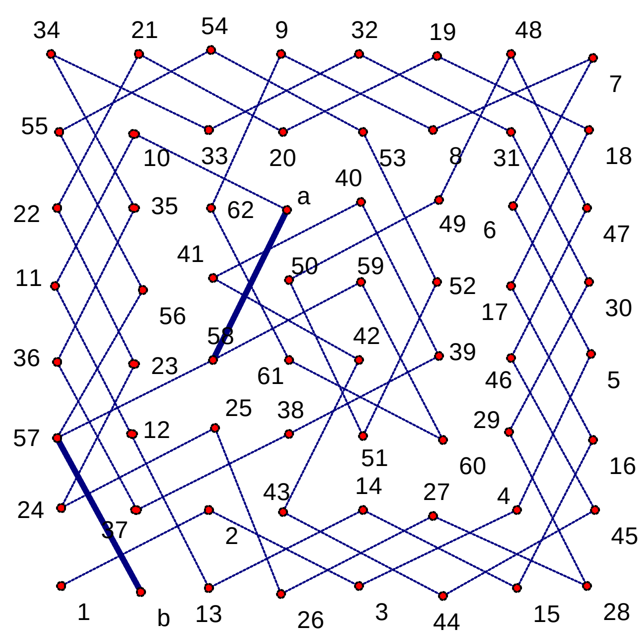
The squares that can be reached from *b* are

57, 25 and 43,

and again we have a pair of consecutive squares, 58 in the first list, which, in the revised path is followed by 57 in the second list. Note the complication, that we have to refer to the new path that Euler writes 1 … 9 – 62 … 10 – *a*, and that in this path, 9 and 10 would not have been consecutive, but 9 and 62 are.

So, Euler makes another transformation. He disconnects square 58 from square 57, and instead connects it to square *a*. Then he traverses the path from *a* to 57 in the opposite direction, and connects 57 to the missing square *b*. He writes this new path as

1 … 9 – 62 … 58 – *a* – 10 … 57 – *b*.

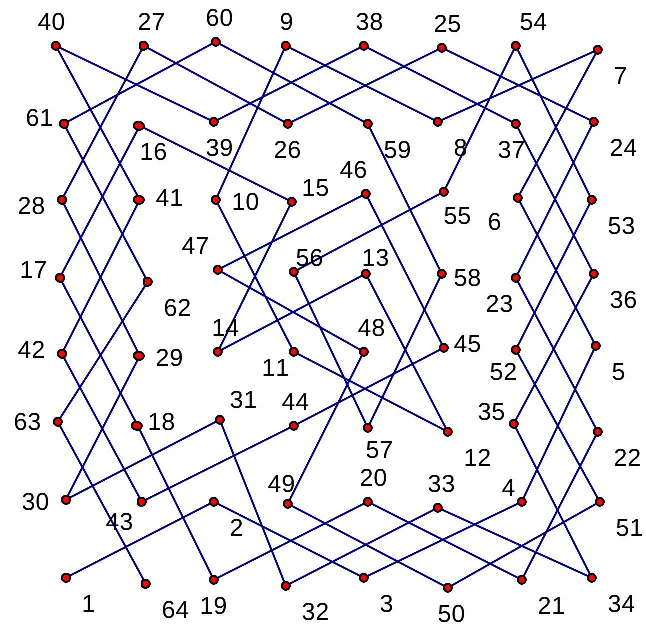


We show the new path at the right, using dashes and thick lines as before.

This example hides some of the difficulties that can arise. For example, we usually see chess boards colored alternating red and black squares. A knight's move always takes it from a square of one color to a square of the other color. If we have a partial path that starts and ends on squares of the same color, then we cannot attempt to complete the path with a move to a square of that same color.

Further, if we have a partial path that starts and ends on squares of opposite colors, then we must try to add a missing square to the appropriately colored end of the path. Euler mentions these parity issues later in the paper when he is talking about tours on rectangles, but he does not mention it in this part of the paper.

Now that Euler has completed his partial tour to construct an open one, he wants to show us how to transform the open tour into a closed tour. His first step is to renumber the squares in their "natural order," that is the order in which this tour visits the squares instead of the order they were visited before he completed the tour. This relabeled tour is at the right.



The process of closing an open tour is quite complicated, and this particular tour is more complicated than some because the starting point and the end point are near a corner, so there aren't as many ways to transform the tour. We will only summarize Euler's calculations.

Euler begins by listing the squares that can be reached from square 64:

63, 31 and 49.

Transposing at 64 does not result in a new tour, so he creates two new tours by transposing 64 first with 31 and then with 49. He names his new tours I and II, and describes them:

I. 1 … 31 – 64 … 32,
II. 1 … 49 – 64 … 50.

Now he reverses these tours, not changing their names but describing them as

I. 32 … 64 – 31 … 1,
II. 50 … 64 – 49 … 1.

The way Euler does it, this makes the calculations slightly easier because the two tours have the same endpoint. Now the last square, 1, connects to 2 and to 18. Transposing at 2 doesn't change anything, so he transposes at 18 and gets two new tours:

A. 32 … 64 – 31 … 18 – 1 … 17,
B. 50 … 64 – 49 … 18 – 1 … 17.

Both of these tours end at square 17, which, in turn, connects to squares

16, 10, 14 and 18.

He makes the four transpositions that change anything, and gets new tours C, D, E and F, starting at squares 32 or 50, and ending at 11 or 15. He makes all possible transformations at square 11 and gets tours G, H, I, K, L, M, N, O, P and Q,[3] and also all transformations at square 15 to get tours g, h, i, k, l, m, n, o, p, q, r and s. Finally, he finds that one of the transformations that arise from tour G leads to a closed tour, and he finds the closed tour:

32 … 42 – 47 … 64 – 31 … 18 – 1 … 10 – 17 … 11 – 46 … 43.

He rewrites this so that it begins with square 1, and gets the final form of his closed tour:

1 … 10 – 17 … 11 – 46 … 43 - 32 … 42 – 47 … 64 – 31 … 18.

This method of closing an open path certainly is not computationally efficient, but that may be because it is a difficult problem, and there may be no better algorithm.

George Jelliss has written a nice history and a plethora of results on the Knight's tour and related topics. Readers who want to know more should surely visit his site.

References:

[E309] Euler, Leonhard, Solution d'une question curieuse que ne paroit soumise à aucune analyse, *Mémoires de l'académie des sciences de Berlin*, **15** (1759) 1766, p. 310-337, also in *Commentationes arithmeticae* **1**, 1849, p. 337-355, reprinted in *Opera Omnia* Series I vol 7, p. 26-56. Available through The Euler Archive at www.EulerArchive.org and at the Digital Library at Berlin-Brandenburgische Akademie der Wissenschaften, http://bibliothek.bbaw.de/bibliothek-digital/digitalequellen/schriften/.

[E530] Euler, Leonhard, Recherches sur un nouvelle espéce de quarrés magiques, *Verhandelingen uitgegeven door het zeeuwsch Genootschap der Wetenschappen te Vlissingen*, **9**, Middelburg, 1782, p. 85-239, also in *Commentationes arithmeticae* **1**, 1849, p. 302-361, reprinted in *Opera Omnia* Series I vol 7, p. 291-392. Available through The Euler Archive at www.EulerArchive.org.

[J] Jelliss, George, "Knight's Tour Notes," http://www.ktn.freeuk.com/index.htm, March 10, 2006.

[J+W] Juskevic, A. P., and E. Winter, eds., *Leonhard Euler und Christian Goldbach: Briefwechsel 1729-1764*, Akademie-Verlag, Berlin, 1965.

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How Euler Did It is updated each month.

[3] Note he skips J because at the time I and J were the same letters.