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How Euler Did It

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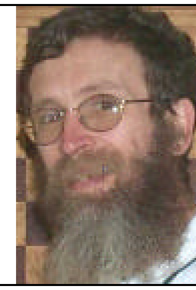

Inexplicable functions

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Imagine my surprise when I was looking at Euler's *Calculi differentialis*. [E212] There, deep into part 2 (the part that John Blanton hasn't translated yet), I saw the odd title of chapter 16, *De differentiatione functionum inexplicabilium*, "On the differentiation of inexplicable functions." That made me curious. What was an "inexplicable function?"

In the nine chapters of part 1, Euler had taught us how to take derivatives of polynomials, of algebraic functions, of transcendental functions, to take higher derivatives, and to solve certain kinds of differential equations. Part 2 is about twice as long as part 1, both in number of pages (278 vs 602) and number of chapters (18 vs 9). The first nine chapters of part 2 mostly involve series. Chapters 10 to 13 are about applications of calculus to finding maxima and minima and to finding roots of equations, and the last five chapters seem to be a kind of grab bag of applications like interpolation and partial fractions. Inexplicable functions are in that grab bag.

Though he defines inexplicable functions as those that are neither algebraic nor transcendental, he doesn't have a very complete idea of what a transcendental function is, and he has only two kinds of examples of inexplicable functions. He might as well have defined inexplicable functions as being functions that are like his examples.

Both of his examples come out of the work on the interpolation of sequences that he presented in his letter to Goldbach of October 13, 1729. That work led to the results on the gamma function and the constant gamma that we have described in the last two columns.

Euler's first example generalizes the partial sums of the harmonic series. In this paper he calls all of his inexplicable functions S, but we'll call this particular one $H(x)$ and write it as

$$H(x) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x},$$

where x is not necessarily a whole number. Even though Euler had shown in 1729 [E20] that this could be defined as a definite integral

$$H(x)=\int_0^1 \frac{1-y^x}{1-y}\,dx,$$

he tells us here that this function "can not be explained in any way." This is probably because the *Calculi differentialis* was designed as a textbook, and its only prerequisite was his *Introductio in analysin infinitorum*. He did not expect that readers would know a result from a twenty-five year old research paper.

Euler also considers related sums like $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots+\frac{1}{2x-1}$ and $1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\cdots+\frac{1}{x^n}$, where, again, x is not necessarily an integer.

It should be no surprise that Euler's second class of examples generalizes the factorial function. It isn't quite the gamma function, because the gamma function is a shift of the factorial function $\Gamma(n+1)=n!$. Euler denoted this by S as well, but we'll call his version $F(x)$. (F is for *factorial*, a word that was not used in Euler's time.) Euler's definition amounts to

$$F(x)=1\cdot 2\cdot 3\cdot 4\cdots x,$$

where, again, x need not be a whole number. He also considers products like $\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdots\frac{2x-1}{2x}$, and a couple of other examples built out of his two basic kinds of inexplicable functions. His fifth example is to differentiate our $F(x)$. It follows three examples based on sums and the one based on products.

Euler had to work without the benefits of subscript notations, which would not become popular for another 50 years, so much of his notation here will seem quite awkward. He took

$$S = A + B + C + D + \ldots + X,$$

where X is the value of the x-th term of the summation. Here, x is allowed to be a fraction, and for the generalized harmonic function, $X = {}^1\!/_x$. He denotes by X', X'', X''', etc., the values of the $(x+1)$-st, $(x+2)$-nd, $(x+3)$-rd, etc., and he denotes the "term at infinity" by $X^{|\infty|}$. Then he writes successive sums as

$$S' = S + X'$$
$$S'' = S + X' + X''$$
$$S''' = S + X' + X'' + X''', \text{and finally}$$
$$S^{|\infty|} = S + X' + X'' + X''' + \cdots + X^{|\infty|}.$$

We take ω to be an infinitely small number. Then, to take the derivative of the sum given by S, Euler wants to compare the sum of the first x terms to Σ, the sum of the first $x + w$ terms. Much like he did with the Ss and the Xs, he takes Z to be the term corresponding to $x + w$, and calls successive terms Z', Z'', Z''', and the term at infinity $Z^{|\infty|}$. Continuing his analogy between S and S, he writes

$$\Sigma' = \Sigma + Z'$$
$$\Sigma'' = \Sigma + Z' + \Sigma''$$
$$\Sigma''' = \Sigma + Z' + Z'' + Z''', \text{ and finally}$$

$$\Sigma^{|\infty|} = \Sigma + Z' + Z'' + Z''' + \cdots + Z^{|\infty|}.$$

Euler is not explicitly concerned with whether the series that give $S^{|\infty|}$ and $\Sigma^{|\infty|}$ converge. In fact, he knows that they diverge, but he is concerned with *how* they diverge. He tells us, "Now, the nature of the series S, S', S'', S''', etc., when it is continued to infinity, will be like an arithmetic progression if the sequence of terms X, X', X'', X''', etc. converges when it is continued to infinity."

That's not very clear, but it means that if $\lim_{n\to\infty} X^{(n)} = b$, then the sequence $X^{|\infty|}, X^{|\infty+1|}, X^{|\infty+2|}$, etc. is like an arithmetic sequence with difference *b*.

Euler denotes the value to which the *X*s converge by $X^{|\infty+1|}$, and claims that $\Sigma^{|\infty|} = S^{|\infty+w|}$. Further, $S^{|\infty+w|}$ ought to lie naturally between $S^{|\infty|}$ and $S^{|\infty+1|}$. Since, at infinity, the *X*s form an arithmetic sequence, Euler is comfortable interpolating $S^{|\infty+w|}$, giving it the value

$$\Sigma^{|\infty|} = S^{|\infty+w|} = S^{|\infty|} + wX^{|\infty+1|}.$$

Euler gathers his tools to get first that

$$\Sigma^{|\infty|} = S^{|\infty+w|} = S + X' + X'' + X''' + \cdots X^{|\infty|} + wX^{|\infty+1|},$$

and then that

$$\Sigma^{|\infty|} = \Sigma + Z' + Z'' + Z''' + \cdots + Z^{|\infty|}.$$

Together, these last two equations relate Σ and S as

(1) $$\Sigma = S + wX^{|\infty+1|} + X' + X'' + X''' + \cdots - Z' - Z'' - Z''' - \text{etc.}$$

Moreover, if the terms X, X', X'', X''' go to zero, that is to say if $X^{|\infty+1|} = 0$, then we get to ignore the term $wX^{|\infty+1|}$ in this expression.

We can do this calculation in the special case of Euler's generalized harmonic sum. Then we get

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{x}.$$

Euler rearranges the terms in his formula for Σ to get

$$\Sigma = S + (X' - Z') + (X'' - Z'') + (X''' - Z''') + \text{etc.}$$

Here, because $X^{|\infty+1|} = 0$, we can leave out the term $wX^{|\infty+1|}$.

Furthermore,

$$X' = \frac{1}{x+1} \qquad Z' = \frac{1}{x+1+w} \qquad X' - Z' = \frac{w}{(x+1)(x+1+w)}$$
$$X'' = \frac{1}{x+2} \qquad Z'' = \frac{1}{x+2+w} \qquad X'' - Z'' = \frac{w}{(x+2)(x+2+w)}$$
$$X''' = \frac{1}{x+3} \qquad Z''' = \frac{1}{x+3+w} \qquad X''' - Z''' = \frac{w}{(x+3)(x+3+w)}$$

etc. etc. etc.

In a transformation that seems to hold absolutely no hope of progress, Euler expands each of the factors $\frac{1}{x+k+w}$ as a Taylor series to get

$$\frac{1}{x+1+w} = \frac{1}{x+1} - \frac{w}{(x+1)^2} + \frac{w^2}{(x+1)^3} - \frac{w^3}{(x+1)^4} + \text{etc.},$$
$$\frac{1}{x+2+w} = \frac{1}{x+2} - \frac{w}{(x+2)^2} + \frac{w^2}{(x+2)^3} - \frac{w^3}{(x+2)^4} + \text{etc.}, \text{ and so forth.}$$

Finally, Euler substitutes these expansions in for $X' - Z'$, $X'' - Z''$, etc. in his expression for Σ, subtracts S, substitutes dS for $\Sigma - S$, and substitutes dx for w to get his final, and rather disappointing answer:

$$\begin{aligned} dS = dx &\left(\frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \frac{1}{(x+4)^2} + \text{etc.} \right) \\ - dx^2 &\left(\frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \frac{1}{(x+4)^3} + \text{etc.} \right) \\ + dx^3 &\left(\frac{1}{(x+1)^4} + \frac{1}{(x+2)^4} + \frac{1}{(x+3)^4} + \frac{1}{(x+4)^4} + \text{etc.} \right) \\ - dx^4 &\left(\frac{1}{(x+1)^5} + \frac{1}{(x+2)^5} + \frac{1}{(x+3)^5} + \frac{1}{(x+4)^5} + \text{etc.} \right) + \text{etc.} \end{aligned}$$

The basic idea here was to make the step from S to Σ by using the terms X', X'', X''', etc. to count up to infinity, to make the step from $X^{|\infty|}$ to $Z^{|\infty|} = X^{|\infty+w|}$ using the properties of the limit of the Xs, and then to count back from infinity using the terms Z', Z'', Z''', etc. It is clever, but outrageous.

As we mentioned earlier, Euler does a few more examples interpolating sums of sequences, including the special series closely related to the Riemann zeta function:

$$S = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots + \frac{1}{x^n}.$$

Before he leads us on to inexplicable functions built from products rather than sums, Euler does a bit more analysis to help us understand what happens if the terms of S neither vanish nor converge to some non-zero value. Formula (1) above describes the difference between S and Σ if the terms converge, and if they converge to zero, then we get to ignore the term $wX^{|\infty+1|}$.

If the terms don't converge, then things are considerably more complicated. Let us look at the special case where the terms don't converge, but the difference between the terms does converge, as happens, for example, with the sequence whose general term is $\ln x$.

Consider three consecutive partial sums "at infinity" that Euler would denote as $S^{|\infty|}$, $S^{|\infty+1|}$ and $S^{|\infty+2|}$. Their first differences will be $S^{|\infty+1|} - S^{|\infty|} = X^{|\infty+1|}$ and $S^{|\infty+2|} - S^{|\infty+1|} = X^{|\infty+2|}$. Now we see that the second differences will be $X^{|\infty+2|} - X^{|\infty+1|}$, and we are assuming that this difference between the terms does converge. Note that the difference between the terms is the *second* difference between the partial sums.

Now we compare $S^{|\infty|}$ to $\Sigma^{|\infty|}$ and get

$$\Sigma^{|\infty|} = S^{|\infty+w|} = S^{|\infty|} + wX^{|\infty+1|} + \frac{w(w-1)}{1\cdot 2}\left(X^{|\infty+2|} - X^{|\infty+1|}\right)$$

From this, we get a kind of second-difference analogue to Formula 1:

$$\begin{aligned}\Sigma = S + {} & X' + X'' + X''' + X'''' + \text{etc.}\\ & + wX^{|\infty+1|} + \frac{w(w-1)}{1\cdot 2}\left(X^{|\infty+2|} - X^{|\infty+1|}\right)\\ & - Z' - Z'' - Z''' - Z'''' - \text{etc.}\end{aligned}$$

For $X^{|\infty+1|}$, Euler substitutes $X' + (X'' - X') + (X''' - X'') + (X'''' - X''') + (X''''' - X'''') + \text{etc.}$, and for $X^{|\infty+2|} - X^{|\infty+1|}$ he writes $X'' - X' + \left((X''' - 2X'' + X') + (X'''' - 2X''' + X'') + (X''''' - 2X'''' + X''') + \text{etc.}\right)$. This gives him the astonishing but awkward formula,

$$\begin{aligned}\Sigma = S + {} & X' + X'' + X''' + X'''' + \text{etc.}\\ & + wX' + w\left((X'' - X') + (X''' - X'') + (X'''' - X''') + (X''''' - X'''') + \text{etc.}\right)\\ & + \frac{w(w-1)}{1\cdot 2}X'' - \frac{w(w-1)}{1\cdot 2}X' \qquad (2)\\ & + \frac{w(w-1)}{1\cdot 2}\left((X''' - 2X'' + X') + (X'''' - 2X''' + X'') + (X''''' - 2X'''' + X''') + \text{etc.}\right)\\ & - Z' - Z'' - Z''' - Z'''' - \text{etc.}\end{aligned}$$

Note that this applies to series for which the general terms need not converge, but the difference between consecutive terms do converge, and that $\ln 1 + \ln 2 + \ln 3 + \ldots + \ln x$ is one such series.

Similar, but even more complicated formulas are possible for sums for which the second differences or third differences of the general terms converge.

Now we are ready to sketch how Euler took the derivative of his version of the gamma function, the function we are writing as

$$F(x) = 1\cdot 2\cdot 3\cdot 4\cdots x.$$

First he takes logarithms of both sides and gets

$$\ln F(x) = \ln 1 + \ln 2 + \ln 3 + \ln 4 + \cdots + \ln x$$

In this series the terms do not vanish, but the differences between the terms do vanish. Euler demonstrates this with the following calculation:

$$\ln(\infty + 1) - \ln(\infty) = \ln\left(1 + \frac{1}{\infty}\right) = \frac{1}{\infty} = 0.$$

This gives Euler license to use his "astonishing but awkward" Formula (2). It seems to lead to mayhem as he substitutes dx for ω and then replaces terms like $X' - Z'$ with Taylor series approximations. Eventually it simplifies a bit and he gets

$$\begin{aligned} \ln F(x) = {} & x\left(\ln\frac{2}{1} + \ln\frac{3}{2} + \ln\frac{4}{3} + \ln\frac{5}{4} + \text{etc.}\right) \\ & - x\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.}\right) \\ & + \frac{1}{2}x^2\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \text{etc.}\right) \\ & - \frac{1}{3}x^3\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \text{etc.}\right) \\ & + \frac{1}{4}x^4\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \text{etc.}\right) + \text{etc.} \end{aligned} \tag{3}$$

Note the first two series on the right. The first series expands as

$$\ln 2 - \ln 1 + \ln 3 - \ln 2 + \ln 4 - \ln 3 + \ln 5 - \ln 4 + \text{etc.}$$

This telescopes to give

$$\ln 1 - \ln(n+1) = -\ln(n+1)$$

Meanwhile, the second series in Formula (3) is just the harmonic series. We know that as n goes to infinity, the difference between the logarithm and the nth partial sum of the harmonic series approaches the Euler-Mascheroni constant, now denoted γ. Euler knew this value to be approximately 0.5772156649015325, and he wrote it out like that rather than denoting it by a symbol. We will use γ.

As an additional concession to modern notation, we'll also note that the other series that appear in Euler's expression for $\ln F(x)$ are values of the Riemann zeta function, and we'll do what Euler couldn't do and write them as $\zeta(2)$, $\zeta(3)$, $\zeta(4)$, etc. With these notations, Euler's expression for $\ln F(x)$ can be written as

$$\ln F(x) = -x\gamma + \frac{1}{2}x^2\zeta(2) - \frac{1}{3}x^3\zeta(3) + \frac{1}{4}x^4\zeta(4) - \text{etc.}$$

Now for the climax. This differentiates to give

$$\frac{dF(x)}{F(x)} = -\gamma\, dx + x\zeta(2)\,dx - x^2\zeta(3)\,dx + x^3\zeta(4) - \text{etc.}$$

Taking $x = 0$ so that $F(0) = 0! = 1$, and this last formula gives the derivative of the factorial function in terms of the constant gamma:

$$\left.\frac{dF(x)}{dx}\right|_{x=0} = -\gamma\ ,$$

or, in terms of the gamma function,

$$\Gamma'(1) = -\gamma\ .$$

This ends the story that has extended over our last three columns. Euler discovered both objects, gamma the function and gamma the constant, early in his career while working on problems in the "interpolation of functions", that is, giving meaningful values to functions that are initially defined only on the integers. Later, Euler showed that the derivative of the function at $x = 1$, is the negative of the constant. Though this result is widely known, it does not seem so well known that the result is due to Euler. Finally, well after Euler's death, the two different objects happed to be given the same name.

What a remarkable coincidence.

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