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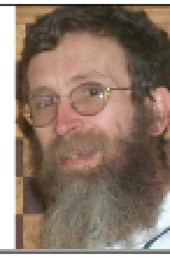
How Euler Did It

by Ed Sandifer



Euler and Gravity

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The popular myth of the discovery of gravity goes something like this: one day, an apple fell on the head of a young Isaac Newton. After pondering this event, Newton wrote down an equation describing an invisible force, which he called gravity. This equation united ideas about the paths of cannon balls and apples (terrestrial motion) with the paths of moons and planets (celestial motion). Once it was written down, it elegantly and easily explained the motion of all the planets and moons, and remained unquestioned, revered, and perfect for centuries (at least until Einstein).

Readers who are familiar with the history of science know better; nothing is ever so simple. In fact, Newton's theory was accepted, tested, clung to, retested, questioned, revised and un-revised, and finally accepted again during the 75 years after the publication of Newton's *Principia*. More importantly for our purposes, Euler was at the center of all of it.

Prelude to a crisis

De Causa Gravitatis

Euler had been thinking about gravity even before the worst of the chaos mentioned above. In 1743, he published an anonymous essay, *De Causa Gravitatis* [Euler 1743]. The essay itself is remarkable for two reasons: first, it was published anonymously. This seems to have been a technique that Euler used to get ideas out to the public without having to publicly defend them, which gave him greater leeway to experiment with bold claims. Furthermore, it was published only two years after he arrived at his new job in Berlin, and written even closer to the time of his move, and we might guess that Euler was especially keen not to publish anything controversial before he had settled in. The second remarkable thing about this paper is that it does not have an Eneström number. Then man who cataloged all of Euler's works early in the 20th century missed this one completely. We only know today that Euler is the author because Euler admitted it in a letter written 22 years later to Georges-Louis Lesage (see [Kleinert 1996] for an interesting introduction to *de Causa* and its discovery).

In this paper Euler examines the standard Newtonian view that gravity is a fundamental property of all bodies, and can't be derived from any more basic principle. Euler is not convinced that this is true, and gives an example of how an inverse square law might be the result of some other property of the universe. Euler takes as a given that the universe is permeated by the *ether*, a subtle fluid existing everywhere that permits light (among other things) to travel the distances between astronomical bodies. Next, since Bernoulli's principle holds that fluid moving quickly relative to a body has lower density than fluid that is stationary, it may be reasonable to assume that the density of the ether near the Earth is less than that far away from the Earth.

If we further assume that the pressure drops off with the reciprocal of the distance to the Earth, we find something interesting. Euler writes:

"Let the absolute compression of the ether (when it is not lessened) be denoted c. And let the distance from the center of the Earth be equal to c minus a quantity reciprocally proportional to x. And so let a compression of the ether in distance x from the center of the Earth be $C = c - cg/x$. From this, if body $AABB$ is in position around the Earth, the surface of it above AA will be pressed downwards by force $= c - cg/CA$, moreover the surface below BB will be pressed upwards by a force $c - cg/CB$, which force, since it is less than before, the body will be pressed downwards by force $= cg(1/CA - 1/CB) = cg.AB/(ACBC)$.

And so if the magnitude of body AB is incomprehensibly less than distance CB, we will have $AC = BC$, from this the gravity of such a body in whatever distance from the center will be as $1/AC^2$; this is reciprocally as a square of the distance from the center." (translation by DeSchmidt and Klyve [Euler 1743])

This is, frankly, rather bizarre. Euler here suggests that the thing that holds us to the ground is a pushing force from the (comparatively) dense ether above us. We should note here that there's no evidence that Euler took this explanation seriously (probably the reason for his anonymous publication in the first place). Rather, he seems to want to use this simple explanation as a reminder that we shouldn't stop looking for a "first cause" of gravity, and that we may be able to derive the inverse square law from an earlier principle.

The gravity crisis

A few years after the publication of *De Causa Gravitatis*, Newton's inverse square law met with challenges considerably greater than those provided by philosophical speculations. New developments in observational technology allowed astronomers to measure the location of the moon and planets to a precision not before achieved. There was just one problem – the new accurate measurements failed to conform to those predicted by the theory. In fact, there were three significant problems with the predictions of Newton's laws, and three attempts to fix them. We turn next to these.

Problem 1: Lunar apsides

It was well known that Newton's laws implied that bodies orbiting Earth would move in an ellipse. When carefully applied, in fact, Newton's laws also show that this ellipse will itself slowly revolve around the Earth, as a consequence of the gravitational force of the sun. The pattern of revolution of the "lunar apsis" is complicated, but averages about 3° per month (see Figure 2), a fact well established through observations in the early eighteenth century.

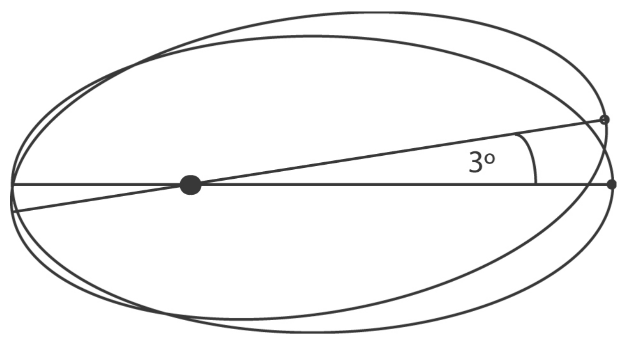

Figure 2: Precession of the apsidal line – exaggerated
(figure courtesy of Rob Bradley)

The problem with this was that the revolution calculated using Newton's inverse-square law for gravitation did not give a figure of 3° per month; it gave only half that. The three leading mathematicians of the time – Jean d'Alembert, Alexis Clairaut, and Euler – had attempted the calculation, and each of them came inescapably to the same conclusion that Newton himself had in the *Principia*: that Newton's laws suggested a monthly apsis revolution of only 1.5°. What was going on? (For an engaging and more detailed account of the issue of the lunar apsis, readers are encouraged to read Rob Bradley's article [Bradley 2007] on this subject.)

Problem 2: Jupiter and Saturn

The interactions of Jupiter and Saturn are devilishly complicated. The basic idea is that when Jupiter starts to catch up to Saturn, it pulls "backward" on Saturn, causing the ringed planet to lose kinetic energy. Saturn then falls to a closer orbit and, a bit surprisingly, speeds up (as a consequence of Kepler's third law). When Jupiter passes Saturn, it pulls "forward", giving Saturn more kinetic energy, moving it to a farther orbit, and slowing it down. The mathematics of all of this is made difficult by the planets' elliptical orbits (see Wilson [Wilson 2007] for details), but it's possible to work out, using calculus, just what the change in Jupiter's and Saturn's orbits should be. As was the case with the moon, however, the theoretically calculated difference did not match the observed difference.

Problem 3: The eclipse of 1748

The first two problems described above were very public problems – all of the major players of the day were aware of them, and historians of science have often discussed them. Another issue that arose in the context of 18th century mechanics, however, is less well known, and was an issue only for Euler.

In 1746, when thirty-nine years old, he published the *Opuscula varii argumenti* [E80], a fascinating book containing eight essays on such topics as light and colors, the material basis for thought, studies on "the nature of the smallest parts of matter," and a discourse describing the frictional force the planets feel as a result of passing through the ether. Also included was his new collection of tables describing the motion of the moon. Deriving the motion as carefully as he could using the calculus, he refined and calibrated the results using observations [Wilson 2007]. These tables became the most accurate available, predicting the location of the moon to an accuracy of 1/2 arc minute.

Flushed with the success of his work, he anxiously awaited a solar eclipse which he had calculated to occur on July 25, 1748. When the eclipse finally happened, Euler found he had correctly predicted the length and the start time of the eclipse, but that he had gotten the length of the "annular" phase (when the sun can be seen as a ring around the moon) completely wrong. Euler wrote "reflections" about this agreement and disagreement during the weeks following the eclipse [E117], and a more detailed paper trying to explain the disagreement a few months later [E141]. In each of these papers he tries, but never fully succeeds, to account for his error, and he must have begun to wonder: is the problem with his calculation the same problem that everyone was finding with the moon, Jupiter, and Saturn?

Desperate for a Solution

At the end of the 1740's, Euler, D'Alembert, and Clairaut, were rapidly running out of ideas. If their mathematics, together with Newton's law of gravity, failed to explain the motion of bodies in the cosmos, something must be wrong. And failing to find any errors in their calculations, they reluctantly turned elsewhere.

Attempted Solution 1: Questioning the inverse square law

One of the boldest attempts to reconcile the observed and theoretical descriptions of the moon's motion was made not by Euler, but Clairaut, who announced in November 1747 at a public session in the French Academy of Sciences that Newton's theory of gravity was wrong. That one of the leading mathematicians in the world would publicly make such a claim is evidence of just how dire the situation had become. Clairaut suggested that the strength of gravity was proportional not to $\frac{1}{r^2}$, but the more complicated

$$\frac{1}{r^2} + \frac{c}{r^4}$$

for some constant c. Over large distances, the c/r^4 term would effectively disappear, accounting for the utility of the inverse square law over large distances. He then began trying to find a value of c which could account for the moon's motion. He would continue to pursue this idea until May 17, 1749, when he made an equally dramatic announcement in which he claimed that Newton was right after all. [Hankins 1985]

Attempted Solution 2: Changing the shape of the moon

During the period when Clairaut was entertaining the change to Newton's law, Euler continued to try to explain the motion of the moon without accepting that Newton may have been wrong. In a surprising letter of January 20, 1748, D'Alembert wrote to Euler [Euler 1980] to suggest a new theory: perhaps the moon (or at least its distribution of mass) was not spherical. If, after all, we only see one side from the Earth, we can't know how far back it truly extends. And perhaps if it extends far enough back, the apsidal motion would indeed be $3°$, as observed. In an even more surprising response written less than four weeks later [Euler 1980], Euler says that he too had considered this idea, and had worked out the details! He found that moon would have to extend back about $2\frac{1}{2}$ Earth diameters in the direction away from us, which seemed untenable (translations and more discussion of this correspondence can be found in Bradley [Bradley 2007]).

Attempted Solution 3: Moving Berlin

A third attempt to fix the discrepancy between prediction and measurements was employed by Euler in his second paper on the solar eclipse. In E141, Euler tries to explain the failure of this predictions about the eclipse by writing

> "Yet, toward correcting the error in the duration of the annulus, I must note that in my calculations I had assumed the latitude of Berlin was $52°$, $36'$; now in fact the last observations that Mr. Kies made with the excellent Quadrant that Mr. de Maupertuis gave to the Academy only gave its elevation to be $52°$, $31'$, $30''$, so I had placed Berlin too far North by $4', 30''$. Just glancing at the chart of this Eclipse published at Nurnberg offers the assurance that if Berlin were situated $4'$ more to the north, the duration of the annulus would have considerably lengthened and would have been very close to my calculation." [E141]

In other words, like lost drivers around the world, Euler blames his maps. Today is it easy, using a software program like Starry Night, to recreate the eclipse, and look at the effect of moving about $4'$ north from Berlin. A arc minute on the surface of the Earth roughly corresponds to a mile, and moving four miles can be seen to have essentially no effect. That is, the discrepancy in Euler's maps cannot be the problem. Euler himself doesn't redo his calculations with the new value, but merely reworks them using a different method, and – hindsight really is easier than foresight – gets the right answer. In the end, though, this is no answer at all. Euler's calculations still fail to predict the motion of the moon.

A Solution at last

When Clairaut announced in 1749 that Newton's inverse-square law was right after all, Euler was anxious to know his solution. He could find no errors in his own calculations, and knew that either Clairaut had erred, or else had found a dramatic new method of calculation. In order to see Clairaut's work, Euler arranged for the annual prize of the Saint Petersburg Academy to be given to the paper which could best "demonstrate whether all the inequalities observed in lunar motion are in accordance with Newtonian theory" (quoted in [Wilson 2007]). Euler was on the prize committee, and copies of all the submissions were sent to him. One can imagine him tearing open the envelope with the "anonymous" submissions, shuffling through them to find Clairaut's (Euler would have recognized his style immediately), and eagerly reading it. After satisfying himself of the correctness of Clairaut's solution, he wrote two long and glowing letters to the French mathematician, praising his work and congratulating him on his success.

What, then, had Euler been doing wrong? The problem turned out to be that he (and Clairaut and d'Alembert) had failed to take into account second-order effects. One way the calculus of the time was done used differentials. Given a variable x, its value an infinitesimal moment later was written $x+\mathrm{d}x$. If $y = x^2$, then a small change in x would change y's value to

$$y + dy = (x + \mathrm{d}x)^2 = x^2 + 2x\mathrm{d}x + (\mathrm{d}x)^2.$$

Since $\mathrm{d}x$ is infinitesimal, $(\mathrm{d}x)^2$ is *really* infinitesimal, and was routinely treated as 0 and eliminated from future calculations (indeed, it is only the coefficient of the $\mathrm{d}x$ term that today we call the derivative of y). It turned out, though, that for 3-body systems (Earth-Moon-Sun or Sun-Jupiter-Saturn), second-order effects are very important. When Clairaut redid his calculations, attempting the tedious task of keeping all the second-order terms, it turned out that Newton's laws precisely matched the observed behavior of the moon, Jupiter, and Saturn.

Back to Philosophy

Letters to a German Princess

By 1750, then, the perceived problems with the law of gravity had been resolved, and Euler could turn his attention back to more philosophical questions. Mathematicians familiar with Euler's work only through today's textbooks are often surprised to discover that Euler was also a writer of "popular science" books – a sort of 18th Century Isaac Asimov or Stephen Jay Gould. His *Letters to a German Princess* [E343] contain explanations of scores of topics spanning the science of the day, from logic to optics, from music to the origin of evil. The letters are available online (in English translation) and are well-worth reading even today. In letters 45-53 (among others), Euler treats gravity.

It is in this work that we see a more mature Euler. He wrote these letters in 1760, when the problems that beset Newton's theory in the 1740's had long since been resolved. Nevertheless, he still shows some of the reservations he had twenty-five years earlier with accepting gravity as a first principle. Indeed, in letter 46 he writes that "Philosophers have warmly disputed, whether there actually exists a power which acts in an invisible manner upon bodies: or whether it be an internal quality inherent in the very nature of the bodies..."

Euler's work is meant to be read by a lay audience, and he gives several visual images for the reader to think about. In one wonderful image, Euler writes: "There is a cellar under my apartment, but the floor supports me, and preserves me from falling into it. Were the floor suddenly to crumble away, and the arch of the cellar to tumble in at the same time, I must infallibly by precipitated into it, because my body is heavy, like all other bodies with which we are acquainted."

Little shows Euler to be product of his time more than his explanation of this last qualifier: "I say, *with which we are acquainted*, for there may, perhaps, be bodies destitute of weight: such as, possibly, light itself, the elementary fire, the electric fluid, or that of the magnet: or such as the bodies of angels which have formerly appeared to men." At the time, light, fire, electricity, and magnetism all were thought by many people to consist of weightless (imponderous) bodies which diffused through other matter, granting it properties like heat or electricity. Euler's claim that angels are weightless (and, for that matter, that they have "appeared to men") was less grounded by scientific evidence.

Euler's later work

Euler would continue to work on problems concerning gravity until the end of his life, even famously working on the orbit of Uranus on the day he died. In 1772 he published a third Lunar Theory, in which he used his new understanding of gravitational calculations to find more accurate descriptions of the motion of the moon. He never did fully satisfy his own curiosity about the cause of gravity but since the scientific community had to wait until Einstein to get an explanation better than Newton's, he can perhaps be forgiven this failure. Had he somehow lived to the 20th century, however, we can be sure that Euler would have found fertile ground for further research in our modern views on gravity as well.

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How Euler Did It is updated each month.