MATH 126 Winter 18: Homework 1

posted on Jan. 12; due by Jan. 19

1. [5 points each]
   (1) Show that the hyperbolic set $\{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2_+ \;:\; x_1 x_2 \geq 1\}$ is convex without using the inequality hint in (2). (Hint: Consider the two cases $(x_1 - y_1)(x_2 - y_2) \geq 0$ and $(x_1 - y_1)(x_2 - y_2) < 0$.)
   (2) Show that the general hyperbolic set $\{\boldsymbol{x} \in \mathbb{R}^d_+ \;:\; \prod_{i=1}^d x_i \geq 1\}$ is convex. (Hint: If $a, b \geq 0$ and $0 \leq \theta \leq 1$, then $a^\theta b^{1-\theta} \leq \theta a + (1-\theta) b$.)

2. [2 points each] Show that the following operators learned in the class (see the lecture note for details):
   (1) intersection
   (2) product
   (3) weighted summation
   (4) affine image
   (5) inverse affine image
   preserve the convexity of sets.

3. [10 points] Show that $f(\boldsymbol{X}) = (\mathsf{det}\, \boldsymbol{X})^{1/d}$ is concave on $\mathrm{dom}\, f = \mathbb{S}^d_{++}$. (Hint: See [1, Section 3.1.5].)

4. [5 points each]
   (1) For given $\boldsymbol{a}_i \in \mathbb{R}^d_{++}$, $b_i \in \mathbb{R}_{++}$ for $i = 1, \ldots, p$, show that the following optimization problem:

$$\min_{\boldsymbol{x} \in \mathbb{R}^d} \max_{i=1,\ldots,p} |\log\big(\boldsymbol{a}_i^\top \boldsymbol{x}\big) - \log(b_i)| \quad \text{subject to } 1 \leq x_i \leq 2, \quad i = 1, \ldots, d,$$

   is equivalent to the following problem:

$$\min_{\boldsymbol{x} \in \mathbb{R}^d} \max_{i=1,\ldots,p} \phi\left(\frac{\boldsymbol{a}_i^\top \boldsymbol{x}}{b_i}\right) \quad \text{subject to } 1 \leq x_i \leq 2, \quad i = 1, \ldots, d, \tag{P}$$

   where $\phi(u) = \max\{u, 1/u\}$ for $u > 0$. (Hint: Here, the problems are called *equivalent* when they have same optimal sets.)
   (2) Show that the problem (P) is convex. (Hint: See [1, Section 3.2.4].)

References

1. Boyd, S., Vandenberghe, L.: Convex optimization. Cambridge, UK (2004). URL http://www.stanford.edu/~boyd/cvxbook.html