MATH 126 Winter 18: Homework 5

posted on Feb. 9; due by Feb. 16

1. [10 points] Show that the proximal operator of the max function $\phi(\boldsymbol{x}) = \max_{i=1,\ldots,m} x_i$ is
$$\text{prox}_\phi(\boldsymbol{x}) = \min\{\boldsymbol{x},\ t_* \mathbf{1}\}$$
where $t_* \in \mathbb{R}$ satisfies $\sum_{i=1}^m [x_i - t_*]_+ = 1$. Use the epigraph form of the proximal operator, and its corresponding KKT conditions.

2. [10 points] Derive a closed-form expression of the proximal operator of the following function:
$$\phi(\boldsymbol{x}) = \begin{cases} 0, & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ \infty, & \text{otherwise,} \end{cases}$$
where $\boldsymbol{A}$ has linearly independent rows. Use the KKT conditions.

3. [5 points each] Consider the problem
$$\begin{aligned} &\min\ -x_1 x_2 x_3 \\ &\text{subject to } x_1 + 3x_2 + 6x_3 \leq 54, \\ &\qquad x_1, x_2, x_3 \geq 0. \end{aligned}$$
   (1) Write the KKT conditions for the problem.
   (2) Find the optimal solution of the problem.

4. [10 points] Consider the problem:
$$\begin{aligned} \min_{\boldsymbol{x} \in \mathbb{R}^d} &\left\{ \frac{1}{2} ||\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}||_2^2 + \frac{\gamma}{2} ||\boldsymbol{x}||_2^2 \right\} \\ \text{subject to } &\boldsymbol{C}\boldsymbol{x} = \boldsymbol{d}. \end{aligned}$$
where we assume that $\boldsymbol{C}$ has linearly independent rows and $\gamma > 0$. Derive the optimal point $\boldsymbol{x}_*$ using the KKT conditions.