

NAME: _____

MATH 1 MIDTERM 2

November 7, 2007

INSTRUCTIONS: This is a closed book, closed notes exam. You are not to provide or receive help from any outside source during the exam.

- Print your name clearly in the space provided.
- You may not use a calculator.

HONOR STATEMENT:

I have neither given nor received help on this exam, and all of the answers are my own.

Signature

| Question | Points | Score |
|----------|--------|-------|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 10 | |
| 4 | 6 | |
| 5 | 4 | |
| 6 | 6 | |
| 7 | 4 | |
| 8 | 4 | |
| 9 | 16 | |
| 10 | 10 | |
| Total: | 100 | |

1. Compute the following derivatives.

(a) [2 points] $\frac{d}{dx}(4x^5 - 3x^3 + 11x^2 - 1 + \sec(x))$

"derivative of a sum is sum of derivatives"

$$\begin{aligned} &= \frac{d}{dx}(4x^5) - \frac{d}{dx}(3x^3) + \frac{d}{dx}(11x^2) - \frac{d}{dx}(1) + \frac{d}{dx}(\sec(x)) \\ &\boxed{= 20x^4 - 9x^2 + 22x - 0 + \sec(x)\tan(x)} \end{aligned}$$

(b) [2 points] $(\sqrt{7}x^3)'$

"constants pull out"

$$= \sqrt{7}(x^3)' = \boxed{\sqrt{7} \cdot 3 \cdot x^2}$$

(c) [2 points] $(\sqrt{7}x^3)'$

$$\begin{aligned} &= (\sqrt{7}\sqrt{x^3})' = (\sqrt{7}(x^3)^{1/2})' = (\sqrt{7}x^{3/2})' = \sqrt{7}(x^{3/2})' \\ &\boxed{= \sqrt{7} \cdot \frac{3}{2} x^{1/2}} \end{aligned}$$

(d) [2 points] $\frac{d}{dx}(x^\pi + x^e)$

$$= \frac{d}{dx}(x^\pi) + \frac{d}{dx}(x^e) = \boxed{\pi \cdot x^{\pi-1} + e \cdot x^{e-1}}$$

(e) [2 points] Use the product rule to compute $\frac{d}{dx}(e^{2x})$.

$$= \frac{d}{dx}(e^x \cdot e^x) = \frac{d}{dx}(e^x) \cdot e^x + e^x \cdot \frac{d}{dx}(e^x)$$

$$\begin{aligned} &= e^x e^x + e^x e^x \\ &= e^{2x} + e^{2x} = \boxed{2e^{2x}} \end{aligned}$$

(f) [2 points] $\left(\frac{e^x}{x^e}\right)' \quad \boxed{\text{quotient rule } \left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}}$

$$= \frac{e^x x^e - e^x e^{e-1} e^x}{(x^e)^2}$$

(g) [2 points] $(e^{32})'$

e^{32} is a constant, so its deriv. is zero
i.e. $(e^{32})' = \boxed{0}$

(h) [2 points] Use the quotient rule to compute $(\csc(x))'$.

$$\csc(x) = \frac{1}{\sin(x)}$$

$$\left(\frac{1}{\sin(x)}\right)' = \frac{0 \cdot \sin(x) - \cos(x) \cdot 1}{\sin^2(x)} = \frac{-\cos(x)}{\sin(x)} \cdot \frac{1}{\sin(x)} \\ = -\cot(x) \cdot \csc(x)$$

(i) [2 points] $(\tan(x) - \cot(x))'$

$$(\tan(x))' = \left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{\cos^2(x) - (-\sin(x))\sin(x)}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

similarly $(\cot(x))' = -\csc^2(x)$

$$= \frac{1}{\cos^2(x)} = \sec^2(x)$$

so $(\tan(x) - \cot(x))' = \sec^2(x) - (-\csc^2(x))$

(j) [2 points] $(\cos(x) \tan(x))' = \boxed{\sec^2(x) + \csc^2(x)}$

$\cos \cancel{\tan}(x) = \cancel{\cos(x)} \frac{\sin(x)}{\cos(x)} = \sin(x)$

$$\Rightarrow \boxed{(\sin(x))' = \cos(x)}$$

2. Find the following limits. Write DNE if the limit does not exist.

(a) [2 points] $\lim_{x \rightarrow -\infty} \frac{1}{x} = \circ$

(b) [2 points] $\lim_{x \rightarrow \infty} \frac{1}{x^3} = \circ$

(c) [2 points] $\lim_{x \rightarrow -\infty} e^x = \circ$

(d) [2 points] $\lim_{x \rightarrow \infty} e^x = \infty$

(e) [2 points] $\lim_{x \rightarrow -\infty} \sin(x)$ DNE

(f) [2 points] $\lim_{x \rightarrow \infty} \frac{4x^2 + 1}{x - 1} = \infty$ (top fcn "wins")

(g) [2 points] $\lim_{x \rightarrow \infty} \frac{x - 1}{4x^2 + 1} = \circ$ (Bottom fcn "wins")

(h) [2 points] $\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 + 1}}{x - 1} = 2$

$$\begin{aligned} &= \lim_{x \rightarrow \infty} \frac{\sqrt{4x^2}}{x} = \lim_{x \rightarrow \infty} \frac{\cancel{\sqrt{4}} \cancel{x}}{\cancel{x}} = 2 \end{aligned}$$

$$(i) [2 \text{ points}] \lim_{x \rightarrow 0} \frac{\sin(x)}{\cos(x) - 1} = \lim_{x \rightarrow 0} \frac{\sin(x)(\cos(x) + 1)}{(\cos(x) - 1)(\cos(x) + 1)}$$

$$= \lim_{x \rightarrow 0} \frac{\sin(x)\cos(x) + \sin(x)}{\cos^2(x) - 1} = \lim_{x \rightarrow 0} \frac{\sin(x)(\cos(x) + 1)}{\sin^2(x)}$$

$$= \lim_{x \rightarrow 0} \frac{\cancel{\sin(x)}}{\cancel{\sin(x)}} \cdot \frac{\cos(x) + 1}{\frac{\sin(x)}{\sin(x)}} = \frac{\lim_{x \rightarrow 0} (\cos(x) + 1)}{\lim_{x \rightarrow 0} \frac{\sin(x)}{\sin(x)}} = \frac{2}{0^+ 0^-} = \boxed{-\infty}$$

$$(j) [2 \text{ points}] \lim_{x \rightarrow 0} \frac{\sin(x)\cos(x) - \sin(x)}{x^2}$$

$$= \lim_{x \rightarrow 0} \frac{\sin(x)}{x} \cdot \lim_{x \rightarrow 0} \frac{\cos(x) - 1}{x}$$

saw in class

$$= 1 \cdot 0 = \boxed{0}$$

3. Give an example of functions with each of the following properties. The examples should all be different for full credit.

(a) [2 points] two or more vertical asymptotes.

$$f(x) = \frac{1}{x^2 - 4} \quad (\text{has vert. asymptotes } @ x = \pm 2)$$

(b) [2 points] exactly one horizontal asymptote.

$$f(x) = \frac{1}{x} \quad (\text{has horiz. asympt. } @ y = 0)$$

(c) [2 points] exactly one vertical asymptote.

$$f(x) = \frac{1}{x^2} \quad (\text{has vert. asympt. } @ x = 0)$$

(d) [2 points] exactly two horizontal asymptotes.

$$f(x) = \begin{cases} \frac{1}{x} & x > 0 \\ \left(\frac{1}{x}\right) - 1 & x < 0 \end{cases} \quad (\text{has horiz. asymptotes } @ y = 0 \text{ & } y = -1)$$

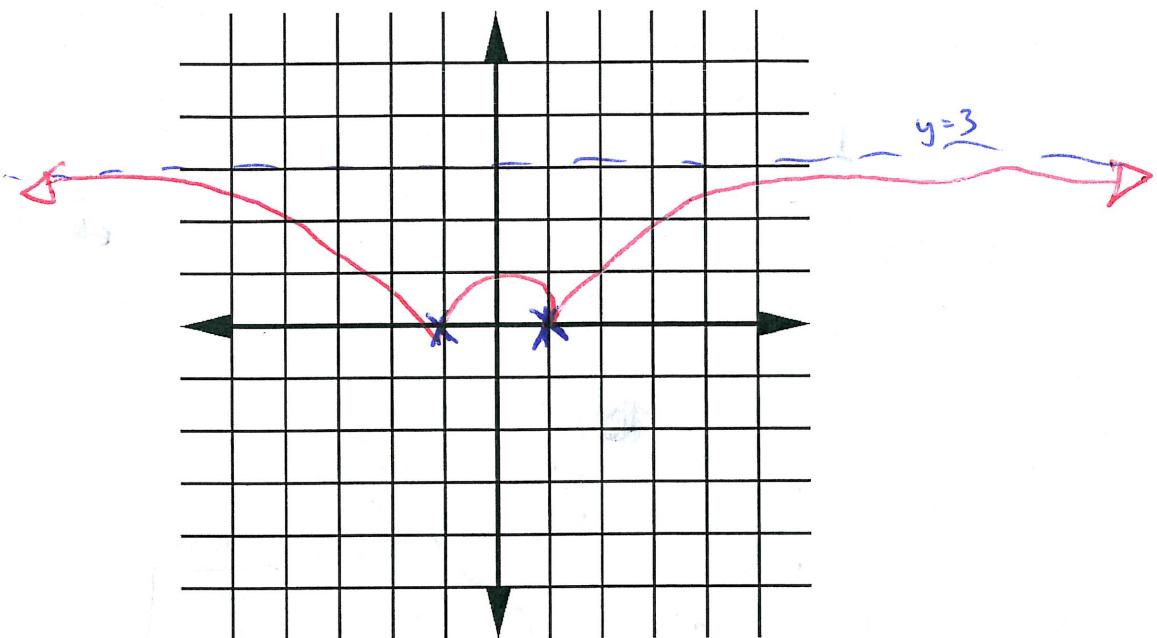
(e) [2 points] exactly one vertical asymptote and one or more horizontal asymptotes.

$$f(x) = \begin{cases} \left(\frac{1}{x}\right) + 1 & x > 0 \\ \left(\frac{1}{x}\right) - 1 & x < 0 \end{cases}$$

has vert. asymp
@ x = 0

has horizontal asymps
@ y = 1 & y = -1

4. [6 points] Sketch a function which has a horizontal asymptote at $y = 3$ and which is continuous everywhere but not differentiable at $x = -1$ and $x = 1$.



5. [4 points] Let f and g be functions with values $f(3) = 4$, $g(3) = 2$, $f'(3) = 10$, and $g'(3) = 1$. Find the derivative of $5f(x) + x^2g(x)$ at $x = 3$.

derivative:

$$\rightarrow 5f'(x) + 2xg(x) + x^2g'(x)$$

plugging in $x = 3$ gives

$$5f'(3) + 2(3) \cdot g(3) + (3)^2 \cdot g'(3)$$

$$= 5 \cdot 10 + 2 \cdot 3 \cdot 2 + 9 \cdot 1$$

$$= 50 + 12 + 9$$

$$= 62 + 9 = 71$$

6. Let $f(x) = \frac{1}{x}$.

(a) [4 points] Compute $f'(x)$ using the limit definition of the derivative.

No credit will be given for using the power or quotient rule.

$$\begin{aligned}
 f'(x) &= \lim_{b \rightarrow 0} \frac{f(x+b) - f(x)}{b} = \lim_{b \rightarrow 0} \frac{\left(\frac{1}{x+b}\right) - \left(\frac{1}{x}\right)}{b} \\
 &= \lim_{b \rightarrow 0} \frac{\left(\frac{1}{x+b}\right)\left(\frac{x}{x}\right) - \left(\frac{1}{x}\right)\left(\frac{x+b}{x+b}\right)}{b} \quad \text{getting a common denominator} \\
 &= \lim_{b \rightarrow 0} \frac{\frac{x}{x^2+xb} - \frac{x+b}{x^2+xb}}{b} = \lim_{b \rightarrow 0} \frac{-b}{x^2+xb} \quad \text{(the } b\text{'s cancel)} \\
 &= \lim_{b \rightarrow 0} \frac{-1}{x^2+xb} = \frac{-1}{x^2} \\
 &= -\boxed{x^{-2}}
 \end{aligned}$$

(b) [2 points] Use your work above to find the slope of f at $x = 4$.

$$\begin{aligned}
 f'(4) &= -\boxed{4^{-2}} \\
 &= -\boxed{\frac{1}{16}}
 \end{aligned}$$

7. [4 points] Find the equation of the line tangent to the curve $y = x^2 e^x + 2 \sin(x)$ at the point $x = 0$.

$$y' = 2xe^x + x^2 e^x + 2 \cos(x)$$

$$\text{At } x=0 \quad y'(0) = 0 \cancel{x^0} + \cancel{0^2 e^0} + 2 \cos(0) = 2 = m$$

This is the slope of the tangent line.

$$\begin{aligned} \text{The point is } (0, y(0)) &= (0, 0^2 e^0 + 2 \sin(0)) \\ &= (0, 0) = (x_1, y_1) \end{aligned}$$

$$\text{Pt-slope } y - y_1 = m(x - x_1) \rightarrow y - 0 = 2(x - 0) \rightarrow \boxed{y = 2x}$$

8. [4 points] Let

$$f(x) = \begin{cases} \sqrt{-x+9} & \text{if } x < 0 \\ 3+x^2 & \text{if } 0 \leq x \leq a \\ (x-3)^2 & \text{if } x > a \end{cases}$$

Find the value of a that makes f continuous everywhere.

We need $3+(a)^2 = (a-3)^2$ to have f conts.

So we solve for " a ".

$$\begin{array}{r} 3+a^2 = a^2 - 6a + 9 \\ -a^2 \quad -a^2 \\ \hline 3 = -6a + 9 \end{array}$$

$$\begin{array}{r} 3 = -6a + 9 \\ -9 \quad -9 \\ \hline -6 = -6a \end{array}$$

$$\begin{array}{r} -6 = -6a \\ \cancel{-6} \quad \cancel{-6} \\ \hline 1 = a \end{array}$$

$$\boxed{1 = a}$$

9. Short answer:

- (a) [2 points] The equation $\lim_{x \rightarrow a} f(x) = f(\lim_{x \rightarrow a} x)$ is true whenever f is continuous at a .

- (b) [2 points] The expression $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ is the derivative of f at the point a .
or "slope of the tangent line"

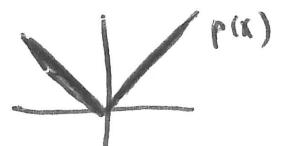
- (c) [4 points] Which of the following statements is true:

True → 1. If a function f is differentiable at the point a , then f is continuous at the point a .

→ 2. If a function f is continuous at the point a , then f is differentiable at the point a .

Give an example of a function which shows the other statement is false.

$$p(x) = |x| = \begin{cases} -x & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases}$$



is conts. @ $x=0$ but not differentiable

- (d) [4 points] What does the derivative of a function at a point represent geometrically?

The slope of the tangent line
the function @ that point

- (e) [4 points] How is the number e defined?

It is the number such that

$$\ln(e) = 1$$

10. [10 points] In what follows, $f(x)$ and $g(x)$ are continuous and differentiable, and c is a constant. Label each statement as true or false. A statement is considered true only if it is always true; otherwise it is false. (one point each)

(a) True $\lim_{x \rightarrow a} cf(x) = c \lim_{x \rightarrow a} f(x)$ (limit law)

(b) False $\lim_{x \rightarrow a} c = a$

(c) False $\left(\frac{f(x)}{g(x)} \right)' = \frac{f(x)g'(x) - g(x)f'(x)}{[g(x)]^2}$ Reverse terms upshifts

(d) True Assume $\lim_{x \rightarrow a} g(x) \neq 0$. Then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$.

(e) False $\sin(x) + \cos(x) = 1$ (The one you're thinking of is $\sin^2(x) + \cos^2(x) = 1$)

(f) False $\lim_{x \rightarrow a+b} f(x) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow b} f(x)$ (This is VERY false.)

(g) True $\lim_{x \rightarrow a} x = a$ $f(x) = x$ is continuous, so you can just "plug in" a for x .

(h) True $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$ limit law

(i) False $\lim_{x \rightarrow a} f(x)g(x) = g(x) \lim_{x \rightarrow a} f(x) + f(x) \lim_{x \rightarrow a} g(x)$

The Derivative Product rule works like this

(j) True $\frac{d}{dx} [f(x)g(x)] = g(x) \frac{d}{dx} [f(x)] + f(x) \frac{d}{dx} [g(x)]$ but NOT the limit law

See previous answer (for (i))