Mathematics 33

Homework Assignment #2

Due Wednesday, April 12

1. (4 points). The Cauchy inequality for any vectors $\mathbf{a}$ and $\mathbf{b}$ from a linear space says that $(\mathbf{a},\mathbf{b}) \leq \|\mathbf{a}\|\,\|\mathbf{b}\|$. Use this inequality to prove the triangle inequality $\|\mathbf{a}+\mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$. Why the latter inequality is called 'triangle'?

Solution. It is easier to prove that $\|\mathbf{a}+\mathbf{b}\|^2 \leq (\|\mathbf{a}\| + \|\mathbf{b}\|)^2$. Expanding the left-hand side of this inequality we obtain

$$\|\mathbf{a}+\mathbf{b}\|^2 = (\mathbf{a}+\mathbf{b},\mathbf{a}+\mathbf{b}) = \|\mathbf{a}\|^2 + 2(\mathbf{a},\mathbf{b}) + \|\mathbf{b}\|^2 .$$

Expanding the right-hand side we obtain

$$(\|\mathbf{a}\| + \|\mathbf{b}\|)^2 = \|\mathbf{a}\|^2 + 2\,\|\mathbf{a}\|\,\|\mathbf{b}\| + \|\mathbf{b}\|^2 .$$

But $(\mathbf{a},\mathbf{b}) \leq \|\mathbf{a}\|\,\|\mathbf{b}\|$ so that $\|\mathbf{a}+\mathbf{b}\|^2 \leq (\|\mathbf{a}\| + \|\mathbf{b}\|)^2$. This inequlity may be interpreted as follows. Let $\mathbf{a}$ and $\mathbf{b}$ be two sides of a triangle and $\mathbf{a}+\mathbf{b}$ be the third side. Then the sum of the lengths of two sides is greater than the length of the third side.

2. (7 points). Five types of operators on R^2 are considered: proportional scaling, general scaling, translation, reflection about x, rotation with angle θ. Using vector/matrix representation, for which operators the product of two operators is of the same type? For which operators the sum of two operators is of the same type?

Solution. All of the above operators can be represented as $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b}$ where $\mathbf{x}$ is the original vector on the plane, $\mathbf{A}$ is a 2×2 matrix and $\mathbf{b}$ is a translation vector ($\mathbf{b} = \mathbf{0}$ for all operators but translation). So, to verify if the sum/product of two operators is of the same type we need to verify whether the sum/product of two matrices is of the same type.

1. Proportional scaling (yes):

$$\mathbf{A} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \tau & 0 \\ 0 & \tau \end{bmatrix}, \mathbf{AB} = \begin{bmatrix} \lambda\tau & 0 \\ 0 & \lambda\tau \end{bmatrix}, \mathbf{A}+\mathbf{B} = \begin{bmatrix} \lambda+\tau & 0 \\ 0 & \lambda+\tau \end{bmatrix}.$$

2. General scaling (yes):

$$\mathbf{A}=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix},\mathbf{B}=\begin{bmatrix}\tau_1 & 0\\ 0 & \tau_2\end{bmatrix},\mathbf{AB}=\begin{bmatrix}\lambda_1\tau_1 & 0\\ 0 & \lambda_2\tau_2\end{bmatrix},\mathbf{A}+\mathbf{B}=\begin{bmatrix}\lambda_1+\tau_1 & 0\\ 0 & \lambda_2+\tau_2\end{bmatrix}.$$

3. Reflection about x (no):

$$\mathbf{A}=\mathbf{B}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\mathbf{AB}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\mathbf{A}+\mathbf{B}=\begin{bmatrix}2 & 0\\ 0 & -2\end{bmatrix}.$$

4. Translation (yes/no). The product of two translations is a translation. This is not true for the sum because $(\mathbf{x}+\mathbf{b})+(\mathbf{x}+\mathbf{c})=2\mathbf{x}+(\mathbf{b}+\mathbf{c})$ is not a translation operator.

5. Rotation (yes/no). The product of two rotations by angle θ_1 and θ_2 is a rotation by angle $\theta_1+\theta_2$. The sum of two rotations is not a rotation because it does not leave the norm the same.

3. (5 points). An $n\times n$ matrix $\mathbf{P}$ is called orthogonal if its vector-columns constitute an orthonormal basis in R^n. Show that the linear transformation with matrix $\mathbf{P}$ does not change the distance.

Solution. For an orthogonal matrix $\mathbf{P}$ we have $\mathbf{P}'\mathbf{P}=\mathbf{I}$, the identity matrix. Let $\mathbf{x}$ and $\mathbf{y}$ be any vectors and $\mathbf{a}=\mathbf{P}\mathbf{x}$ and $\mathbf{b}=\mathbf{P}\mathbf{y}$ be the according transformed vectors. We want to show that $\|\mathbf{x}-\mathbf{y}\|=\|\mathbf{a}-\mathbf{b}\|$. We have

$$\|\mathbf{a}-\mathbf{b}\|^2=\|\mathbf{P}\mathbf{x}-\mathbf{P}\mathbf{y}\|^2=\|\mathbf{P}(\mathbf{x}-\mathbf{y})\|^2=(\mathbf{x}-\mathbf{y})'\mathbf{P}'\mathbf{P}(\mathbf{x}-\mathbf{y})=(\mathbf{x}-\mathbf{y})'(\mathbf{x}-\mathbf{y})=\|\mathbf{x}-\mathbf{y}\|^2.$$

4. (4 points). Let $\mathbf{p}_1,\mathbf{p}_2,...,\mathbf{p}_n$ are orthogonal vectors in R^k $(k\geq n)$ and $\mathbf{x}\in R^k$. Define the angle between $\mathbf{x}$ and $\{\mathbf{p}_i\}$ and provide the formula. Is it true that the angle is zero only if $\mathbf{x}$ is one of $\mathbf{p}_i$?

Solution. We define the angle between $\mathbf{x}$ and $\{\mathbf{p}_i\}$ as the angle between $\mathbf{x}$ and the projection of $\mathbf{x}$ on the linear subspace spanned by the vectors $\{\mathbf{p}_i\}$. The projection vector, $\widehat{\mathbf{x}}$ is a linear combination of $\{\mathbf{p}_i\}$, or more precisely, $\widehat{\mathbf{x}}=\mathbf{P}\boldsymbol{\lambda}$ where $\boldsymbol{\lambda}=(\mathbf{P}'\mathbf{P})^{-1}\mathbf{P}'\mathbf{x}$, i.e. $\widehat{\mathbf{x}}=(\mathbf{P}'\mathbf{P})^{-1}\mathbf{P}'\mathbf{x}$. Then the angle is

$$\arccos\frac{\|\widehat{\mathbf{x}}\|}{\|\mathbf{x}\|}=\arccos\sqrt{\frac{\mathbf{x}'\mathbf{P}(\mathbf{P}'\mathbf{P})^{-1}\mathbf{P}'\mathbf{x}}{\mathbf{x}'\mathbf{x}}}.$$