Antiderivatives and Initial Value Problems

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Definition

- An *antiderivative* of a function f on an interval I is another function F such that $F'(x) = f(x)$ for all $x \in I$.

Theorem. *Suppose that h is differentiable in an interval I and $h'(x) = 0$ for all $x \in I$. Then h is a constant function; i.e. $h(x) = C$ for all $x \in I$, where C is a constant.*

- If $F(x)$ is one antiderivative of $f(x)$, then any other antiderivative must be of the form $F(x) + C$, where C is a constant.

- We refer to $F(x) + C$ as the *general antiderivative* and denote it by

$$\int f(x)\mathrm{d}x.$$

- We call this the *indefinit integral* of f

- The indefinite integral is the general antiderivative of f

$$\int f(x)\mathsf{d}x = F(x) + C$$

Examples

$$\int x^r dx = \frac{x^{r+1}}{r+1} + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int e^x \, dx = e^x + C$$

$$\int \frac{1}{x} \, dx = \ln |x| + C$$

Theorem. *Suppose the functions f and g both have antiderivatives on the interval I. Then for any constant a, the function $af + g$ has an antiderivative on I and*

$$\int (af + g)\,\mathsf{d}x = a \int f(x)\,\mathsf{d}x + \int g(x)\,\mathsf{d}x$$

Differential Equations

- Finding an antiderivative of f can be thought of as solving the equation $\frac{dy}{dx} = f(x)$ for the unknown function y.
- Such equations that involve one or more derivatives of an unknown function are called *differential equations*.
- Solving a differential equation means finding a function $f(x)$ that satisfies the equation identically when substituted for the unknown function y.

Examples

- Solve the differential equation $y' = 2x + sinx$.

- Solve the second-order differential equation

$$\frac{d^2y}{dx^2} + y = 0.$$

Definition

- An *initial-value problem* is a differential equation together with enough additional conditions to specify the constants of integration that appear in the general solution.

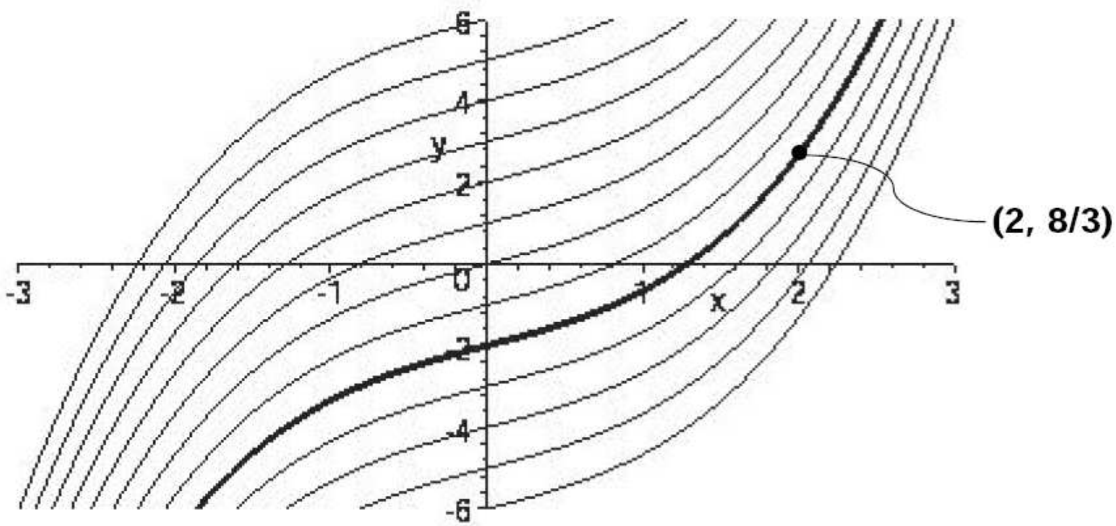
- The *particular solution of the problem* is then a function that satisfies both the differential equation and also the additional conditions.

Example

- Solve the initial value problem $\frac{dx}{dt} = 2t + \sin t$ subject to the initial condition $x(0) = 0$.

Example

- Solve the differential equation $y' = x^2 + 1$ subject to the additional condition $y(2) = 8/3$.



Example

- Solve the initial-value problem

$$y^{''} = \cos x,\ y^{'}(\frac{\pi}{2}) = 2, y(\frac{\pi}{2}) = \pi.$$

Velocity and Acceleration

An object dropped from a cliff has acceleration $a = -9.8\,m/sec^2$ under the influence of gravity. Let its height at time t be given by $s(t)$. Then its motion is described by the initial-value problem

$$\frac{d^2s}{dt^2} = -9.8,\ s(0) = s_0,\ s^{'}(0) = v_0.$$

Example

- Suppose that a baseball is thrown upward from the roof of a 100 meter high building. It hits the street below eight seconds later. What was the initial velocity of the baseball, and how high did it rise above the street before beginning its descent?

Example

- Suppose an object moves on the x-axis so that its position at time t is given by $x(t) = 2t^3 - 9t^2 + 12t + 6$, $-1 < t < 1$. When is the object at rest? When is it moving to the right? When is it speeding up? When is it slowing down?