Case Study: Torricelli's Law

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Objective

- To determine how long it would take a tank of given dimensions to empty its liquid contents through a bottom outlet hole.

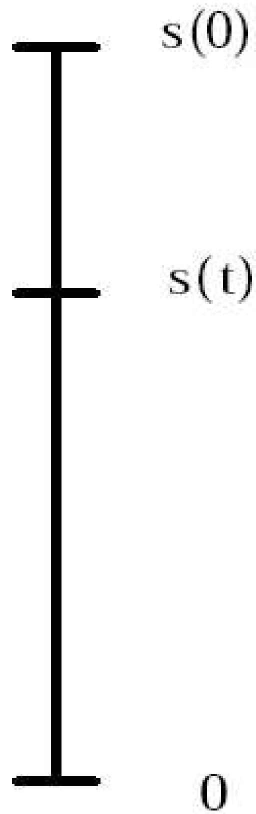
- We will take note of certain *physical laws.*

- Use these laws to set up a differential equation that can be solved using methods that we have learned.

Background: Falling Objects

- Let m be the mass of a falling object
- let g be the acceleration due to gravity.
- Then $g = 9.8$ meters per second squared, or $g = 32.2$ feet per second squared.

- Let $s(t), s(t) \geq 0$ for all t, be the position of the object above the ground at time t according to the following coordinate axis:



- If a is the acceleration and $v(t)$ is the velocity of the object at time t, then we have:

$$\begin{aligned} a &= -g \\ v &= -gt \\ s &= -\frac{gt^2}{2} + s_0 \end{aligned}$$

- We can find the final velocity v_f at the final time t_f

$$v_f = -\sqrt{2gs_0}$$

- We can also write the Initial Value Problem that gives the velocity $v = \frac{ds}{dt}$ as a function of s:

$$\frac{ds}{dt} = -\sqrt{2gs}, \quad s(0) = s_0.$$

The CSC: Torricelli's Law

Water in an open tank will flow out through a small hole in the bottom with the velocity it would acquire in falling freely from the water level to the hole.

Setup

- Suppose a cylindrical tank with cross-sectional area A has an outlet hole in the bottom,
- suppose that $h(t)$ is the height of water above the outlet at time t,
- $a$is the area of the outlet hole,
- $V(t)$is the remaining fluid volume at time t.

- Consider now the change in volume of water in the tank from time t to time $t + \Delta t$.

- This equals the amount of water that flows out through the outlet hole in time $[t, t + \Delta t]$.

- By Torricelli, the initial velocity of a drop is $\sqrt{2gh}$.

- Thus, the volume of the tube in the interval $[t, t + \Delta t]$ is approximately $a\sqrt{2gh}\Delta t$.

- We get

$$\begin{aligned} V(t) - V(t+\Delta t) &\simeq a\sqrt{2gh}\Delta t \\ \frac{dV}{dt} &= -a\sqrt{2gh} \end{aligned}$$

- We also have that the volume V of remaining water at time t equals the cross-sectional area A of the tank times the height h of water at time t.

- The rate of change of volume with respect to time is the cross-sectional area of the tank times the rate of change of the height of water with respect to time.

- Putting these comments together with the above equation for $\frac{dV}{dt}$ will give a differential equation involving $\frac{dh}{dt}$ and h.